

BANACH ALGEBRAS OF VECTOR-VALUED FUNCTIONS

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ABSTRACT. We introduce the concept of an E -valued function algebra, a type of Banach algebra that consist of continuous E -valued functions on some compact Hausdorff space, where E is a Banach algebra. We present some basic results about such algebras, having to do with the Shilov boundary and the set of peak points of some commutative E -valued function algebras. We give some specific examples.

1. INTRODUCTION AND PRELIMINARIES

We consider only algebras over the field of complex numbers, \mathbb{C} . A Banach algebra is an algebra equipped with a submultiplicative norm with respect to which it is complete. See [4, 15] for background on Banach algebras.

1.1. E -valued Function Algebras. Let X be a nonempty compact Hausdorff space, E be a unital Banach algebra and $C(X, E)$ be the space of all continuous maps from X into E . We define the *uniform norm* on $C(X, E)$ by

$$\|f\|_X := \sup_{x \in X} \|f(x)\|, \quad \forall f \in C(X, E).$$

For $f, g \in C(X, E)$ and $\lambda \in \mathbb{C}$, the pointwise operations λf , $f + g$ and fg in $C(X, E)$ are defined as usual. It is easy to see that $C(X, E)$, equipped with the norm $\|\cdot\|_X$, is a Banach algebra. If $E = \mathbb{C}$ we get the ordinary uniform function algebra $C(X) := C(X, \mathbb{C})$ of all continuous

2010 *Mathematics Subject Classification.* 46H99.

Key words and phrases. Banach algebra, Function algebra, Shilov boundary, peak point, vector-valued.

complex-valued functions on X . See any of [2, 4, 6, 13] for background on uniform algebras.

Definition 1.1. *By an E -valued function algebra on X we mean a subalgebra $A \subseteq C(X, E)$, equipped with some norm that makes it complete, such that (1) A has as an element the constant function $x \mapsto 1_E$, (2) A separates points on X , i.e. given distinct points $a, b \in X$, there exists $f \in A$ such that $f(a) \neq f(b)$, and (3) the evaluation map*

$$e_x : \begin{cases} A & \rightarrow E, \\ f & \mapsto f(x) \end{cases}$$

is continuous, for each $x \in X$.

We remark that, as it stands, condition (3) is the very weak assumption that the inclusion map $A \hookrightarrow E^X$ is continuous, where E^X is given the cartesian product topology, but it follows from the Closed Graph Theorem that if A is an E -valued function algebra on X , then the inclusion map $A \hookrightarrow C(X, E)$ is continuous, so there exists some constant $M > 0$ such that

$$\|f\|_X \leq M\|f\|_A, \quad \forall f \in A.$$

Normally, we shall use the same notation a for the element $a \in E$ and the constant function $x \mapsto a$ on X . The map $a \mapsto (x \mapsto a)$ imbeds E isomorphically as a subalgebra of each E -valued function algebra A , and we normally identify E with its image. Notice that A is commutative if and only if E is commutative.

The classical concept of a function algebra (cf. [6, 4]) corresponds, in our terminology, to a \mathbb{C} -valued function algebra. Note however that some authors (e.g. [2]) have used the term function algebra to refer only to *closed* subalgebras of $C(X)$. We do not assume that an E -valued function algebra on X is closed in the uniform norm.

An important class of examples is afforded by taking a compact set $X \subset \mathbb{C}^n$ and a commutative unital Banach algebra E , and defining the

algebra $P(X, E)$ to be the uniform closure of $E[z]|X$ in $C(X, E)$, where $E[z] = E[z_1, \dots, z_n]$ is the algebra of all polynomials in the coordinate functions z_1, \dots, z_n with coefficients in E . We can also form the algebra $R(X, E)$, defined to be the uniform closure on X of the algebra of functions of the form $p(z)/q(z)$, where $p(z) \in E[z]$, $q(z) \in E[z]$, and $q(x) \in E^{-1}$ whenever $x \in X$.

Johnson [9] considered the rather similar concept of the convolution algebra $L^1(G, A)$ of A -valued Bochner-integrable functions from a locally-compact abelian group G into a commutative Banach algebra A . The abstract Fourier transform maps such an $L^1(G, A)$ isomorphically to an A -valued algebra of continuous functions on the dual group \hat{G} .

There is also work [1] on operator-valued Fourier-Stieltjes algebras, and operator-valued maps occur in applications such as homotopy theory, but in this paper we are going to concentrate on algebras of functions into commutative algebras E . Specifically, we shall study boundaries. We proceed to define the terms.

1.2. Characters. For a commutative unital Banach algebra A , let $M(A)$ denote the set of all characters (nonzero complex-valued multiplicative linear functionals) on A . It is well-known that $M(A)$ is nonempty and that its elements are automatically continuous, with norm 1. Endowed with the weak-star topology, $M(A)$ becomes a compact Hausdorff space. The Gelfand transform of $f \in A$ is the complex-valued function \hat{f} defined by $\hat{f}(\varphi) = \varphi(f)$ on $M(A)$. Let $\hat{A} = \{\hat{f} : f \in A\}$. The algebra \hat{A} consists of \mathbb{C} -valued continuous functions on $M(A)$. Hence it is a \mathbb{C} -valued function algebra on $M(A)$ when endowed with the quotient norm. However, we shall use the notation $\|\hat{f}\|$ to denote the uniform norm of \hat{f} on $M(A)$, and with respect to this norm \hat{A} may or may not be complete.

The kernel of the map

$$\hat{\cdot} : \begin{cases} A & \rightarrow \hat{A} \\ f & \mapsto \hat{f} \end{cases}$$

is the Jacobson radical of A . The characters on $C(X)$ are exactly the evaluations $e_x : f \mapsto f(x)$, with $x \in X$, and X is homeomorphic to $M(C(X))$ with its relative weak-star topology as a subset of the dual A^* .

When A is a \mathbb{C} -valued function algebra on X , the map $x \mapsto e_x|_A$ imbeds X homeomorphically as a compact subset of $M(A)$. When this map is surjective, one calls A a *natural* \mathbb{C} -valued function algebra on X [4].

The basic example $C(X, E)$ itself was studied by Hausner, who showed that its maximal ideal space is homeomorphic to $M(E) \times X$. More precisely, he showed [7, Lemma 2]:

Lemma 1.1 (Hausner). *For each commutative Banach algebra E with identity and each compact Hausdorff space X , the map $(\phi, x) \mapsto \phi \circ e_x$ is a homeomorphism from $M(E) \times X$ onto $M(C(X, E))$.*

1.3. Shilov Boundary and Peak Points.

Definition 1.2. *A closed boundary for a commutative Banach algebra A is a closed subset $F \subseteq M(A)$ such that for each $a \in A$,*

$$\sup_{\varphi \in M(A)} |\hat{a}(\varphi)| = \sup_{\varphi \in F} |\hat{a}(\varphi)|.$$

The Shilov boundary of A is the intersection

$$\Gamma(A) = \bigcap \{F : F \text{ is a closed boundary for } A\}.$$

It can be shown [15, Theorem 15.2] or [13] that $\Gamma(A)$ is the unique minimal closed boundary for A .

Definition 1.3. *Let A be a unital commutative Banach algebra. A closed subset $S \subseteq M(A)$ is called a peak set if there exists an element*

$a \in A$ such that $\hat{a}(\varphi) = 1$ for $\varphi \in S$ and $|\hat{a}(\psi)| < 1$ for $\psi \in M(A) \setminus S$. A point $\varphi \in M(A)$ is a peak point for A if $\{\varphi\}$ is a peak set. We write $S_0(A)$ for the set of peak points for A .

Obviously, $S_0(A) \subseteq \Gamma(A)$. If $M(A)$ is metrisable, then (cf. [4, Cor. 4.3.7]) $\Gamma(A)$ is the closure of $S_0(A)$.

1.4. Main Result. Our results are about commutative algebras.

In Section 2 we introduce the concept of an *admissible quadruple* (X, E, B, \tilde{B}) , which formalises the idea of an E -valued function algebra \tilde{B} that is organically connected to a \mathbb{C} -valued function algebra B on the same space X . To such a quadruple we associate an injective map $\pi : M(E) \times X \rightarrow M(\tilde{B})$, and we say that the quadruple is *natural* when π is bijective. We prove the following result about the relation between the three Shilov boundaries that are in play:

Theorem 1.2. *Let (X, E, B, \tilde{B}) be a natural admissible quadruple. Then the associated map π maps $\Gamma(E) \times \Gamma(B)$ homeomorphically onto $\Gamma(\tilde{B})$*

We give some specific examples, and other results.

2. ADMISSIBLE QUADRUPLES

Definition 2.1. *By an admissible quadruple we mean a quadruple (X, E, B, \tilde{B}) , where*

- (1) X is a compact Hausdorff space,
- (2) E is a commutative Banach algebra with unit,
- (3) $B \subseteq C(X)$ is a natural \mathbb{C} -valued function algebra on X ,
- (4) $\tilde{B} \subseteq C(X, E)$ is an E -valued function algebra on X ,
- (5) $B \cdot E \subseteq \tilde{B}$, and
- (6) $\{\lambda \circ f, f \in \tilde{B}, \lambda \in M(E)\} \subseteq B$.

We remark that if we assume that the linear span of $B \cdot E$ is dense in \tilde{B} , then (6) is automatically true.

Condition (6) is undemanding if the Jacobson radical $J(E)$ of E is large. In fact, we are mainly interested in semisimple algebras. The meat of Theorem 1.2 is really about the quotient $E/J(E)$.

Given an admissible quadruple (X, E, B, \tilde{B}) , we define the *associated map*

$$\pi : \begin{cases} M(E) \times X & \rightarrow M(\tilde{B}) \\ (\psi, x) & \mapsto \psi \circ e_x. \end{cases}$$

Lemma 2.1. *Let (X, E, B, \tilde{B}) be an admissible quadruple. Then the associated map π is a continuous injection.*

Proof. π is injective from $M(E) \times X$ into $M(\tilde{B})$, since \hat{E} separates points on $M(E)$ and \hat{B} separates points on X . To see that π is continuous, observe that it is the composition of the (weak-star continuous) restriction map $C(X, E)^* \rightarrow (\tilde{B})^*$ with Hausner's homeomorphism $M(E) \times X \rightarrow M(C(X, E))$. \square

Corollary 2.2. *Let (X, E, B, \tilde{B}) be an admissible quadruple. Then the following are equivalent:*

- (1) *The associated map π is surjective.*
- (2) *The associated map π is bijective.*
- (3) *The associated map π is a homeomorphism of $M(E) \times X$ onto $M(\tilde{B})$*

Definition 2.2. *We say that an admissible quadruple (X, E, B, \tilde{B}) is natural if the associated map π is bijective.*

For instance, if B is a natural \mathbb{C} -valued function algebra on X , then (X, \mathbb{C}, B, B) is a natural admissible quadruple, so this terminology is a reasonable extension of the usual use of 'natural'. Further, if (X, E, B, \tilde{B}) is an admissible quadruple and E is semisimple, then \hat{E}

(with the induced norm given by $\|\hat{h}\| = \|h\|_E$) is a natural \mathbb{C} -valued function algebra on $M(E)$, so \tilde{B} is isometrically isomorphic to a \mathbb{C} -valued function algebra on $M(E) \times X$, and it is a natural \mathbb{C} -valued function algebra if and only if the quadruple is natural.

Tomiyama [14] showed that if A and B are commutative Banach algebras with identity, and some completion of C of $A \otimes B$ is also a Banach algebra, then the natural map $M(A) \times M(B) \rightarrow M(C)$ is a homeomorphism. Thus if (X, E, B, \tilde{B}) is an admissible quadruple, and the linear span of $B \cdot E$ is dense in \tilde{B} , we may apply Tomiyama's theorem with $A = E$ and $B = B$ and deduce that the quadruple is natural.

In view of the corollary, when given a natural admissible quadruple (X, E, B, \tilde{B}) , we often identify $M(E) \times X$ with $M(\tilde{B})$.

Proof of Theorem 1.2. First, we show that the image of π is a boundary for \tilde{B} : Let $f \in \tilde{B}$. Fix a character $\phi \in M(\tilde{B})$. Then ϕ is of the form $\psi \circ e_x$ for some $x \in X$ and some $\psi \in M(E)$, and then $\hat{f}(\phi) = (\psi \circ f)(x)$. Now $\psi \circ f \in B$, so there exists a point $y \in \Gamma(B)$ such that $|(\psi \circ f)(x)| \leq |(\psi \circ f)(y)|$. Next, $(\psi \circ f)(y) = \widehat{f(y)}(\psi)$, and $f(y) \in E$, so there exists a point $\chi \in \Gamma(E)$ such that $|\widehat{f(y)}(\psi)| \leq |\widehat{f(y)}(\chi)|$. Thus

$$|\hat{f}(\phi)| \leq |\hat{f}(\pi(\chi, y))|.$$

This shows that for each $f \in \tilde{B}$, \hat{f} attains its maximum modulus on the image of π , so that image is a boundary, and

$$\Gamma(\tilde{B}) \subseteq \pi(\Gamma(E) \times \Gamma(B)).$$

To see the opposite inclusion, fix $x \in \Gamma(B)$ and $\psi \in \Gamma(E)$. Let U be any neighborhood of x in X and V be any neighborhood of ψ in $M(E)$. There exists $f \in B$ such that $\|\hat{f}\| = 1$ and $|f(y)| < 1$ for all $y \in X \setminus U$. In addition there exists $v \in E$ such that $\|\hat{v}\| = 1$ and $|\phi(v)| < 1$ for all

$\phi \in M(E) \setminus V$. Now define $g : X \rightarrow E$ by $g = vf$. We have $g \in \tilde{B}$ and

$$\begin{aligned} \|\hat{g}\| &= \sup_{\phi \in M(E)} \sup_{y \in X} |\widehat{vf}(\phi \circ e_y)| \\ &= \sup_{\phi \in M(E)} \sup_{y \in X} |f(y)\phi(v)| \\ &= \sup_{\phi \in M(E)} |\phi(v)| \cdot \sup_{y \in X} |f(y)| \\ &= \|\hat{v}\| \cdot \|\hat{f}\| = 1. \end{aligned}$$

On the other hand every $\varphi' \in \pi(M(E) \times X \setminus (U \times V))$ is of the form $\varphi' = \phi \circ e_y$ with $y \in X \setminus U$ or $\phi \in M(E) \setminus V$ (or both). Therefore,

$$|\hat{g}(\varphi')| = |\varphi'(vf)| = |\phi(v)f(y)| < 1.$$

Since U and V were arbitrary neighbourhoods, it follows from [15, Theorem 15.3] that $\psi \circ e_x \in \Gamma(\tilde{B})$. Therefore $\pi(\Gamma(B) \times \Gamma(E)) \subseteq \Gamma(\tilde{B})$ and so the proof is complete. \square

2.1. Examples. (i) Let X be a compact Hausdorff space and E be a unital commutative Banach algebra. Then $(X, E, C(X), C(X, E))$ is an admissible quadruple. It is natural by Lemma 1.1, and this case of Theorem 1.2 is Hausner's theorem [7] that the Shilov boundary of $C(X, E)$ is equal to the cartesian product $X \times \Gamma(E)$.

(ii) Let (X, d) be a compact metric space and E be a commutative unital Banach algebra. For a constant $0 < \alpha \leq 1$ and a function $f : X \rightarrow E$, the Lipschitz constant of f is defined as

$$p_\alpha(f) := \sup_{\substack{x, y \in X \\ x \neq y}} \frac{\|f(x) - f(y)\|}{d(x, y)^\alpha},$$

and the E -valued big Lipschitz algebra or simply E -valued Lipschitz algebra (of order α) is defined by

$$\text{Lip}^\alpha(X, E) = \{f : X \rightarrow E : p_\alpha(f) < \infty\}.$$

Similarly, for $0 < \alpha < 1$, the E -valued little Lipschitz algebra (of order α) is defined by

$$\text{lip}^\alpha(X, E) = \left\{ f \in \text{Lip}^\alpha(X, E) : \frac{\|f(x) - f(y)\|}{d(x, y)^\alpha} \rightarrow 0 \text{ as } d(x, y) \rightarrow 0 \right\}.$$

For each $f \in \text{Lip}^\alpha(X, E)$ we define a norm by

$$\|f\|_\alpha = \|f\|_X + p_\alpha(f).$$

In [3] it was shown that $(\text{Lip}^\alpha(X, E), \|\cdot\|_\alpha)$ is a Banach algebra having $\text{lip}^\alpha(X, E)$ as a closed subalgebra. It is relatively straightforward to check that $(X, E, \text{Lip}^\alpha(X, \mathbb{C}), \text{Lip}^\alpha(X, E))$ is an admissible quadruple, for each $\alpha \in (0, 1]$, and that $(X, E, \text{lip}^\alpha(X, \mathbb{C}), \text{lip}^\alpha(X, E))$ is an admissible quadruple, for each $\alpha \in (0, 1)$. (The result that the maximal ideal space of $\text{Lip}^\alpha(X)$ is X is due originally to Sherbert [12, 4].)

The scalar-valued Lipschitz algebras are normal, because if F and K are disjoint nonempty closed subsets of X , then the function $f : x \mapsto \frac{d(x, F)}{d(x, F) + d(x, K)}$ belongs to $\text{Lip}^1(X)$. It follows [4, p. 413] that they have partitions of unity subordinate to any open covering. Applying partitions of unity and a method similar to Hausner's in [7, Lemma 1], one can see that each of these E -valued Lipschitz algebras (and little Lipschitz algebras) is dense in $C(X, E)$. Then, given a character ϕ on $\text{Lip}^\alpha(X, E)$ and a function $f \in C(X, E)$, we may choose a sequence $(f_n) \in \text{Lip}^\alpha(X, E)$ such that $\|f - f_n\|_X \rightarrow 0$. Since

$$\|\hat{h}\|_{M(\text{Lip}^\alpha(X, E))} \leq \|h\|_X$$

for each $h \in \text{Lip}^\alpha(X, E)$, the sequence $(\phi(f_n))$ is Cauchy, so we may define $\tilde{\phi}(f) = \lim_n \phi(f_n)$. Clearly $\tilde{\phi}(f)$ does not depend on the choice of (f_n) , and $\tilde{\phi}$ is a well-defined character on $C(X, E)$, extending ϕ . Thus, by Lemma 1.1, $\phi = \psi \circ e_x$ for some $\psi \in M(E)$ and some $x \in X$. A similar argument works for $\text{lip}^\alpha(X, E)$. Thus Theorem 1.2 applies, and the Shilov boundary of $\text{Lip}^\alpha(X, E)$ (or $\text{lip}^\alpha(X, E)$) is equal to the cartesian product $X \times \Gamma(E)$ in the product topology.

(iii) Let X be a compact set in \mathbb{C}^n and E be a unital commutative Banach algebra, and consider the algebra $P(X, E)$. The algebra $P(X) = P(X, \mathbb{C})$ has character space naturally identified with \hat{X} , the

polynomially-convex hull of X [2, 6, 11, 13], and $P(X, E)$ may be regarded as an E -valued function algebra on \hat{X} . Using this, it is easy to see that $(\hat{X}, E, P(X), P(X, E))$ is an admissible quadruple: In fact, each $f \in P(X, E)$ is the limit in norm of a sequence $\{g_n\}$ with each g_n of the form $\sum_{j=1}^{m_n} a_j p_j$ where $m_n \in \mathbb{N}$, $a_j \in E$ and $p_j \in P(X)$ depend on n . Then by a method similar to [10, Proposition 1.5.6] one sees that $P(X, E) = P(X) \check{\otimes} E$. Thus, since $P(X) = P(\hat{X})$ [6, Chapter II, Theorem 1.4], we have

$$P(X, E) = P(X) \check{\otimes} E = P(\hat{X}) \check{\otimes} E = P(\hat{X}, E).$$

We note that in the particular case when X is a compact plane set, \hat{X} is obtained by “filling in the holes” in X , and $\Gamma(P(X))$ is the topological boundary of \hat{X} in \mathbb{C} . In higher dimensions, the Shilov boundary $\Gamma(P(X))$ is some closed subset of $\text{bdy}(X)$.

By a method similar to [7, Lemma 2], one sees that every character ϕ on $P(X, E)$ is of the form $\phi = \psi \circ e_x$ for some $x \in \hat{X}$ and some $\psi \in M(E)$. Therefore the theorem applies, and the Shilov boundary of $P(X, E)$ is equal to the cartesian product $\Gamma(E) \times \Gamma(P(X))$

(iv) Let $X \subset \mathbb{C}$ be compact, E be a commutative unital Banach algebra and E^* be the dual space of E . The algebra of E -valued analytic functions is defined as follows,

$$A(X, E) = \{f \in C(X, E) : \Lambda \circ f \in A(X), \Lambda \in E^*\},$$

where $A(X)$ is the algebra of all continuous functions on X into \mathbb{C} which are holomorphic on the interior of X . It is clear that $A(X, E)$ is a closed subalgebra of $(C(X, E), \|\cdot\|_X)$. Arens showed [6] that $M(A(X))$ is naturally identified with X , and so one sees at once that $(X, E, A(X), A(X, E))$ is an admissible quadruple. By a method similar to the one given in [5, Theorem 2], we can deduce that when E is a unital Banach algebra then every character ϕ on $A(X, E)$ is of the

form $\phi = \psi \circ e_x$ for some $x \in X$ and some $\psi \in M(E)$. So the Shilov boundary of $A(X, E)$ is equal to the cartesian product $\Gamma(A(X)) \times \Gamma(E)$ in the product topology, by Theorem 1.2.

(v) Theorem 1.2 also applies to the algebra $R(X, E)$, for any commutative unital Banach algebra E . The characters on $R(X)$ are the evaluations at the points of the rationally-convex hull \check{X} of X , which is the set of points $a \in \mathbb{C}^n$ such that each polynomial $p(z) \in \mathbb{C}[z]$ that vanishes at a also vanishes at some point of X . In dimension $n = 1$, $\check{X} = X$, but in higher dimensions it may be a larger set. So $R(X)$ is a natural \mathbb{C} -valued function algebra on \check{X} .

We claim that every character ϕ on $R(X, E)$ is of the form $\phi = \psi \circ e_x$, for some $\psi \in M(E)$ and some $x \in \check{X}$.

To see this, let $\phi \in M(R(X, E))$. The restriction of ϕ to $P(X, E)$ is a character, so there exists $x_0 \in \hat{X}$ and $\psi \in M(E)$ such that $\phi = \psi \circ e_{x_0}$ on $P(X, E)$. Given $g = p/q$ where $p, q \in E[z]$ and $q(x) \in E^{-1}$ for each $x \in X$, we get $p = gq$, $\phi(p) = \phi(g)\phi(q)$, and hence (since $\phi(q) = \psi(q(x_0)) \neq 0$)

$$\phi(g) = \frac{\psi(p(x_0))}{\psi(q(x_0))} = \psi(g(x_0)).$$

Thus, by continuity, $\phi = \psi \circ e_{x_0}$ on all $R(X, E)$. Since $R(X)1 \subseteq R(X, E)$, it follows that $x_0 \in \check{X}$ (cf. [6, Theorem 5, p. 86]). Thus the claim holds.

Hence if X is rationally-convex, then $(X, E, R(X), R(X, E))$ is a natural admissible quadruple.

There seems no reason to suppose that $R(X, E) = R(\check{X}, E)$ for general X , except when E is a uniform algebra. In general, one sees readily that there is a contractive algebra homomorphism

$$R(X, E) \rightarrow C(\check{X}, C(M(E))),$$

and that if E is a uniform algebra then this gives an isometric isomorphism from $R(X, E)$ onto $R(\check{X}, E)$. We do not, however, know an example in which the restriction map $R(\check{X}, E) \rightarrow R(X, E)$ is not onto.

(vi) The bidisk algebra [2] may (in view of Hartogs' theorem) be regarded as $\tilde{B} = A(X, A(X))$, where X is the closed unit disk in \mathbb{C} . The quadruple $(X, A(X), A(X), \tilde{B})$ is admissible, the theorem applies, and reduces to the classical fact that the Shilov boundary of \tilde{B} is the torus. More generally, one gets the (known) result that the Shilov boundary of $A(X, A(Y))$ is $\text{bdy}X \times \text{bdy}Y$ whenever $X \subset \mathbb{C}$ and $Y \subset \mathbb{C}$ are compact.

(vii) Let $0 < \alpha < 1$. The subalgebra of $\text{lip}^\alpha(X, E)$ which is the closure of $E[z]|X$ in $\text{Lip}^\alpha(X, E)$ norm, where $X \subset \mathbb{C}^n$, is denoted by $\text{Lip}_P^\alpha(X, E)$. It is easy to see that $\text{Lip}_P^\alpha(X, E)$ is dense in $P(X, E)$. Now by [8, p.15], $M(\text{Lip}_P^\alpha(X, \mathbb{C})) = \hat{X}$. Thus if X is polynomially-convex, then the quadruple $(X, E, \text{Lip}_P^\alpha(X, \mathbb{C}), \text{Lip}_P^\alpha(X, E))$ is admissible and natural.

(viii) Also for $0 < \alpha < 1$, the subalgebra of $\text{lip}^\alpha(X, E)$ which is the closure of the algebra of functions of the form $p(z)/q(z)$ in $\text{Lip}^\alpha(X, E)$ where $X \subset \mathbb{C}^n$, $p(z) \in E[z]$, $q(z) \in E[z]$, and $q(x) \in E^{-1}$ whenever $x \in X$, is denoted by $\text{Lip}_R^\alpha(X, E)$. It is easy to see that $\text{Lip}_R^\alpha(X, E)$ is dense in $R(X, E)$ Now by [8, p.15], $M(\text{Lip}_R^\alpha(X, \mathbb{C})) = \check{X}$. Thus if X is rationally-convex, then the quadruple $(X, E, \text{Lip}_R^\alpha(X, \mathbb{C}), \text{Lip}_R^\alpha(X, E))$ is admissible and natural.

2.2. By similar arguments, one obtains the following:

Theorem 2.3. *Let (X, E, B, \tilde{B}) be a natural admissible quadruple. Then the set of peak points of \tilde{B} is equal to the cartesian product $S_0(B) \times S_0(E)$ in the product topology, that is,*

$$S_0(\tilde{B}) = S_0(B) \times S_0(E).$$

2.3. Acknowledgments. This paper was conceived while the first-named author was visiting National University of Ireland, Maynooth, whose hospitality is acknowledged with thanks.

The authors are grateful to the referee, whose observations and suggestions greatly improved the paper.

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