A NOVEL APPROACH TO BOUNDING THE INTEGRATOR OUTPUTS OF SECOND ORDER SIGMA-DELTA CONVERTERS

Ronan Farrell *Orla Feely*
Dept of Electronic & Electrical Engineering,
University College Dublin, Dublin 4, Ireland.
Ronan.Farrell@ucd.ie
Orla.Feely@ucd.ie

ABSTRACT

The development, to date, of the bounds for the outputs of the integrators of second order sigma-delta modulators has relied on a mixture of theoretical analysis and extensive use of simulation. This paper presents a new approach that uses a combination of geometrical techniques and the characterization of the trajectories of the system in state space. With this approach it is possible to develop flexible but tight results for the standard second order modulator with constant input, without the need for simulation. This approach is flexible and can be adapted for other second order architectures.

1. INTRODUCTION

Sigma-delta ($\Sigma\Delta$) modulators are playing an increasing role in modern analog-to-digital converters. This increasing role is driven by the affinity between VLSI technology and $\Sigma\Delta$ modulators. Much work [1] has been done to develop analytical results for many of the design parameters of the second order converter: parameters such as the voltage span required for the integrators, the range of stability, and the effect of dither and non-unity pole positions for the integrators. The area on which this paper will focus is that of bounding the outputs of the integrators.

Second order $\Sigma\Delta$ modulators have two discrete-time integrators, the output of the second integrator generally being the input for the quantizer.

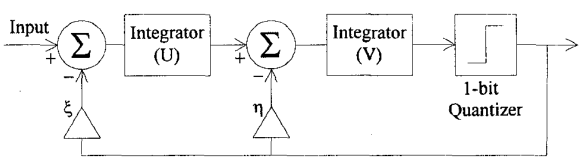

Figure 1: Standard second order sigma-delta 1-bit modulator with variable feedback gains ξ, η.

Many attempts have been made to develop tight bounds on the values that the outputs of the integrators may take. These bounds are important as the maximum required voltage is a significant design consideration, for example, dictating capacitor areas and device saturation levels. The techniques used for the development of upper bounds have included computational algorithms [2], non-linear dynamics [3], and geometric analyses [4], [5]. The approach that this paper will present uses an exact characterization of the system's trajectories within a half-plane. This is used in conjunction with geometric analysis to express an upper bound on the maximum of the output of the second integrator, V_{max}, in terms of the maximum of the first, U_{max}. To develop the maximum of the output of the first integrator, a feature of the behaviour of the system upon transition is applied and an upper bound is obtained.

The paper is organized as follows. Section 2 will give a brief overview of the approach being used. Section 3 will describe the characterization of the behaviour of the system within a half-plane. Section 4 will apply a geometric feature of the trajectories to reduce the problem to finding an upper bound on the first integrator. Section 5 will describe the characterization of the behaviour of the system upon transition, and hence derive the upper bound on the output of the first integrator. Section 6 will discuss the derived bounds and compare them to results from simulation. The final section will comment on some further developments of this technique.

2. OVERVIEW OF THEORY

The approach used depends on developing an exact characterization for the behaviour of the trajectories in a half-plane defined by the sign of the second integrator. It has long been noted that the trajectories have a parabolic nature [5]. Using the fact that the maximum of such a curve occurs when the rate of change of that curve equals zero, a relationship between the maximum of the output of the second integrator and the maximum of the output of the first integrator can be developed. Analysis of the behaviour of the system at transition yields an approach for developing an upper bound on the maximum output of the first integrator.

The approach used to obtain the bounds on the second integrator, V, was based upon the parabolic nature of the trajectory. In any parabolic curve there must be a

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point where the rate of change of the curve changes sign. It was possible to develop a rate function representing the rate of change of V and then determine the point at which this function changes sign. The maximum value of the curve will be the value of the curve on the iteration before the rate function changes sign. The rate function, and hence the location of V_{max}, is highly dependent on the maximum value of U.

The approach used to obtain the bounds on the first integrator, U, was founded upon the observation that the integrator outputs stay predominantly in the half-plane in which the output of the quantizer is of the same sign as the input. When the trajectory leaves this half-plane, it takes at most a certain number of iterations to return to the dominant half-plane. For a given input it is possible to determine the maximum number of iterations that will be required for the trajectories to return to the dominant half-plane. Using this observation, which is proven in [6], it is possible to develop conditions upon U under which the number of iterations needed to return to the dominant half-plane can be obtained. With these conditions, and by calculating how much the value of U will change while outside the dominant half-plane, it is possible to obtain the maximum permissible value for U.

3. CHARACTERISING THE TRAJECTORIES

It is essential to have expressions for the values of the integrators' outputs for future iterations, with respect to some starting value. These can be developed from the state equations for the standard second order modulator, (Figure 1), with input, x_n, in the range [$-\xi$, $+\xi$].

$$\begin{aligned} U_{n+1} &= U_n + x_n - \xi\,\mathrm{sgn}(V_n) \\ V_{n+1} &= V_n + U_n - \eta\,\mathrm{sgn}(V_n) \end{aligned} \tag{1}$$

When V lies in one half-plane, say the positive half, and the input is constant, these equations reduce to a much simpler linear form.

$$\begin{aligned} U_{n+1} &= U_n + x - \xi \\ V_{n+1} &= V_n + U_n - \eta \end{aligned} \tag{2}$$

The first pair of values, U_s and V_s, in the upper half-plane, where V is positive, can be chosen as our reference positions. Using these, closed form equations predicting the values of U_n and V_n, for successive iterations can be developed, provided that the trajectories stay in the dominant half-plane.

$$\begin{aligned} U_{s+n} &= U_s + n(x-\xi) \\ V_{s+n} &= V_s + n\left[\frac{(x-\xi)(n+1)}{2} + U_s - \eta\right] \end{aligned} \tag{3}$$

It is clear that the maximum U_s is also the maximum value of U possible, U_{max}, as the magnitude of the output of U decays while within a half-plane. If V_s is examined with respect to the state equations, it is possible to develop an upper bound on V_s, dependent only on U_s.

$$V_s < U_s + \eta \tag{4}$$

4. GEOMETRIC ANALYSIS

If we look at the plot of successive values of the integrator outputs, we can identify some features. While in the upper half-plane, where V is positive, the output of the first integrator, U, decays linearly at a fixed rate, and that the output of the second integrator, V, is parabolic in nature (Fig 2). From these observations it is possible to identify the iteration at which the maximum of V occurs.

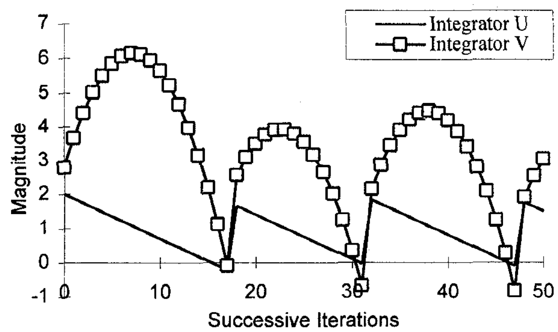

Figure 2: Plot of outputs of integrators on successive iterations, input value = 0.8

To identify the point at which the rate of change of V changes sign it is necessary to develop a rate of change function for V.

$$\begin{aligned} \Delta V_j &= V_{j+1} - V_j \\ &= U_j - \eta \end{aligned} \tag{5}$$

From this it is easy to see that the point at which the maximum occurs is at the iteration before U_j drops below η. It is possible to determine the number of iterations, N_{max}, after our starting value, U_s, when this condition is satisfied.

$$N_{\max} = \mathrm{int}\left[\frac{U_s - \eta}{\xi - x}\right] \tag{6}$$

As can be seen, N_{max} is dependent on U_s. By looking at (3) it is clear that V_{max} is dependent on N_{max} and U_s. Thus to obtain the upper bound on V_{max}, it will be necessary to use the maximum value of U_s, which is U_{max}. The result is that given U_{max} it is possible to identify the iteration upon which the maximum of V will occur, and using (3) it is possible to determine an upper bound on the value of V by determining U_{max} only.

5. SYSTEM BEHAVIOUR

In this section the behaviour of the system will be examined with a view to determining the maximum of the first integrator U. One of the characteristics of the behaviour of the trajectories is that when a trajectory leaves the dominant half-plane, it takes at most a certain number of iterations before it returns to the dominant half-plane. For different input values the maximum required number of iterations will vary. For the ideal second order system, with all feedback gains equal to one, the number required varies from two to three [3], [6].

The importance of this behaviour is that it can be shown that the number of iterations the system experiences before returning to the dominant half-plane is dependent on the value of the output of the first integrator upon the trajectory's return to the dominant half-plane, U_s. The greater the number of iterations required, the larger the value of U_s, and hence V_{max}. It is this behaviour that results in the step changes in the maxima of both U and V, for low values of input, visible in figure 3. Due to the dependence of the bounds on the number of iterations required, it is important to identify the maximum required number for a given input. Wang [3] identified this behaviour and gave some indications on where the transition between two and three iterations occurred, for the ideal system. A different technique, used by Farrell and Feely [6], was able to identify these transition points exactly. For the case of the ideal modulator, the regions of behaviour can be identified:

3 Iterations Required:	$0.0000 \le x \le 0.1111$
	$0.1667 \le x \le 0.1905$
2 Iterations Required:	elsewhere

and similarly for negative inputs, for a range of [-1,+1].

It is possible to use this characteristic to determine the maximum value of U. The key is to determine the maximum value of U, on the last iteration before leaving the dominant half-plane, that will result in obtaining the maximum possible number of iterations in the other half-plane. This will maximise U_s and V_s upon returning to the dominant half-plane.

Consider the case for values of input which require a maximum of two iterations before returning to the dominant half-plane:

Let V_n be the value of V on the last iteration of the trajectory in the dominant half-plane. Assume a positive input x, and V_n positive.

$$\begin{aligned} V_{n+1} &= V_n + U_{n+1} - \eta\,\mathrm{sgn}(V_n) \\ &= V_n + U_n + x - (\xi + \eta) \end{aligned} \tag{7}$$

$$\begin{aligned} V_{n+2} &= V_{n+1} + U_{n+2} - \eta\,\mathrm{sgn}(V_{n+1}) \\ &= V_{n+1} + U_{n+2} + \eta \\ &= V_n + 2U_n + 3x - \xi \end{aligned} \tag{8}$$

Now using the knowledge that V must spend two iterations in the negative plane, then V_{n+2} must be negative.

$$\begin{aligned} V_{n+2} &= V_n + 2U_n + 3x - \xi < 0 \\ U_n &< \tfrac{1}{2}\xi - \tfrac{3}{2}x - \tfrac{1}{2}V_n \end{aligned} \tag{9}$$

But V_n is positive, therefore

$$U_n < \tfrac{1}{2}\xi - \tfrac{3}{2}x \tag{10}$$

Given U_n and combined with the fact that the trajectory will only spend two iterations in the negative half-plane, we can say that U_{m2}, the maximum U given two iterations in the negative half-plane.

$$\begin{aligned} U_{m2} &= U_s = U_{n+3} \\ U_{m2} &< (\tfrac{1}{2}\xi - \tfrac{3}{2}x) + 3x + \xi \\ &< \tfrac{3}{2}(\xi + x) \end{aligned} \tag{11}$$

Similarly for three iterations in the negative half-plane, we get another maximum, U_{m3}.

$$U_{m3} < 2\xi + \tfrac{1}{3}\eta + 2x \tag{12}$$

It is easy to prove that the maximum associated with the maximum number of iterations allowed for a given input will result in the greatest maximum for U. Combining the correct maximum for U with (6) we are able to compute the iteration upon which the maximum of V will occur and hence using (3) the actual value of the maximum.

6. RESULTS

The results of the equations were checked against the maxima obtained by simulation. The method used to obtain the maxima was to randomly select initial conditions for the integrators, ignore the first one hundred cycles and take the maximum of the next eight hundred. This was repeated two hundred times, each with randomly selected initial conditions. Towards the higher end of the input range it was difficult to obtain the maxima. Extending the simulation run lengths partially solves the problem but there appears to be a fine structure present that limits the maxima for certain input values.

In Figure 3 the predicted bounds are plotted alongside the simulated maxima. As can be seen there is a step nature to the bounds. The steps occur at the values where there is a transition between regions requiring two and three iterations before returning to the dominant half-plane. These step changes occur at the values predicted. For lower values of input, the match between the predicted bounds and the simulated maxima is very good. The predicted bounds usually exceed the simulated maxima by less than a percent. The maxima found become asymptotically larger as the run lengths of the simulations are extended, but never exceed the predicted upper bound. Thus longer simulations may give greater indication of the tightness of the bounds.

At higher input values, greater than 0.6, there appears to be some fine structure superimposed on the general behaviour which produces a sawtooth form for the simulated maxima. The bounds presented in this paper have included no mechanism that would explain or predict this behaviour, but the peaks of the pattern come close to but do not exceed the predicted bounds. This sawtooth behaviour is one reason why care must be taken when finding maxima by simulation.

Another feature of the curve is the existence of step changes in the slope of the curve of the maxima of V. At input values of approximately 0.42 and 0.49, significant, discrete, changes in the slope of the curve can be seen. This is due to the step change in N_{max} from one integer value to another.

7. CONCLUSIONS

This paper has presented a new method for developing bounds on the standard second order sigma-delta modulator. A combination of non-linear dynamics and geometric techniques has been used. This approach has been proven to be flexible and has been used to develop bounds for other second order architectures and for some chaotic systems [6].

Further developments of this work may include the extension of this approach to a large range of systems, and possible higher order modulators. Other work may include adapting aspects of this approach for stability analysis.

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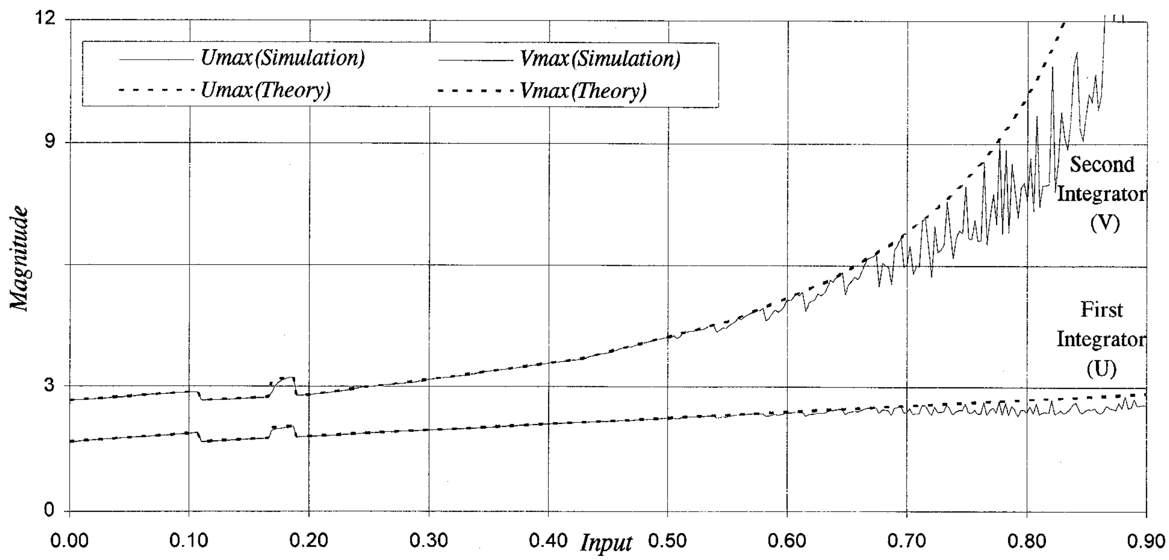

Figure 3: A comparison of the theoretical bounds against the maxima obtained by simulation.