Hiding Symbols and Functions: New Metrics and Constructions for Information-Theoretic Security

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Abstract

We present information-theoretic definitions and results for analyzing symmetric-key encryption schemes beyond the perfect secrecy regime, i.e. when perfect secrecy is not attained. We adopt two lines of analysis, one based on lossless source coding, and another akin to ratedistortion theory. We start by presenting a new information-theoretic metric for security, called ϵ -symbol secrecy, and derive associated fundamental bounds. This metric provides a parameterization of secrecy that spans other information-theoretic metrics for security, such as weak secrecy and perfect secrecy. We then introduce list-source codes (LSCs), which are a general framework for mapping a key length (entropy) to a list size that an eavesdropper has to resolve in order to recover a secret message. We provide explicit constructions of LSCs, and show that LSCs that achieve high symbol secrecy also achieve a favorable tradeoff between key length and uncertainty list size. We also demonstrate that, when the source is uniformly distributed, the highest level of symbol secrecy for a fixed key length can be achieved through a construction based on minimum-distance separable (MDS) codes. Using an analysis related to rate-distortion theory, we then show how symbol secrecy can be used to determine the probability that an eavesdropper correctly reconstructs functions of the original plaintext. More specifically, we present lower bounds for the minimum-mean-squared-error of estimating a target function of the plaintext given that a certain set of functions of the plaintext is known to be hard (or easy) to infer, either by design of the security system or by restrictions imposed on the adversary. We illustrate how these bounds can be applied to characterize security properties of symmetric-key encryption schemes, and, in particular, extend security claims based on symbol secrecy to a functional setting. Finally, we discuss the application of our methods in key distribution, storage and privacy.

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1 Introduction

The security properties of a communication scheme can, in general, be evaluated from two fundamental perspectives: information theoretic and computational. For a noiseless setting, unconditional (i.e. perfect) information-theoretic secrecy can only be attained when the communicating parties share a random key with entropy at least as large as the message itself [3]. Consequently, usual information-theoretic approaches focus on physically degraded models [4], where the goal is to maximize the secure communication *rate* given that the adversary has a noisier observation of the message than the legitimate receiver. On the other hand, computationally secure cryptosystems have thrived both from a theoretical and a practical perspective. Such systems are based on yet unproven hardness assumptions, but nevertheless have led to cryptographic schemes that are widely adopted (for an overview, see [5]). Currently, computationally secure encryption schemes are used millions of times per day, in applications that range from online banking transactions to digital rights management.

Computationally secure cryptographic constructions do not necessarily provide an informationtheoretic guarantee of security. For example, one-way permutations and public-key encryption cannot be deemed secure against an adversary with unlimited computational resources. This is not to say that such primitives are not secure in practice – real-world adversaries are indeed computationally bounded. There are, however, cryptographic schemes that are believed to be computationally secure and simultaneously provide *some* security guarantee against computationally unbounded adversaries, albeit such guarantee is not absolute secrecy. This was noted by Shannon [3] and later by Hellman [6] in a companion paper to his and Diffie's work "New directions in Cryptography" [7].

Our goal in this work is to characterize the fundamental information-theoretic security properties of cryptographic schemes when perfect secrecy is not attained. We follow the footsteps of Shannon and Hellman and study symmetric-key encryption with small keys, i.e. when the length of the key is smaller than the length of the message. In this case, the best a computationally unrestricted adversary can do is to decrypt the ciphertext with all possible keys, resulting in a list of possible plaintext messages. The adversary's uncertainty regarding the original message is then represented by a probability distribution over this list. This distribution, in turn, depends on both the distribution of the key and the distribution of the plaintext messages.

We evaluate the information-theoretic security in this setting through two complementary lines of analysis: (i) one based on lossless source coding, where the security properties of the uncertainty list are measured using mutual information-based metrics and secure communication schemes are provided based on linear code constructions, and (ii) another akin to rate-distortion theory, where the mutual information-based metrics are translated into restrictions on the inference capabilities of the adversary through converse results. We describe each approach below.

1.1 Lossless Source Coding Approach

If perfect secrecy is not achieved, then meaningful metrics are required to quantify the level of information-theoretic security provided by a cryptographic scheme. We define a new metric for characterizing security, ϵ -symbol secrecy, which quantifies the uncertainty of specific source symbols given an encrypted source sequence. This metric subsumes traditional rate-based information-theoretic measures of secrecy which are generally asymptotic [4]. However, our definition is not asymptotic and, indeed, we provide a construction that achieves fundamental symbol secrecy bounds, based on maximum distance separable (MDS) codes, for finite-length sequences. We note that there has been a long exploration of the connection between coding and cryptography [8], and our work is inscribed in this school of thought.

We also introduce a general source coding framework for analyzing the fundamental informationtheoretic properties of symmetric-key encryption, called *list-source codes* (LSCs). LSCs compress a source sequence *below* its entropy rate and, consequently, a message encoded by an LSC is decoded to a list of possible source sequences instead of a unique source sequence. We demonstrate how any symmetric-key encryption scheme can be cast as an LSC, and prove that the best an adversary can do is to reduce the set of possible messages to an exponentially sized list with certain properties, where the size of the list depends on the length of the key and the distribution of the source. Since the list has a size exponential in the key length, it cannot be resolved in polynomial time in the key length, offering a certain level of computational security. We characterize the achievable ϵ -symbol secrecy of LSC-based encryption schemes, and provide explicit constructions using algebraic coding.

1.2 Rate-Distortion Approach

While much of information-theoretic security has considered the hiding of the plaintext, cryptographic metrics of security seek to hide also functions thereof [9]. More specifically, cryptographic metrics characterize how well an adversary can (or cannot) infer functions of a hidden variable, and are stated in terms of lower bounds for average estimation error probability. This contrasts with standard information-theoretic metrics of security, which are concerned with the average number of bits that an adversary learns about the plaintext. Nevertheless, as shown here, restrictions on the average mutual information can be mapped to lower bounds on average estimation error probability through rate-distortion formulations.

Using a rate-distortion based approach, we extend the definition of ϵ -symbol secrecy in order to quantify not only the information that an adversary gains about individual symbols of the source sequence, but also the information gained about *functions* of the encrypted source sequence. We prove that ciphers with high symbol secrecy guarantee that certain functions of the plaintext are provably hidden regardless of computational assumptions. In particular, we show that certain one-bit function of the plaintext (i.e. predicates) cannot be reliably inferred by the adversary.

We illustrate the application of our results both for hiding the source data and functions thereof. We provide an extension of the one-time pad [3] to a functional setting, demonstrating how certain classes of functions of the plaintext can be hidden using a short key. We also consider the privacy against statistical inference setup studied in [10], and show how the analysis introduced here sheds light on the fundamental privacy-utility tradeoff.

From a practical standpoint, we investigate the problem of secure content caching and distribution. We propose a hybrid encryption scheme based on list-source codes, where a large fraction of the message can be encoded and distributed using a key-independent list-source code. The information necessary to resolve the decoding list, which can be much smaller than the whole message, is then encrypted using a secure method. This scheme allows a significant amount of content to be distributed and cached *before* dealing with key generation, distribution and management issues.

1.3 Related work

Shannon's seminal work [3] introduced the use of statistical and information-theoretic metrics for analyzing secrecy systems. Shannon characterized several properties of conditional entropy (equivocation) as a metric for security, and investigated the effect of the source distribution on the security of a symmetric-key cipher. Shannon also considered the properties of "random ciphers", and showed that, for short keys and sufficiently long, non-uniformly distributed messages, the random cipher is (with high probability) breakable: only one message is very likely to have produced a given ciphertext. Shannon defined the length of the message required for a ciphertext to be uniquely produced by a given plaintext as the *unicity distance*.

Hellman extended Shannon's approach to cryptography [6] and proved that Shannon's random cipher model is conservative: A randomly chosen cipher is likely to have small unicity distance, but does not preclude the existence of other ciphers with essentially infinite unicity distance (i.e. the plaintext cannot be uniquely determined from the ciphertext). Indeed, Hellman argued that carefully designed ciphers that match the statistics of the source can achieve high unicity distance. Ahlswede [11] also extended Shannon's theory of secrecy systems to the case where the exact source statistics are unknown.

The problem of quantifying not only an eavesdropper's uncertainty of the entire message but of individual symbols of the message was studied by Lu in the context of additive-like instantaneous block ciphers (ALIB) [12–14]. The results presented here are more general since we do not restrict ourselves to ALIB ciphers. More recently, the design of secrecy systems with distortion constraints on the adversary's reconstruction was studied by Schieler and Cuff [15]. We adopt here an alternative approach, quantifying the information an adversary gains on average about the individual symbols of the message, and investigate which functions of the plaintext an adversary can reconstruct. Our results and definitions also hold for the finite-blocklength regime.

Tools from algebraic coding have been widely used for constructing secrecy schemes [8]. In addition, the notion of providing security by exploiting the fact that the adversary has incomplete access to information (in our case, the key) is also central to several secure network coding schemes and wiretap models. Ozarow and Wyner [16] introduced the wiretap channel II, where an adversary can observe a set k of his choice out of n transmitted symbols, and proved that there exists a code that achieves perfect secrecy. A generalized version of this model was investigated by Cai and Yeung in [17], where they introduce the related problem of designing an information-theoretically secure linear network code when an adversary can observe a certain number of edges in the network. Their results were later extended in [18–21]. A practical approach was presented by Lima *et al.* in [22]. For a survey on the theory of secure network coding, we refer the reader to [23].

The list-source code framework introduced here is related to the wiretap channel II in that a fraction of the source symbols is hidden from a possible adversary. Oliveira *et al.* investigated in [24] a related setting in the context of data storage over untrusted networks that do not collude, introducing a solution based on Vandermonde matrices. The MDS coding scheme introduced in this paper is similar to [24], albeit the framework developed here is more general.

List decoding techniques for channel coding were first introduced by Elias [25] and Wozencraft [26], with subsequent work by Shannon *et al.* [27,28] and Forney [29]. Later, algorithmic results for list decoding of channel codes were discovered by Gurusuwami and Sudan [30]. We refer the reader to [31] for a survey of list decoding results. List decoding has been considered in the context of source coding in [32]. The approach is related to the one presented here, since we may view a secret key as side information, but [32] did not consider source coding and list decoding together for the purposes of security.

The use of rate-distortion formulations in security and privacy settings was studied by Yamamoto [33] and Reed [34]. Information-theoretic approaches to privacy that take distortion into account were also considered in [10,35–37].

1.4 Notation

Throughout the paper capital letters (e.g. X and Y) are used to denote random variables, and calligraphic letters (e.g. \mathcal{X} and \mathcal{Y}) denote sets. All the random variables in this paper have a discrete support set, and the support set of the random variables X and Y are denoted by \mathcal{X} and \mathcal{Y} , respectively. For a positive integer $j, k, n, j \leq k$, $[n] \triangleq \{1, \ldots, n\}, [j, k] \triangleq \{j, j + 1, \ldots, k\}$. Matrices are denoted in bold capital letters (e.g. **H**) and vectors in bold lower-case letters (e.g. **h**). A sequence of n random variables X_1, \ldots, X_n is denoted by X^n . Furthermore, for $\mathcal{J} \subseteq [n]$, $X^{\mathcal{J}} \triangleq \left(X_{i_1}, \ldots, X_{i_{|\mathcal{J}|}}\right)$ where $i_k \in \mathcal{J}$ and $i_1 < i_2 < \cdots < i_{|\mathcal{J}|}$. Equivalently, for a vector $\mathbf{x} = (x_1, \ldots, x_n), \ \mathbf{x}^{\mathcal{J}} \triangleq \left(x_{i_1}, \ldots, x_{i_{|\mathcal{J}|}}\right)$. For two vectors $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$, we denote by $\mathbf{x} \leq \mathbf{z}$ the set of inequalities $\{x_i \leq z_i\}_{i=1}^n$. Furthermore, we denote by $\mathcal{I}_n(t)$ the set of all subsets of [n] of size t, i.e. $\mathcal{J} \in \mathcal{I}_n(t) \Leftrightarrow \mathcal{J} \subseteq [n]$ and $|\mathcal{J}| = t$.

All the logarithms in the paper are in base 2. We denote the binary entropy function as

$$h_b(x) \triangleq -x \log x - (1-x) \log(1-x).$$

The inverse of the binary entropy function is the mapping $h_b^{-1}: [0,1] \to [0,1/2]$ where

$$h_b^{-1}(h(x)) = \begin{cases} x, & 0 \le x \le 1/2\\ 1-x, & \text{otherwise.} \end{cases}$$

The set of all unit variance functions of a random variable X with distribution p_X (denoted by $X \sim p_X$) is given by

$$\mathcal{L}_2(p_X) \triangleq \{ \phi : \mathcal{X} \to \mathbb{R} \text{ such that } \| \phi(X) \|_2 = 1, \ X \sim p_X \},\$$

where $\|\phi(X)\|_2 \triangleq \sqrt{\mathbb{E}[\phi(X)^2]}$.

The operators T_X and T_Y denote conditional expectation and, in particular, $(T_X \circ g)(x) = \mathbb{E}[g(Y)|X = x]$ and $(T_Y \circ f)(y) = \mathbb{E}[f(X)|Y = y]$, respectively. For two random variables X and Y, the minimum-mean-squared error (MMSE) of estimating X from an observation of Y is given by

$$\mathsf{mmse}(X|Y) \triangleq \min_{X \to Y \to \hat{X}} \mathbb{E}\left[(X - \hat{X})^2 \right].$$

1.5 Communication and threat model

A transmitter (Alice) wishes to transmit confidentially to a legitimate receiver (Bob) a sequence of length n produced by a discrete source X with alphabet \mathcal{X} and probability distribution p_X . We assume that the communication channel shared by Alice and Bob is noiseless, but is observed by a passive, computationally unbounded eavesdropper (Eve). Both Alice and Bob have access to a shared secret key K drawn from a discrete alphabet \mathcal{K} , such that $H(K) < H(X^n)$, and encryption/decryption functions $\mathsf{Enc} : \mathcal{X}^n \times \mathcal{K} \to \mathcal{M}$ and $\mathsf{Dec} : \mathcal{M} \times \mathcal{K} \to \mathcal{X}^n$, where \mathcal{M} is the set encrypted messages. Alice observes the source sequence X^n , and transmits an encrypted message $M = \mathsf{Enc}(X^n, K)$. Bob then recovers X^n by decrypting the message using the key, producing $\hat{X}^n = \mathsf{Dec}(M, K)$. The communication is successful if $\hat{X}^n = X^n$. We consider that the encryption is closed [3, pg. 665], so $\mathsf{Dec}(c, k_1) \neq \mathsf{Dec}(c, k_2)$ for $k_1, k_2 \in \mathcal{K}, k_1 \neq k_2$. We assume Eve knows the functions Enc and Dec , but does not know the secret key, K. Eve's goal is to gain knowledge about the original source sequence.

1.6 Organization of the paper

1.6.1 Symbol secrecy

We introduce the definitions of absolute and ϵ -symbol secrecy in Section 2. Symbol secrecy quantifies the uncertainty that an eavesdropper has about individual symbols of the message.

1.6.2 Encryption with key entropy smaller than the message entropy

We present the definition of list-source codes (LSCs), together with fundamental bounds, in Section 3. Practical code constructions of LSCs are introduced in Section 4. We then analyze the symbol secrecy properties of LSCs in Section 5.

1.6.3 A Rate-Distortion View of Symbol Secrecy

In Section 6 we introduce results for characterizing the information leakage of a security system in terms of functions of the original source data. In particular, we derive converse bounds for the minimum-mean-squared error (MMSE) of estimating a target function of the plaintext given that certain functions of the plaintext are known to be hard (or easy) to infer. We illustrate the application of these bounds in a generalization of the one-time pad. We also use these results to bound the probability of error of estimating predicates of the plaintext given that a certain level of symbol secrecy is achieved.

1.6.4 Further applications and practical considerations

Section 7 presents further applications of our results to security and privacy, together with practical considerations of the proposed secrecy framework. Finally, Section 8 presents our concluding remarks.

2 Symbol Secrecy

In this section we define ϵ -symbol secrecy, an information-theoretic metric for quantifying the information leakage from security schemes that do not achieve perfect secrecy. Given a source sequence X^n and a random variable Z dependent of X^n , ϵ -symbol secrecy is the largest fraction t/n such that, given Z, at most ϵ bits can be learned on average from any t-symbol subsequence of X^n . We also prove an ancillary lemma that bounds the average mutual information between X^n and Z in terms of symbol secrecy.

Definition 1. Let X^n be a random variable with support \mathcal{X}^n , and Z be the information that leaks from a security system (e.g. the ciphertext). Denoting $X^{\mathcal{J}} = \{X_i\}_{i \in \mathcal{J}}$, we say that $p_{X^n,Z}$ achieves an ϵ -symbol secrecy of $\mu_{\epsilon}(X^n|Z)$ if

$$\mu_{\epsilon}(X^{n}|Z) \triangleq \max\left\{\frac{t}{n} \left| \frac{I(X^{\mathcal{J}};Z)}{|\mathcal{J}|} \le \epsilon \quad \forall \mathcal{J} \subseteq [n], 0 < |\mathcal{J}| \le t \right\}.$$
(1)

In particular, the *absolute symbol secrecy* of X^n from Y is given by

$$\mu_0(X^n|Z) \triangleq \max\left\{\frac{t}{n} \left| I(X^{\mathcal{J}};Z) = 0 \quad \forall \mathcal{J} \subseteq [n], 0 < |\mathcal{J}| \le t \right\}.$$
(2)

We also define the dual function of symbol-secrecy for X^n and Z as:

$$\epsilon_t^*(X^n|Z) \triangleq \inf \left\{ \epsilon \ge 0 \, | \, \mu_\epsilon(X^n|Z) \ge t/n \right\}.$$
(3)

The next examples illustrate a few use cases of symbol secrecy.

Example 1. Symbol secrecy encompasses other definitions of secrecy, such as weak secrecy [38], strong secrecy [39] and perfect secrecy. For example, given two sequences of random variables

 X^n and Z^n , if $\mu_{\epsilon}(X^n|Z^n) \to 1$ for all $\epsilon > 0$, then $\frac{I(X^n;Z^n)}{n} \to 0$. The converse is not true, as demonstrated in Example 3 below. Furthermore, $I(X^n;Z^n) = 0$ if and only if $\mu_0(X^n|Z^n) = 1$. Finally, the reader can verify that $I(X^n;Z^n) \to 0$ if and only if there exists a sequence $\epsilon_n = o(n)$ such that $\mu_{\epsilon_n}(X^n|Z^n) \to 1$.

Example 2. Consider the case where $\mathcal{X} = \{0, 1\}$, X^n is uniformly drawn from \mathcal{X}^n , and Z is the result of sending X^n through a discrete memoryless erasure channel with erasure probability α . Then, for any $\mathcal{J} \subseteq [n], \ \mathcal{J} \neq \emptyset$,

$$\frac{I(X^{\mathcal{J}};Z)}{|\mathcal{J}|} = (1-\alpha),$$

and, consequently,

$$\mu_{\epsilon}(X^{n}|Z) = \begin{cases} 0, & \text{for } 0 \le \epsilon < 1 - \alpha, \\ 1, & \epsilon \ge 1 - \alpha. \end{cases}$$

Example 3. Now assume again that X^n is a uniformly distributed sequence of n bits, but now $Z = X_1$. This corresponds to the case where one bit of the message is always sent in the clear, and all the other bits are hidden. Then, for any $\mathcal{J} \subseteq [n]$ such that $\{1\} \in \mathcal{J}$,

$$I(X^{\mathcal{J}}; Z) = 1,$$

and, for $0 \leq \epsilon < 1$,

 $\mu_{\epsilon}(X^n|Z) = 0.$

Consequently, a non-trivial symbol-secrecy cannot be achieved for $\epsilon < 1$. In general, if a symbol X_i is sent in the clear, then a non-trivial symbol secrecy cannot be achieved for $\epsilon < H(X_i)$. Note that $I(X^n; Z)/n \to 0$, so weak secrecy is achieved.

Example 4. We now illustrate how symbol secrecy does not necessarily capture the information that leaks about functions of X^n . We address this issue in more detail in Section 6. Still assuming that X^n is a uniformly distributed sequence of n bits, let Y be the parity bit of X^n , i.e. $Z = \prod_{i=1}^{n} (-1)^{X_i}$. Then, for any $\mathcal{J} \subsetneq [n]$,

$$I(X^{\mathcal{J}}; Z) = 0,$$

and, for $0 \leq \epsilon < 1$,

$$\mu_{\epsilon}(X^n|Z) = \frac{n-1}{n},$$

and, for $\epsilon \geq 1$, $\mu_{\epsilon}(X^n | Z) = 1$.

The following lemma provides an upper bound for $I(X^n; Z)$ in terms of $\mu_{\epsilon}(X^n | Z)$ when X^n is the output of a discrete memoryless source.

Lemma 1. Let X^n be the output of a discrete memoryless source X, and Z a noisy observation of X^n . For any ϵ such that $0 \leq \epsilon \leq H(X)$, if $\mu_{\epsilon}(X^n|Z) = u^*$, then

$$\frac{1}{n}I(X^n;Z) \le H(X) - u^*(H(X) - \epsilon).$$
(4)

Proof. Let $\mu_{\epsilon}(X^n|Z) = u^* \triangleq t/n, \ \mathcal{J} \in \mathcal{I}_n(t) \text{ and } \overline{\mathcal{J}} = [n] \setminus \mathcal{J}$. Then

$$\frac{1}{n}I(X^n;Z) = \frac{1}{n}I(X^{\mathcal{J}};Z) + \frac{1}{n}I(X^{\bar{\mathcal{J}}};Z|X^{\mathcal{J}})$$
$$\leq \frac{t}{n}\left(\epsilon + \frac{1}{t}I(X^{\bar{\mathcal{J}}};Z|X^{\mathcal{J}})\right)$$
$$\leq u^*\epsilon + \frac{(n-t)}{n}H(X)$$
$$= H(X) - u^*(H(X) - \epsilon),$$

where the first inequality follows from the definition of symbol secrecy, and the second inequality follows from the assumption that the source is discrete and memoryless and, consequently, $I(X^{\bar{\mathcal{J}}}; Z|X^{\mathcal{J}}) \leq H(X^{\bar{\mathcal{J}}}|X^{\mathcal{J}}) = (n-t)H(X).$

The previous result implies that when $\mu_{\epsilon}(X^n|Z)$ is large, only a small amount of information about X^n can be gained from Z on average. However, even if $I(X^n;Z)$ is large, as long as $\mu_{\epsilon}(X^n|Z)$ is non-zero, the uncertainty about X^n given Z will be spread throughout the individual symbols of the source sequence. This property is desirable for symmetric-key encryption and, as we shall show in Section 6, can be extended to determine which functions of X^n can or cannot be reliably inferred from Z. Furthermore, in Section 5 we introduce explicit constructions for symmetrickey encryption schemes that achieve a provable level of symbol secrecy using the list-source code framework introduced next.

3 LSCs

In this section we present the definition of LSCs and derive fundamental bounds. We also demonstrate how any symmetric-key encryption scheme can be mapped to a corresponding list-source code.

3.1 Definition and Fundamental Limits

We introduce the definition of list-source codes is given below.

Definition 2. A $(2^{nR}, |\mathcal{X}|^{nL}, n)$ -LSC $(f_n, g_{n,L})$ consists of an encoding function $f_n : \mathcal{X}^n \mapsto [2^{nR}]$ and a list-decoding function $g_{n,L} : [2^{nR}] \mapsto \mathcal{P}(\mathcal{X}^n) \setminus \emptyset$, where $\mathcal{P}(\mathcal{X}^n)$ is the power set of \mathcal{X}^n and $|g_{n,L}(w)| = |\mathcal{X}|^{nL} \quad \forall w \in [2^{nR}]$. The value R is that *rate* of the LSC, L is the *normalized list size*, and $|\mathcal{X}|^{nL}$ is the *list size*.

Note that $0 \leq L \leq 1$. From an operational point of view, L is a parameter that determines the size of the decoded list. For example, L = 0 corresponds to traditional lossless compression, i.e., each source sequence is decoded to a unique sequence. Furthermore, L = 1 represents the trivial case when the decoded list corresponds to \mathcal{X}^n .



Figure 1: Rate list region for normalized list size L and code rate R.

For a given LSC, an error is declared when a string generated by a source is not contained in the corresponding decoded list. The average error probability is given by

$$\mathbf{e}(f_n, g_{n,L}) \triangleq \Pr(X^n \notin g_{n,L}(f_n(X^n))).$$
(5)

Definition 3. For a given discrete memoryless source X, the rate list size pair (R, L) is said to be *achievable* if for every $\delta > 0$, $0 < \epsilon < 1$ and sufficiently large n there exists a sequence of $(2^{nR_n}, |\mathcal{X}|^{nL_n}, n)$ -list-source codes $\{(f_n, g_{n,L_n})\}_{n=1}^{\infty}$ such that $R_n < R + \delta$, $|L_n - L| < \delta$ and $\mathbf{e}(f_n, g_{n,L_n}) \leq \epsilon$. The rate list region is the closure of all rate list pairs (R, L).

Definition 4. The rate list function R(L) is the infimum of all rates R such that (R, L) is in the rate list region for a given normalized list size $0 \le L \le 1$.

Theorem 1. For any discrete memoryless source X, the rate list function is given by

$$R(L) = H(X) - L\log|\mathcal{X}| .$$
(6)

Proof. Let $\delta > 0$ be given and $\{(f_n, g_{n,L_n})\}_{n=1}^{\infty}$ be a sequence of codes with (normalized) list size L_n such that $L_n \to L$ and for any $0 < \epsilon < 1$ and n sufficiently large $0 \le e(f_n, g_{n,L_n}) \le \epsilon$. Then

$$\Pr\left(X^{n} \in \bigcup_{w \in \mathcal{W}^{n}} g_{n,L_{n}}(w)\right) \ge \Pr\left(X^{n} \in g_{n,L_{n}}(f_{n}(X^{n}))\right)$$
(7)

$$\geq 1 - \epsilon \tag{8}$$

where $\mathcal{W}^n = [2^{nR_n}]$ and R_n is the rate of the code (f_n, g_{n,L_n}) . There exists $n_0(\delta, \epsilon, |\mathcal{X}|)$ where if

 $n \geq n_0(\delta, \epsilon, |\mathcal{X}|)$, then

$$R_{n} + L_{n} \log|\mathcal{X}| = \frac{1}{n} \log \left(2^{nR_{n}} |\mathcal{X}|^{nL_{n}} \right)$$
$$= \frac{1}{n} \log \left(\sum_{w \in \mathcal{W}^{n}} |g_{n,L_{n}}(w)| \right)$$
$$\geq \frac{1}{n} \log \left| \bigcup_{w \in \mathcal{W}^{n}} g_{n,L_{n}}(w) \right|$$
$$\geq H(X) - \delta, \tag{9}$$

where the last inequality follows from [40, Lemma 2.14]. Since this holds for any $\delta > 0$, it follows that $R(L) \ge H(X) - L \log |\mathcal{X}|$ for all *n* sufficiently large.

We prove achievability next. Let 0 < L < 1 be given, and let $L_n \triangleq \lfloor nL \rfloor$. Furthermore, let X^n be a sequence of n source symbols, and denote X^{nL_n} the first nL_n source symbols and $X^{[nL_n+1,n]}$ the last $n(1-L_n)$ source symbols where we assume, without loss of generality, that nL is an integer. Then, from standard source coding results [41, pg. 552], for any $\epsilon > 0$ and n sufficiently large, and denoting $\alpha_n \triangleq \lceil nL_n(H(X) + \epsilon) \rceil / n$, $\beta_n \triangleq \lceil n(1-L_n)(H(X) + \epsilon) \rceil / n$, there are (surjective) encoding functions

$$f_{nL}^1: \mathcal{X}^{nL_n} \to [2^{n\alpha_n}] \text{ and } f_{n(1-L_n)}^2: \mathcal{X}^{n(1-L_n)} \to [2^{n\beta_n}],$$

and corresponding (injective) decoding functions

$$g_{n,1}^1: [2^{n\alpha_n}] \to \mathcal{X}^{nL_n} \text{ and } g_{n,1}^2: [2^{n\beta_n}] \to \mathcal{X}^{nL_n}$$

such that $\Pr(g_{n,1}^1(f_{nL_n}^1(X^{nL_n})) \neq X^{nL_n}) \leq O(\epsilon)$ and $\Pr(g_{n,1}^2(f_{n(1-L_n)}^2(X^{(1-L_n)n})) \neq X^{(1-L_n)n}) \leq O(\epsilon)$.

For $w \in [2^{n\beta_n}]$ and $\mathbf{x} \in \mathcal{X}^n$, let the list-source coding and decoding functions be given by $f_n(\mathbf{x}) \triangleq f_{n(1-L_n)}^2(\mathbf{x}^{[nL_n+1,n]})$ and

$$g_{n,\tilde{L}_n}(w) \triangleq \{ \mathbf{x} \in \mathcal{X}^n : \exists v \in [2^{n\alpha_n}] \text{ such that } (f_{nL}^1(\mathbf{x}^{[nL]}), f_{n(1-L)}^2(\mathbf{x}^{[nL+1,n]})) = (v,w) \},$$

respectively. Then

$$\begin{split} \Pr\left(X^n \in g_{n,\widetilde{L}_n}(f_n(X^n))\right) &\geq \Pr\left(g_{n,1}^1(f_{nL}^1(X^{Ln})) = X^{Ln} \wedge g_{n,1}^2(f_{n(1-L)}^2(X^{(1-L)n})) = X^{(1-L)n}\right) \\ &\geq 1 - O(\epsilon). \end{split}$$

Observe that the rate-list pair achieved by (f_n, g_{n,\tilde{L}_n}) is $(R_n, \tilde{L}_n) = (\beta_n, \alpha_n/\log|\mathcal{X}|)$. Consequently,

$$R_n \le (1 - L_n)(H(X) + \epsilon) + n^{-1}$$
$$\le H(X) + \epsilon - \alpha_n$$

$$= H(X) + \epsilon - \tilde{L}_n \log |\mathcal{X}|,$$

where the second inequality follows from $\alpha_n \leq L_n(H(X) + \epsilon) + n^{-1}$. Observe that $R_n \to n(1 - L)H(X) + \epsilon \triangleq R$. Since $\tilde{L}_n \to L(H(X) + \epsilon)/\log|\mathcal{X}| \triangleq \tilde{L}$ as $n \to \infty$, by choosing *n* sufficiently large the rate-list pair (R, \tilde{L}) can be achieved, where *R* and \tilde{L} satisfy

$$R \le H(X) + \epsilon - L \log|\mathcal{X}|.$$

Since ϵ is arbitrary and \tilde{L} can span any value in $[0, H(X)/\log|\mathcal{X}|]$, it follows that $R(L) \leq H(X) - L\log|\mathcal{X}|$.

3.2 Symmetric-Key Ciphers as LSCs

Let (Enc, Dec) be a symmetric-key cipher where, without loss of generality, $\mathcal{M} = [2^{nR}]$ and Enc : $\mathcal{X}^n \times \mathcal{K} \to \mathcal{M}$ and Dec : $\mathcal{M} \times \mathcal{K} \to \mathcal{X}^n$. Then an LSC can be designed based on this cipher by choosing k' from \mathcal{K} and setting the encoding function $f_n(\mathbf{x}) = \text{Enc}(\mathbf{x}, k')$, where $\mathbf{x} \in \mathcal{X}^n$, and

$$g_{n,L}(f_n(\mathbf{x})) = \{ \mathbf{z} \in \mathcal{X}^n : \exists k \in \mathcal{K} \text{ such that } \mathsf{Enc}(\mathbf{z}, k) = f_n(\mathbf{x}) \},\$$

where L satisfies $|\mathcal{K}| = |\mathcal{X}|^{nL}$. If the key is chosen uniformly from \mathcal{K} then the decoded list corresponds set of possible source sequences that could have generated the ciphertext. The adversary's uncertainty will depend on the distribution of the source sequence X^n .

Alternatively, symmetric-key ciphers can also be constructed based on an $(2^{nR}, |\mathcal{X}|^{nL}, n)$ -listsource code. Let $(f_n, g_{n,L})$ be the corresponding encoding/decoding function of the LSC, and assume that the key is drawn uniformly from $\mathcal{K} = [|\mathcal{X}|^{nL}]$, where the normalized list size L determines the length of the key. Without loss of generality, we also assume that Alice and Bob agree on an ordering of \mathcal{X} and, consequently, \mathcal{X}^n can be ordered using the corresponding dictionary ordering. We denote $\mathsf{pos}(\mathbf{x})$ the position of the source sequence $\mathbf{x} \in \mathcal{X}$ in the corresponding list $g_{n,L}(f_n(\mathbf{x}))$, where $\mathsf{pos} : \mathcal{X}^n \to [|\mathcal{X}|^{nL}]$.

The cipher can then be constructed by letting the message set be $\mathcal{M}' = [2^{nR}] \times [|\mathcal{X}|^{nL}]$ and, for $\mathbf{x} \in \mathcal{X}^n$ and $k \in \mathcal{K}$,

$$\operatorname{Enc}(\mathbf{x},k) = (f_n(\mathbf{x}), (\operatorname{pos}(\mathbf{x}) + k) \mod |\mathcal{K}|)$$

For $(a, b) \in \mathcal{M}'$, the decryption function is given by

$$\mathsf{Dec}((a,b),k) = \{\mathbf{x} : f_n(\mathbf{x}) = a, \mathsf{pos}(\mathbf{x}) = (b-k) \mod |\mathcal{K}|\}.$$

In this case, an eavesdropper that does not know the key k cannot recover the function $pos(\mathbf{x})$ and, consequently, her uncertainty will correspond to the list $g_{n,L}(f_n(\mathbf{x}))$.

4 LSC design

In this section we discuss how to construct LSCs that achieve the rate-list tradeoff (6) in the finite block length regime. As shown below, an LSC that achieves good rate-list tradeoff does not necessarily lead to good symmetric-key encryption schemes. This naturally motivates the constructions of LSCs that achieve high symbol secrecy.

4.1 Necessity for code design

Assume that the source X is uniformly distributed in \mathbb{F}_q , i.e., $\Pr(X = x) = 1/q \ \forall x \in \mathbb{F}_q$. In this case $R(L) = (1 - L) \log q$. A trivial scheme for achieving the list-source boundary is the following. Consider a source sequence $X^n = (X^p, X^s)$, where X^p denotes the first $p = n - \lfloor Ln \rfloor$ symbols of X^n and X^s denotes the last $s = \lfloor Ln \rfloor$ symbols. Encoding is done by discarding X^s , and mapping the prefix X^p to a binary codeword Y^{nR} of length $nR = \lceil n - \lfloor Ln \rfloor \log q \rceil$ bits. This encoding procedure is similar to the achievability scheme used in the proof of Theorem 1.

For decoding, the codeword Y^{nR} is mapped to X^p , and the scheme outputs a list of size q^s composed by X^p concatenated with all possible combinations of suffixes of length s. Clearly, for n sufficiently large, $R \approx (1-L) \log q$, and we achieve the optimal list-source size tradeoff.

The previous scheme is inadequate for security purposes. An adversary that observes the codeword Y^{nR} can uniquely identify the first p symbols of the source message, and the uncertainty is concentrated over the last s symbols. Assuming that all source symbols are of equal importance, we should spread the uncertainty over all symbols of the message. Given the encoding $f(X^n)$, a sensible security scheme would provide $I(X_i; f(X^n)) \leq \epsilon \ll \log q$ for $1 \leq i \leq n$. We can naturally extend this notion for groups of symbols or functions over input symbols, which is what symbol secrecy captures.

4.2 A construction based on linear codes

Let X be an i.i.d. source with support \mathcal{X} and entropy H(X), and (s_n, r_n) a source code for X with encoder $s_n : \mathcal{X}^n \to \mathbb{F}_q^{m_n}$ and decoder $r_n : \mathbb{F}_q^{m_n} \to \mathcal{X}^n$. Furthermore, let \mathcal{C} be a (m_n, k_n, d) linear code¹ over \mathbb{F}_q with an $(m_n - k_n) \times m_n$ parity check matrix \mathbf{H}_n (i.e. $\mathbf{c} \in \mathcal{C} \Leftrightarrow \mathbf{H}_n \mathbf{c} = 0$). Consider the following scheme, where we assume

$$k_n \triangleq nL_n \log |\mathcal{X}| / \log q$$

is an integer, $0 \leq L_n \leq 1$ and $L_n \to L$ as $n \to \infty$.

Scheme 1. Encoding: Let $\mathbf{x}_n \in \mathcal{X}^n$ be an *n*-symbol sequence generated by the source. Compute the syndrome $\boldsymbol{\sigma}_n$ through the matrix multiplication

$$\boldsymbol{\sigma}_n \triangleq \mathbf{H}_n s_n(\mathbf{x}_n)$$

¹For an overview of linear codes an related terminology, we refer the reader to [42].

and map each syndrome to a distinct sequence of $nR = \lceil (m_n - k_n) \log q \rceil$ bits, denoted by \mathbf{y}_{nR} .

Decoding: Map the binary codeword \mathbf{y}_{nR} to the corresponding syndrome $\boldsymbol{\sigma}_n$. Output the list

$$g_{n,L_n}(\boldsymbol{\sigma}_n) = \left\{ r_n(\mathbf{z}) \middle| \mathbf{z} \in \mathbb{F}_q^{m_n}, \ \boldsymbol{\sigma}_n = \mathbf{H}_n \mathbf{z} \right\}.$$

Theorem 2. If a sequence of source codes $\{(s_n, r_n)\}_{n=1}^{\infty}$ is asymptotically optimal for source X, i.e. $m_n/n \to H(X)/\log q$ with vanishing error probability, scheme 1 achieves the rate list function R(L) for source X.

Proof. Since the cardinality of each coset corresponding to a syndrome σ_n is exactly

$$|g_{n,L_n}(\boldsymbol{\sigma}_n)| = q^{k_n},$$

the normalized list size is

$$L_n = \log_{|\mathcal{X}|} q^{k_n} = (k_n \log q) / (n \log |\mathcal{X}|).$$

By assumption, $L_n \to L$ as $n \to \infty$. Denoting $m_n/n = H(X)/\log q + \delta_n$, where $\delta_n \to 0$ since the source code is assumed to be asymptotically optimal, it follows that the rate of the LSC is

$$R_n = \lceil (m_n - k_n) \log q \rceil / n$$

= $\lceil (H(X) + \delta_n \log q)n - L_n n \log |\mathcal{X}| \rceil / n$
 $\rightarrow H(X) - L \log |\mathcal{X}|,$

which is arbitrarily close to the rate in (6) for sufficiently large n.

The source coding scheme used in the proof of Theorem 2 can be any asymptotically optimal scheme. Note that if the source X is uniformly distributed in \mathbb{F}_q , then $L_n = k_n/n$ and any message in the coset indexed by σ_n is equally likely. Hence, $R_n = (n-k) \log q/n = H(X) - L \log q$, which matches the upper bound in (6). Scheme 1 provides a constructive way of hiding information, and we can take advantage of the properties of the underlying linear code to make precise assertions regarding the security of the scheme.

With the syndrome in hand, how can we recover the rest of the message? One possible approach is to find a $k_n \times n$ matrix \mathbf{D}_n that has full rank such that the rows of \mathbf{D}_n and \mathbf{H}_n form a basis of $\mathbb{F}_q^{m_n}$. Such a matrix can be easily found, for example, using the Gram-Schmidt process with the rows of \mathbf{H}_n as a starting point. Then, for a source sequence \mathbf{x}_n , we simply calculate $\mathbf{t}_n = \mathbf{D}_n \mathbf{x}_n$ and forward \mathbf{t}_n to the receiver through a secure channel. The receiver can then invert the system

$$\begin{pmatrix} \mathbf{H}_n \\ \mathbf{D}_n \end{pmatrix} \mathbf{x}_n = \begin{pmatrix} \boldsymbol{\sigma}_n \\ \mathbf{t}_n \end{pmatrix},\tag{10}$$

and recover the original sequence \mathbf{x}_n . This property allows list-source codes to be deployed in

practice using well known linear code constructions, such as Reed-Solomon [42, Chap. 5] or Random Linear Network Codes [43, Chap. 2].

Remark 1. This approach is valid for general linear spaces, and holds for any pair of full rank matrices \mathbf{H}_n and \mathbf{D}_n with dimensions $(m_n - k_n) \times m_n$ and $k_n \times m_n$, respectively, such that $\operatorname{rank}([\mathbf{H}_n^T \mathbf{D}_n^T]^T) = m_n$. However, here we adopt the nomenclature of linear codes since we make use of known code constructions to construct LSCs with provable symbol secrecy properties in the next section.

Remark 2. The LSC described in scheme 1 can be combined with other encryption methods, providing, for example, an additional layer of security in probabilistic encryption schemes ([5,9]). A more detailed discussion of practical applications is presented in Section 7.

5 Symbol Secrecy of LSCs

We next present fundamental bounds for the amount of symbol secrecy achievable by any LSC considering a discrete memoryless source. Since any encryption scheme can be cast as an LSC, these results quantify the amount of symbol secrecy achievable by any symmetric-key encryption scheme that encrypts a discrete memoryless source.

Lemma 2. Let $\{(f_n, g_n)\}_{n=1}^{\infty}$ be a sequence of list-source codes that achieves a rate-list pair (R, L) and an ϵ -symbol secrecy of $\mu_{\epsilon}(X^n|Y^{nR_n}) \to \mu_{\epsilon}$ as $n \to \infty$. Then $0 \le \mu_{\epsilon} \le \min\left\{\frac{L\log|\mathcal{X}|}{H(X)-\epsilon}, 1\right\}$.

Proof. We denote $\mu_{\epsilon}(X^n|Y^{nR}) = \mu_{\epsilon,n}$. Note that, for $\mathcal{J} \subseteq [n]$ and $|\mathcal{J}| = n\mu_{\epsilon,n}$,

$$I(X^{\mathcal{J}}; Y^{nR_n}) = H(X^{\mathcal{J}}) - H(X^{\mathcal{J}}|Y^{nR_n})$$

= $n\mu_{\epsilon,n}H(X) - H(X^{\mathcal{J}}|Y^{nR_n})$
 $\leq n\mu_{\epsilon,n}\epsilon,$

where the last inequality follows from the definition of symbol secrecy and $I(X^{\mathcal{J}}; Y^{nR_n}) \leq |\mathcal{J}|\epsilon = n\mu_{\epsilon,n}\epsilon$. Therefore

$$\mu_{\epsilon,n}(H(X) - \epsilon) \leq \frac{1}{n} H(X^{\mathcal{J}} | Y^{nR_n})$$
$$\leq L_n \log |\mathcal{X}|.$$

The result follows by taking $n \to \infty$.

The previous result bounds the amount of information an adversary gains about particular source symbols by observing a list-source encoded message. In particular, for $\epsilon = 0$, we find a meaningful bound on what is the largest fraction of input symbols that is *perfectly* hidden.

The next theorem relates the rate-list function with ϵ -symbol secrecy through the upper bound in Theorem 2.

Theorem 3. If a sequence of list-source codes $\{(f_n, g_{n,L_n})\}_{n=1}^{\infty}$ achieves a point (R', L) with $\mu_{\epsilon}(X^n|Y^{nR_n}) \to \frac{L\log|\mathcal{X}|}{H(X)-\epsilon} \triangleq c_{\epsilon}$ for some ϵ , where $R' = \lim_{n \to \infty} \frac{1}{n}H(Y^{nR_n})$, then R' = R(L).

Proof. Assume that $\{(f_n, g_{n,L_n})\}_{n=1}^{\infty}$ satisfies the conditions in the theorem and $\delta > 0$ is given. Then for *n* sufficiently large, we have from (4):

$$\frac{1}{n}H(Y^{nR_n}) = \frac{1}{n}I(X^n; Y^{nR_n})$$

$$\leq H(X) - c_{\epsilon}(H(X) - \epsilon) + \delta$$

$$= H(X) - L\log|\mathcal{X}| + \delta.$$

Since this holds for any δ , then $R' \leq H(X) - L \log |\mathcal{X}|$. However, from Theorem 1, $R' \geq H(X) - L \log |\mathcal{X}|$, and the result follows.

5.1 A scheme based on MDS codes

We now prove that for a uniform i.i.d. source X in \mathbb{F}_q , using scheme 1 with an MDS parity check matrix **H** achieves μ_0 . Since the source is uniform and i.i.d., no source coding is used.

Proposition 1. If **H** is the parity check matrix of an (n, k, d) MDS code and the source X^n is uniform and i.i.d., then Scheme 1 achieves the upper bound $\mu_0 = L$, where L = k/n.

Proof. Let \mathcal{C} be the set of codewords of an (n, k, n - k + 1) MDS code over \mathbb{F}_q with parity matrix **H**, and let $\mathbf{x} \in \mathcal{C}$. Fix a set $\mathcal{J} \in \mathcal{I}_n(k)$ of k positions of \mathbf{x} , denoted $\mathbf{x}^{\mathcal{J}}$. Since the minimum distance of \mathcal{C} is n - k + 1, for any other codeword in $\mathbf{z} \in \mathcal{C}$ we have $\mathbf{z}^{\mathcal{J}} \neq \mathbf{x}^{\mathcal{J}}$. Denoting by $\mathcal{C}^{\mathcal{J}} = {\mathbf{x}^{\mathcal{J}} \in \mathbb{F}_q^k : x \in \mathcal{C}}$, then $|\mathcal{C}^{\mathcal{J}}| = |\mathcal{C}| = q^k$. Therefore, $\mathcal{C}^{\mathcal{J}}$ contains all possible combinations of k symbols. Since this property also holds for any coset of **H**, the result follows.

6 A Rate-Distortion View of Symbol Secrecy

Symbol secrecy provides a fine-grained metric for quantifying the amount of information that leaks from a security system. However, standard cryptographic definitions of security are concerned not only with what an eavesdropper learns about individual symbols of the plaintext, but also which functions of the plaintext an adversary can reliably infer. In order to derive analogous informationtheoretic metrics for security, in this section we take a step back from the symmetric-key encryption setup and study the general estimation problem of inferring properties of a hidden variable X from an observation Y. More specifically, we derive lower bounds for the error of estimating functions of X from an observation of Y. By using standard converse results (e.g. Fano's inequality [41, Chap. 2]), symbol secrecy guarantees are then translated to guarantees on how well certain functions of the plaintext can or cannot be estimated.

We first derive converse bounds for the minimum-mean-squared-error (MMSE) of estimating a function ϕ of the hidden variable X given Y. We assume that the MMSE of estimating a set of

functions $\Phi \triangleq \{\phi_j(X)\}_{i=1}^m$ given Y is known, as well as the correlation between $\phi_j(X)$ and $\phi(X)$. Bounds for the MMSE of $\phi(X)$ are then expressed in terms of the MMSE of each $\phi_j(X)$ and the correlation between $\phi(X)$ and $\phi_j(X)$. We also apply this result to the setting where ϕ and ϕ_j are binary functions, and present bounds for the probability of correctly guessing $\phi(X)$ given Y. These results are of independent interest, and are particularly useful in the security setting considered here.

The set of functions Φ can be used to model known properties of a security system. For example, when X is a plaintext and Y is a ciphertext, the functions ϕ_j may represent certain predicates of X that are known to be hard to infer given Y. In privacy systems, X may be a user's data and Y a distorted version of X generated by a privacy preserving mechanism. The set Φ could then represent a set of functions that are known to be easy to infer from Y due to inherent utility constraints of the setup. In particular, as will be shown in Section 6.4, we will consider the functions in Φ as the individual symbols of the plaintext. In this case, the results introduced in this section are used to derive bounds on the MMSE of reconstructing a target function of the plaintext in terms of the symbol-secrecy achieved by the underlying list-source code given by the encryption scheme. This result extends symbol secrecy to a broader setting.

6.1 Lower Bounds for MMSE

The results introduced in this section are based on the following Lemma.

Lemma 3. Let $z_n : (0, \infty)^n \times [0, 1]^n \to \mathbb{R}$ be given by

$$z_n(\mathbf{a}, \mathbf{b}) \triangleq \max\left\{\mathbf{a}^T \mathbf{y} \middle| \mathbf{y} \in \mathbb{R}^n, \|y\|_2 \le 1, \mathbf{y} \le \mathbf{b}\right\}.$$
 (11)

Let π be a permutation of (1, 2, ..., n) such that $b_{\pi(1)}/a_{\pi(1)} \leq ... \leq b_{\pi(n)}/a_{\pi(n)}$. If $b_{\pi(1)}/a_{\pi(1)} \geq 1$, $z_n(\mathbf{a}, \mathbf{b}) = \|\mathbf{a}\|_2$. Otherwise,

$$z_{n}(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^{k^{*}} a_{\pi(i)} b_{\pi(i)} + \sqrt{\left(\left\| \mathbf{a} \right\|_{2}^{2} - \sum_{i=1}^{k^{*}} a_{\pi(i)}^{2} \right) \left(1 - \sum_{i=1}^{k^{*}} b_{\pi(i)}^{2} \right)}$$
(12)

where

$$k^* \triangleq \max\left\{k \in [n] \; \left| \; \frac{b_{\pi(k)}}{a_{\pi(k)}} \le \sqrt{\frac{\left(1 - \sum_{i=1}^{k-1} b_{\pi(i)}^2\right)^+}{\|\mathbf{a}\|_2^2 - \sum_{i=1}^{k-1} a_{\pi(i)}^2}} \right\}.$$
(13)

Proof. The proof is given in the appendix.

Throughout this section we assume $\Phi \subseteq \mathcal{L}_2(p_X)$ and $\mathbb{E}[\phi_i(X)\phi_j(X)] = 0$ for $i \neq j$. Furthermore,

let Y be an observed variable that is dependent of X, and for a given ϕ_i the inequality

$$\max_{\psi \in \mathcal{L}_2(p_Y)} \mathbb{E}\left[\phi_i(X)\psi(Y)\right] = \|\mathbb{E}\left[\phi_i(X)|Y\right]\|_2 \le \lambda_i$$

is satisfied, where $0 \le \lambda_i \le 1$. This is equivalent to $\mathsf{mmse}(\phi_i(X)|Y) \ge 1 - \lambda_i^2$.

Theorem 4. Let $|\mathbb{E}[\phi(X)\phi_i(X)]| = \rho_i > 0$. Denoting $\rho \triangleq (|\rho_1|, \dots, |\rho_m|)$, $\lambda \triangleq (\lambda_1, \dots, \lambda_m)$, $\rho_0 \triangleq \sqrt{1 - \sum_{i=1}^k \rho_i^2}$, $\lambda_0 = 1 \ \rho_0 \triangleq (\rho_0, \rho)$ and $\lambda_0 \triangleq (\lambda_0, \lambda)$, then

$$\|\mathbb{E}\left[\phi(X)|Y\right]\|_{2} \le B_{|\Phi|}(\boldsymbol{\rho}_{0},\boldsymbol{\lambda}_{0}),\tag{14}$$

where

$$B_{|\Phi|}(\boldsymbol{\rho}_{0},\boldsymbol{\lambda}_{0}) \triangleq \begin{cases} z_{|\Phi|+1}\left(\boldsymbol{\rho}_{0},\boldsymbol{\lambda}_{0}\right), & \text{if } \rho_{0} > 0, \\ z_{|\Phi|}(\boldsymbol{\rho},\boldsymbol{\lambda}), & \text{otherwise.} \end{cases}$$
(15)

and z_n is given in (11). Consequently,

$$\mathsf{mmse}(\phi(X)|Y) \ge 1 - B_{|\Phi|}(\boldsymbol{\rho}_0, \boldsymbol{\lambda}_0)^2.$$
(16)

Proof. Let $h(X) \triangleq \rho_0^{-1}(\phi(X) - \sum_i \rho_i \phi_i(X))$ if $\rho_0 > 0$, otherwise h(X) = 0. Note that $h(X) \in \mathcal{L}_2(p_X)$. Then for $\psi \in \mathcal{L}_2(p_Y)$

$$|\mathbb{E} \left[\phi(X)\psi(Y)\right]| = \left|\rho_0 \mathbb{E} \left[h(X)\psi(Y)\right] + \sum_{i=1}^m \rho_i \mathbb{E} \left[\phi_i(X)\psi(Y)\right]\right|$$
$$\leq \rho_0 \left|\mathbb{E} \left[h(X)\psi(Y)\right]\right| + \sum_{i=1}^m \left|\rho_i \mathbb{E} \left[\phi_i(X)\psi(Y)\right]\right|$$
$$= \rho_0 \left|\mathbb{E} \left[h(X)(T_X\psi)(X)\right]\right| + \sum_{i=1}^m \left|\rho_i \mathbb{E} \left[\phi_i(X)(T_X\psi)(X)\right]\right|$$

Denoting $|\mathbb{E}[h(X)(T_X\psi)(X)]| \triangleq x_0$, $|\mathbb{E}[\phi_i(X)(T_X\psi)(X)]| \triangleq x_i$, $\mathbf{x} \triangleq (x_0, x_1, \dots, x_m)$, and $\boldsymbol{\rho} \triangleq (\rho_0, |\rho_1|, \dots, |\rho_m|)$, the last inequality can be rewritten as

$$|\mathbb{E}\left[\phi(X)\psi(Y)\right]| \le \boldsymbol{\rho}_0^T \mathbf{x}.$$
(17)

Observe that $\|\mathbf{x}\|_2 \leq 1$ and $x_i \leq \lambda_i$ for i = 0, ..., m, and the right hand side of (17) can be maximized over all values of \mathbf{x} that satisfy these constraints. We assume, without loss of generality, that $\rho_0 > 0$ (otherwise set $x_0 = 0$). The left-hand side of (17) can be further bounded by

$$|\mathbb{E}\left[\phi(X)\psi(Y)\right]| \le z_{m+1}(\rho_0, \lambda_0),\tag{18}$$

where $\boldsymbol{\lambda} = (1, \lambda_1, \dots, \lambda_m)$ and z_{m+1} is defined in (11). The result follows directly from Lemma 3 and noting that $\max_{\psi \in \mathcal{L}_2(p_Y)} \mathbb{E}\left[\phi(X)\psi(Y)\right] = \|\mathbb{E}\left[\phi(X)|Y\right]\|_2$.

Denote $\psi_i \triangleq T_Y \phi_i / ||T_Y \phi_i||_2$ and $\phi_0(X) \triangleq (\phi(X) - \sum_{i=1}^m \rho_i \phi_i(X)) / \rho_0^{-1}$. The previous bound can be further improved when $\mathbb{E}[\psi_i(Y)\phi_j(X)] = 0$ for $i \neq j, j \in \{0, \dots, m\}$.

Theorem 5. Let $|\mathbb{E}[\phi(X)\phi_i(X)]| = \rho_i > 0$ for $\phi_i \in \Phi$. In addition, assume $\mathbb{E}[\psi_i(Y)\psi_j(Y)] = 0$ for $i \neq j, i \in [t]$ and $j \in \{0, \ldots, |\Phi|\}$, where $0 \leq t \leq |\Phi|$. Then

$$\|\mathbb{E}\left[\phi(X)|Y\right]\|_{2} \leq \sqrt{\sum_{k=1}^{t} \lambda_{i}^{2} \rho_{i}^{2} + B_{|\Phi|-t} \left(\widetilde{\boldsymbol{\rho}}, \widetilde{\boldsymbol{\lambda}}\right)^{2}},\tag{19}$$

where $\tilde{\rho} = (\rho_0, \rho_t, \dots, \rho_m)$, $\tilde{\lambda} = (1, \lambda_t, \dots, \lambda_m)$ and B_m is defined in (15) (considering $B_0 = 0$). In particular, if t = m,

$$\|\mathbb{E}[\phi(X)|Y]\|_{2} \leq \sqrt{\rho_{0}^{2} + \sum_{k=1}^{|\Phi|} \lambda_{i}^{2} \rho_{i}^{2}},$$
(20)

and this bound is tight when $\rho_0 = 0$. Furthermore,

$$\mathsf{mmse}(\phi(X)|Y) \ge 1 - \sum_{k=1}^{t} \lambda_i^2 \rho_i^2 - B_{|\Phi|-t} \left(\widetilde{\boldsymbol{\rho}}, \widetilde{\boldsymbol{\lambda}}\right)^2.$$
(21)

Proof. For any $\psi \in \mathcal{L}_2(p_Y)$, let $\alpha_i \triangleq \mathbb{E}[\psi(Y)\psi_i(Y)]$ and $\psi_0(Y) \triangleq (\psi(Y) - \sum_{i=1}^t \alpha_i\psi_i(Y))\alpha_0^{-1}$, where $\alpha_0 = (1 - \sum_{i=i}^t \alpha_i^2)^{-1/2}$. Observe that $\psi_0 \in \mathcal{L}_2(p_Y)$ and $\mathbb{E}[\phi_i(X)\psi_j(Y)] = \mathbb{E}[\psi_i(Y)\psi_j(Y)] = 0$ for $i \neq j, i \in \{0, \dots, |\Phi|\}$ and $j \in [t]$. Consequently

$$\mathbb{E}\left[\phi(X)\psi(Y)\right] = \mathbb{E}\left[\left(\sum_{i=0}^{|\Phi|} \rho_i\phi_i(X)\right)\left(\sum_{j=0}^t \alpha_j\psi_j(Y)\right)\right]$$
$$= \sum_{i=0}^{|\Phi|} \sum_{j=0}^t \rho_i\alpha_j\mathbb{E}\left[\phi_i(X)\psi_j(Y)\right]$$
$$\leq \left|\alpha_0\sum_{i=0,i\notin[n]}^{|\Phi|} \rho_i\mathbb{E}\left[\phi_i(X)\psi_0(Y)\right]\right| + \sum_{i=1}^t |\lambda_i\rho_i\alpha_i|$$
$$\leq |\alpha_0|B_{|\Phi|-t}\left(\widetilde{\rho},\widetilde{\lambda}\right) + \sum_{i=1}^t |\lambda_i\rho_i\alpha_i| \tag{22}$$
$$\leq \left[\sum_{i=0}^t \lambda_i^2 \rho_i^2 + B_{i+1} - \left(\widetilde{\rho},\widetilde{\lambda}\right)^2\right]$$

$$\leq \bigvee \sum_{i=1}^{\infty} \lambda_i^{-} \rho_i^{-} + B_{|\Phi|-t} \left(\boldsymbol{\rho}, \boldsymbol{\lambda} \right)$$
(23)

Inequality (22) follows from the bound (14), and (23) follows by observing that $\sum_{i=0}^{t} \alpha_i^2 = 1$ and applying the Cauchy-Schwarz inequality.

Finally, when $\rho_0 = 0$, (23) can be achieved with equality by taking $\psi = \sum_i \frac{\lambda_i \rho_i}{\sqrt{\sum_i \lambda_i^2 \rho_i^2}} \psi_i$.

The following three, diverse examples illustrate different usage cases of Theorems 4 and 5. Example 5 illustrates Theorem 5 for the binary symmetric channel. In this case, the basis Φ can be conveniently expressed as the parity bits of the input to the channel. Example 6 illustrates how Theorem 5 can be applied to the q-ary symmetric channel, and demonstrates that bound (20) is sharp. Finally, Example 7 then illustrates Theorem 4 for the specific case where all the values ρ_i and λ_i are equal.

Example 5 (Binary Symmetric Channel). Let $\mathcal{X} = \{-1, 1\}$ and $\mathcal{Y} = \{-1, 1\}$, and Y^n be the result of passing X^n through a memoryless binary symmetric channel with crossover probability ϵ . We also assume that X^n is composed by n uniform and i.i.d. bits. For $\mathcal{S} \subseteq [n]$, let $\chi_{\mathcal{S}}(X^n) \triangleq \prod_{i \in \mathcal{S}} X_i$. Any function $\phi : \mathcal{X} \to \mathbb{R}$ can then be decomposed in terms of the basis of functions $\chi_{\mathcal{S}}(X^n)$ as [44]

$$\phi(X^n) = \sum_{\mathcal{S} \subseteq [n]} c_{\mathcal{S}} \chi_{\mathcal{S}}(X^n),$$

where $c_{\mathcal{S}} = \mathbb{E}[\phi(X^n)\chi_{\mathcal{S}}(X^n)]$. Furthermore, since $\mathbb{E}[\chi_{\mathcal{S}}(X^n)|Y^n] = (1-2\epsilon)^{|\mathcal{S}|}$, it follows from Theorem 5 that

$$\mathsf{mmse}(\phi(X^n)|Y^n) = 1 - \sum_{\mathcal{S}\subseteq[n]} c_{\mathcal{S}}^2 (1-2\epsilon)^{2|\mathcal{S}|}.$$
(24)

This result can be generalized for the case where $X^n = Y^n \otimes Z^n$, where the operation \otimes denotes bit-wise multiplication, Z^n is drawn from $\{-1, 1\}^n$ and X^n is uniformly distributed. In this case

$$\mathsf{mmse}(\phi(X^n)|Y^n) = 1 - \sum_{\mathcal{S}\subseteq[n]} c_{\mathcal{S}}^2 \mathbb{E}\left[\chi_{\mathcal{S}}(Z^n)\right]^2.$$
(25)

This example will be revisited in Section 6.3, where we restrict ϕ to be a binary function.

Example 6 (q-ary symmetric channel). For $\mathcal{X} = \mathcal{Y} = [q]$, an (ϵ, q) -ary symmetric channel is defined by the transition probability

$$p_{Y|X}(y|x) = (1-\epsilon)\mathbb{1}_{y=x} + \epsilon/q.$$
 (26)

Any function $\phi_i \in \mathcal{L}_2(p_X)$ such that $\mathbb{E}[\phi_i(X)] = 0$ satisfies

$$\psi_i(Y) = T_Y \phi(X) = (1 - \epsilon)\phi(Y),$$

and, consequently, $||T_Y\phi(X)||_2 = (1-\epsilon)$. We shall use this fact to show that the bound (20) is sharp in this case.

Observe that for $\phi_i, \phi_j \in \mathcal{L}_2(p_X)$, if $\mathbb{E}[\phi_i(X)\phi_j(X)] = 0$ then $\mathbb{E}[\psi_i(Y)\psi_j(Y)] = 0$. Now let $\phi \in \mathcal{L}_2(p_X)$ satisfy $\mathbb{E}[\phi(X)] = 0$ and $\mathbb{E}[\phi(X)\phi_i(X)] = \rho_i$ for $\phi_i \in \Phi$, where $|\Phi| = m$, Φ satisfies the conditions in Theorem 5, and $\sum_i \rho_i^2 = 1$. In addition, $\|\psi_i\|_2 = (1 - \epsilon) = \lambda_i$. Then, from (20),

$$||T_Y\phi(X)||_2 \le \sqrt{\sum_{i=1}^m \lambda_i^2 \rho_i^2}$$

$$= (1 - \epsilon) \sqrt{\sum_{i} \rho_i^2}$$
$$= 1 - \epsilon,$$

which matches $||T_Y \phi(X)||_2$, and the bound is tight in this case.

Example 7 (Equal MMSE and correlation). We now turn our attention to Theorem 4. Consider the case when the correlations of ϕ with the references functions ϕ_i are all the same, and each ϕ_i can be estimated with the same MMSE, i.e. $\lambda_1 = \cdots = \lambda_m = \lambda$ and $\rho_1^2 = \cdots = \rho_m^2 = \rho^2$, $\rho \ge 0$ and $\lambda^2 \le \rho^2 \le 1/m$. Then bound (14) becomes

$$\|\mathbb{E}\left[\phi(X)|Y\right]\|_{2} \leq m\lambda\rho + \sqrt{(1-m\rho^{2})(1-m\lambda^{2})}.$$

6.2 One-Bit Functions

Let X be a hidden random variable and Y be a noisy observation of X. Here we denote $\Phi = \{\phi_i\}_{i=1}^m$ a collection of m predicates of X, where $F_i = \phi_i(X), \phi_i : \mathcal{X} \to \{-1, 1\}$ for $i \in [m]$ and, without loss of generality $\mathbb{E}[F_i] = b_i \ge 0$.

We denote by \hat{F}_i an estimate of F_i given an observation of Y, where $F_i \to X \to Y \to \hat{F}_i$. We assume that for any \hat{F}_i

$$\left|\mathbb{E}[F_i\hat{F}_i]\right| \le 1 - 2\alpha_i$$

for some $0 \le \alpha_i \le (1-b_i)/2 \le 1/2$. This condition is equivalent to imposing that $\Pr\{F_i \ne \hat{F}_i\} \ge \alpha_i$, since

$$\mathbb{E}\left[F_i\hat{F}_i\right] = \Pr\{F_i = \hat{F}_i\} - \Pr\{F_i \neq \hat{F}_i\}$$
$$= 1 - 2\Pr\{F_i \neq \hat{F}_i\}.$$

In particular, this captures how well F_i can be guessed based solely on an observation of Y.

Now assume there is a bit $F = \phi(Y)$ such that $\mathbb{E}[FF_i] = \rho_i$ for $i \in [m]$ and $\mathbb{E}[F_iF_j] = 0$ for $i \neq j$. We can apply the same method used in the proof of Theorem 4 to bound the probability of F being guessed correctly from an observation of Y.

Corollary 1. For $\lambda_i = 1 - 2\alpha_i$,

$$\Pr(F \neq \hat{F}) \ge \frac{1}{2} \left(1 - B_{|\Phi|}(\boldsymbol{\rho}, \boldsymbol{\lambda}) \right).$$
(27)

Proof. The proof follows the same steps as Theorem 4, $\phi(Y) \in \mathcal{L}_2(p_Y)$.

In the case m = 1, we obtain the following simpler bound, presented in Proposition 2, which depends on the following Lemma.

Lemma 4. For any random variables A, B and C

$$\Pr(A \neq B) \le \Pr(A \neq C) + \Pr(B \neq C).$$

Proof.

$$Pr(A \neq B) = Pr(A \neq B \land B = C) + Pr(A \neq B \land B \neq C)$$
$$= Pr(A \neq C \land B = C) + Pr(B \neq C) Pr(A \neq B | B \neq C)$$
$$\leq Pr(A \neq C) + Pr(B \neq C).$$

Proposition 2. If $Pr(F_1 \neq \hat{F}_1) \geq \alpha$ for all \hat{F}_1 and $\mathbb{E}[FF_1] = \rho \geq 0$. Then for any estimator \hat{F}

$$\Pr(F \neq \hat{F}) \ge \left(\frac{1-\rho}{2} - \alpha\right)^+.$$
(28)

Proof. From Lemma 4:

$$\Pr(F \neq \hat{F}) \ge \left(\Pr(F_1 \neq F) - \Pr(F_1 \neq \hat{F})\right)^+$$
$$\ge \left(\frac{1-\rho}{2} - \alpha\right)^+.$$

6.3 One-Time Pad Encryption of Functions with Boolean Inputs

We return to the setting where a legitimate transmitter (Alice) wishes to communicate a plaintext message X^n to a legitimate receiver (Bob) through a channel observed by an eavesdropper (Eve). Both Alice and Bob share a secret key K that is not known by Eve. Alice and Bob use a symmetric key encryption scheme determined by the pair of encryption and decryption functions (Enc, Dec), where $Y^n = \text{Enc}(X^n, K)$ and $X^n = \text{Dec}(Y^n, K)$. Here we assume that both the ciphertext and the plaintext have the same length.

We use the results derived in the previous section to assess the security properties of the onetime pad with non-uniform key distribution when no assumptions are made on the computational resources available to Eve. In this case, perfect secrecy (i.e. $I(X^n; Y^n) = 0$) can only be achieved when $H(K) \ge H(X^n)$ [3], which, in turn, is challenging in practice. Nevertheless, as we shall show in this section, information-theoretic security claims can still be made in the short key regime, i.e. $H(K) < H(X^n)$. We first prove the following ancillary result.

Lemma 5. Let F be a Boolean random variable and $F \to X \to Y \to \hat{F}$, where $|\mathcal{Y}| \ge 2$. Furthermore, $\Pr\{F \neq \hat{F}\} \ge \alpha$ for all $Y \to \hat{F}$. Then $I(F;Y) \le 1 - 2\alpha$.

Proof. The result is a direct consequence of the fact that the channel with binary input and finite output alphabet that maximizes mutual information for a fixed error probability is the erasure channel, proved next. Assume, without loss of generality, that $\mathcal{Y} = [m]$ and $p_{F,Y}(-1, y) \ge p_{F,Y}(1, y)$ for $y \in [k]$ and $p_{F,Y}(-1, y) \le p_{F,Y}(1, y)$ for $y \in \{k + 1, \ldots, m\}$, where $k \in [m]$. Now let \widetilde{Y} be a random variable that takes values in [2m] such that

$$p_{F,\tilde{Y}}(b,y) = \begin{cases} p_{F,Y}(b,y) - p_{F,Y}(1,y) & y \in [k], \\ p_{F,Y}(b,y) - p_{F,Y}(-1,y) & y \in \{k+1,\dots,m\}, \\ p_{F,Y}(1,y) & y - m \in [k], \\ p_{F,Y}(-1,y) & y - m \in \{k+1,\dots,m\}. \end{cases}$$

Note that $F \to \tilde{Y} \to Y$, since $Y = \tilde{Y} - m\mathbb{1}_{\{\tilde{Y} > m\}}$ and, consequently, $I(F; \tilde{Y}) \ge I(F; Y)$. Furthermore, the reader can verify that

$$\min_{Y \to \hat{F}} \Pr\{F \neq \hat{F}\} = \min_{\widetilde{Y} \to \hat{F}} \Pr\{F \neq \hat{F}\} = \alpha.$$

In particular, given the optimal estimator $\tilde{Y} \to \hat{F}$, a detection error can only occur when $\tilde{Y} \in \{k+1,\ldots,m\}$, in which case $\hat{F} = F$ with probability 1/2.

Finally,

$$\begin{split} H(F|\widetilde{Y}) &= -\sum_{\substack{b \in \{-1,1\}\\y \in [2m]}} p_{\widetilde{Y}}(y) p_{F|\widetilde{Y}}(b|y) \log p_{F|\widetilde{Y}}(b|y)} \\ &= \sum_{y \in \{m+1,2m\}} p_{\widetilde{Y}}(y) \\ &\geq 2\alpha. \end{split}$$

Consequently, $I(F; \widetilde{Y}) = H(F) - H(F|\widetilde{Y}) \le 1 - 2\alpha$. The result follows.

Let X^n be a plaintext message composed by a sequence of n bits drawn from $\{-1,1\}^n$. The plaintext can be perfectly hidden by using a one-time pad: A ciphertext Y^n is produced as $Y^n = X^n \otimes Z^n$, where the key $K = Z^n$ is a uniformly distributed sequence of n i.i.d. bits chosen independently from X^n . The one-time pad is impractical since, as mentioned, it requires Alice and Bob to share a very long key.

Instead of trying to hide the entire plaintext message, assume that Alice and Bob wish to hide only a set of functions of the plaintext from Eve. In particular, we denote this set of functions as $\Phi = \{\phi_1, \ldots, \phi_m\}$ where $\phi_i : \{-1, 1\}^n \to \{-1, 1\}, \mathbb{E}[\phi_i(X^n)] = 0$ and $\mathbb{E}[\phi_i(X^n)\phi_j(X^n)] = 0$. The set of functions Φ is said to be hidden $I(\phi_i(X^n); Y^n) = 0$ for all $\phi_i \in \Phi$. Can this be accomplished with a key that satisfies $H(K) \ll H(X^n)$? The answer is positive, but it depends on Φ . We denote the Fourier expansion of $\phi_i \in \Phi$ as

$$\phi_i = \sum_{\mathcal{S} \subseteq [n]} \rho_{i,\mathcal{S}} \chi_{\mathcal{S}}.$$

The following result shows that ϕ_i is perfectly hidden from Eve if and only if $I(\chi_{\mathcal{S}}(X^n); Y^n) = 0$ for all $\chi_{\mathcal{S}}$ such that $\rho_{i,\mathcal{S}} > 0$.

Lemma 6. If $I(\phi_i(X^n); Y^n) = 0$ for all $\phi_i \in \Phi$, then $I(\chi_{\mathcal{S}}(X^n); Y^n) = 0$ for all \mathcal{S} such that $\rho_{i,\mathcal{S}} > 0$ for some $i \in [m]$.

Proof. Assume that $I(\chi_{\mathcal{S}}(X^n); Y^n) > 0$ for a given $\rho_{i,\mathcal{S}} > 0$. Then there exists $b : \mathcal{Y}^n \to \{-1,1\}$ such that $\mathbb{E}[b(Y^n)\chi_{\mathcal{S}}(X^n)] = \lambda > 0$. Consequently, from (20), $\mathbb{E}[b(Y^n)\phi_1(X^n)] \ge \lambda \rho_{i,\mathcal{S}} > 0$, and $\phi_1(X^n)$ is not independent of Y^n .

The previous result shows that hiding a set of functions perfectly, or even a single function, might be as hard as hiding X^n . Indeed, if there is a $\phi_i \in \Phi$ such that $\mathbb{E}[\phi_i(X^n)\mathcal{X}_{\mathcal{S}}(X^n)] > 0$ for all $\mathcal{S} \subseteq [n]$ where $|\mathcal{S}|= 1$, then perfectly hiding this set of functions can only be accomplished by using a one-time pad. Nevertheless, if we step back from perfect secrecy, a large class of functions can be hidden with a comparably small key, as in the next example.

Example 8 (BSC revisited). Let Z^n be a sequence of n i.i.d. bits such that $\Pr\{Z_i = -1\} = \epsilon$, and consider once again the one-time pad $Y^n = X^n \otimes Z^n$. Furthermore, denote

$$\Phi_k = \{ \phi : \{-1,1\}^n \to \{-1,1\} \mid \mathbb{E}\left[\phi(X^n)\chi_{\mathcal{S}}(X^n)\right] = 0 \; \forall |\mathcal{S}| < k \}$$

Let $\phi \in \Phi_k$ and $\phi(X^n) = \sum_{\mathcal{S}:|\mathcal{S}| \ge k} \rho_{\mathcal{S}} \chi_{\mathcal{S}}(X^n)$. Then, from Theorem 5 and Corollary 1, for any $\hat{b}: \mathcal{Y}^n \to \{-1, 1\},$

$$\Pr\{\phi(X^n) \neq \hat{b}(Y^n)\} \ge \frac{1}{2} \left(1 - \sqrt{\sum_{|\mathcal{S}| > T} \rho_{\mathcal{S}}^2 (1 - 2\epsilon)^{2|\mathcal{S}|}} \right)$$
$$\ge \frac{1}{2} \left(1 - (1 - 2\epsilon)^k \right).$$

Consequently, from Lemma 5, $I(\phi(X^n); Y^n) \leq (1 - 2\epsilon)^k$ for all $\phi \in \Phi_k$. Note that $H(Z^n) = nh(\epsilon)$, which can be made very small compared to n. Therefore, even with a small key, a large class of functions can be almost perfectly hidden from the eavesdropper through this simple one-time pad scheme. The BSC setting discussed in Example 5 is generalized in the following theorem which, in turn, is a particular case of the analysis in [45].

Theorem 6 (Generalized One-time Pad). Let $Y^n = X^n \otimes Z^n$, $X^n \perp Z^n$, X^n be uniformly distributed, $\phi : \{-1,1\}^n \to \{-1,1\}$ and $\phi(X^n) = \sum_{S \subseteq [n]} \rho_S \chi_S(X^n)$. We define $c_S \triangleq \mathbb{E}[\chi_S(Z^n)]$ for $S \subseteq [n]$. Then

$$I(\phi(X^n); Y^n) \le \sqrt{\sum_{\mathcal{S}\subseteq[n]} (c_{\mathcal{S}}\rho_{\mathcal{S}})^2}.$$
(29)

In particular, $I(\phi(X^n); Y^n) = 0$ if and only if $c_{\mathcal{S}} = 0$ for all \mathcal{S} such that $\rho_{\mathcal{S}} \neq 0$.

Proof. Let $\psi : \{-1,1\}^n \to \{-1,1\}$ and $\psi(Y^n) = \sum_{\mathcal{S} \subseteq [n]} d_{\mathcal{S}} \chi_{\mathcal{S}}(Y^n)$. Note that $\sum_{\mathcal{S} \subseteq [n]} d_{\mathcal{S}}^2 = 1$. Then

$$\mathbb{E}\left[\phi(X^{n})\psi(Y^{n})\right] = \mathbb{E}\left[\phi(X^{n})\mathbb{E}\left[\psi(Y^{n})|X^{n}\right]\right]$$

$$= \mathbb{E}\left[\phi(X^{n})\sum_{S\subseteq[n]}d_{S}\mathbb{E}\left[\chi_{S}(Y^{n})|X^{n}\right]\right]$$

$$= \mathbb{E}\left[\phi(X^{n})\sum_{S\subseteq[n]}d_{S}\mathbb{E}\left[\chi_{S}(X^{n}\otimes Z^{n})|X^{n}\right]\right]$$

$$= \mathbb{E}\left[\phi(X^{n})\sum_{S\subseteq[n]}d_{S}\mathbb{E}\left[\chi_{S}(X^{n})\chi_{S}(Z^{n})|X^{n}\right]\right]$$

$$= \sum_{S\subseteq[n]}d_{S}\mathbb{E}\left[\phi(X^{n})\chi_{S}(X^{n})\right]\mathbb{E}\left[\chi_{S}(Z^{n})\right]$$

$$= \sum_{S\subseteq[n]}d_{S}\rho_{S}c_{S}$$

$$\leq \sqrt{\sum_{S\subseteq[n]}(c_{S}\rho_{S})^{2}},$$
(31)

where (31) follows from the Cauchy-Schwarz inequality. The inequality (29) then follows from Lemma 5. Finally, assume there exists $S \subseteq [n]$ such that both $c_S \neq 0$ and $\rho_S \neq 0$. Then setting $\psi(Y^n) = \chi_S(Y^n)$, it follows from (30) that $\mathbb{E}[\phi(X^n)\psi(Y^n)] = \rho_S c_S \neq 0$ and, consequently, $I(\phi(X^n);Y^n) > 0.$

6.4 From Symbol Secrecy to Function Secrecy

Symbol secrecy captures the amount of information that an encryption scheme leaks about individual symbols of a message. A given encryption scheme can achieve a high level of (weak) information-theoretic security, but low symbol secrecy. As illustrated in Section 4.1, by sending a constant fraction of the message in the clear, the average amount of information about the plaintext that leaks relative to the length of the message can be made arbitrarily small, nevertheless the symbol secrecy performance is always constant (i.e. does not decrease with message length).

When X is uniformly drawn from \mathbb{F}_q for which an (n, k, n - k + 1) MDS code exists, then an absolute symbol secrecy of k/n can always be achieved using the encryption scheme suggested in Proposition 1. If X is a binary random variable, then we can map sequences of plaintext bits of length $\lfloor \log_2 q \rfloor$ to an appropriate symbol in \mathbb{F}_q , and then use the parity check matrix of an MDS code to achieve a high symbol secrecy. Therefore, we may assume without loss of generality that X^n is drawn from $\{-1,1\}^n$. We also make the assumption that X^n is uniformly distributed. This can be regarded as an approximation for the distribution of X^n when it is, for example, the output of an optimal source encoder with sufficiently large blocklength.

Theorem 7. Let X^n be a uniformly distributed sequence of n bits, $Y = \text{Enc}_n(X^n, K)$, and u_{ϵ} and ϵ_t^* the corresponding symbol secrecy and dual symbol secrecy of Enc_n , defined in (1) and (3), respectively. Furthermore, for $\phi : \{-1, 1\}^n \to \{-1, 1\}$ and $\mathbb{E}[\phi(X^n)] = 0$, let $\phi(X^n) = \sum_{S \subseteq [n]} \rho_S \chi_S(X^n)$. Then for any $\hat{\phi} : \mathcal{Y} \to \{-1, 1\}$

$$\Pr\{\phi(X^n) \neq \hat{\phi}(Y)\} \ge \frac{1}{2} \left(1 - B_{|\Phi|}(\boldsymbol{\rho}, \boldsymbol{\lambda})\right), \tag{32}$$

where $\Phi = \{\chi_{\mathcal{S}} : \rho_{\mathcal{S}} \neq 0\}, \ \lambda(t) \triangleq h_b^{-1}((1 - \epsilon_t^* t)^+), \ \boldsymbol{\lambda} = \{\lambda(|\mathcal{S}|)\}_{\mathcal{S}\subseteq[n]} \ and \ \boldsymbol{\rho} = \{|\rho_{\mathcal{S}}|\}_{\mathcal{S}\subseteq[n]}.$ In particular,

$$\Pr\{\phi(X^n) \neq \hat{\phi}(Y)\} \ge \frac{1}{2} \left(1 - \sqrt{\sum_{|\mathcal{S}| > n\mu_0} \rho_{\mathcal{S}}^2} \right).$$
(33)

Proof. From the definition of symbol secrecy, for any $S \subseteq [n]$ with |S| = t

$$I(\chi_{\mathcal{S}}(X^n);Y) \le I(X^{\mathcal{S}};Y) \le \epsilon_t^* t,$$

and, consequently,

$$H(\chi_{\mathcal{S}}(X^n)|Y) \ge (1 - \epsilon_t^* t)^+.$$

From Fano's inequality, for any binary \hat{F} where $Y \to \hat{F}$

$$\Pr\{\chi_{\mathcal{S}}(X^n) \neq \hat{F}\} \ge h_b^{-1}((1 - \epsilon_t^* t)^+),$$

where $h_b^{-1}: [0,1] \to [0,1/2]$ is the inverse of the binary entropy function. In particular, from the definition of absolute symbol secrecy, if $\epsilon_t^* = 0$, then

$$\Pr\{\chi_{\mathcal{S}}(X^n) \neq \hat{F}\} = 1/2 \; \forall |\mathcal{S}| \le n\mu_0.$$

The result then follows directly from Theorem 5, the fact that $\phi(X^n) = \sum_{S \subseteq [n]} \rho_S \chi_S(X^n)$ and letting $\lambda(t) \triangleq h_b^{-1}((1 - \epsilon_t^* t)^+)$.

7 Discussion

In this section we discuss the application of our results to different settings in privacy and cryptography.

7.1 The Correlation-Error Product

We momentarily diverge from the cryptographic setting and introduce the *error-correlation product* for the privacy setting considered by Calmon and Fawaz in [10]. Let W and X be two random

variables with joint distribution $p_{W,X}$. W represents a variable that is supposed to remain private, while X represents a variable that will be released to an untrusted data collector in order to receive some utility based on X. The goal is to design a randomized mapping $p_{Y|X}$, called the privacy assuring mapping, that transforms X into an output Y that will be disclosed to a third party.

The goal of a privacy assuring mechanism is to produce an output Y, derived from X according to the mapping $p_{Y|X}$, that will be released to the data collector in the place of X. The released variable Y is chosen such that W cannot be inferred reliably given an observation of Y. Simultaneously, given an appropriate distortion metric, X should be close enough to Y so that a certain level of utility can still be provided. For example, W could be a user's political preference, and X a set of movie ratings released to a recommender system in order to receive movie recommendations. Y is chosen as a perturbed version of the movie recommendations so that the user's political preference is obscured, while meaningful recommendations can still be provided.

Given $W \to X \to Y$ and $p_{W,X}$, a privacy assuring mapping is given by the conditional distribution $p_{Y|X}$. The choice of $p_{Y|X}$ determines the tradeoff between privacy and utility. If $p_{Y|X} = p_Y$, then perfect privacy is achieved (i.e. W and Y are independent), but no utility can be provided. Conversely, if $p_{Y|X}$ is the identity mapping, then no privacy is gained, but the highest level of utility can be provided.

When $W = \phi(X)$ where $\phi \in \mathcal{L}_2(p_X)$, the bounds from Section 6.1 shed light on the fundamental privacy-utility tradeoff. Returning to the notation of Section 6.1, let $W = \phi(X)$ be correlated with a set of functions $\Phi = \{\phi_i\}_{i=1}^m$. The next result is a direct corollary of Theorem 5.

Corollary 2. Let $\mathbb{E}[W\phi_i(X)] = \rho_i$, $\sum_{i=1}^{|\Phi|} \rho_i^2 = 1$, $\psi_i(Y) = \mathbb{E}[\phi_i(X)|Y]$ and, for $i \neq j$, $\mathbb{E}[\phi_i(X)\phi_j(X)] = 0$ and $\mathbb{E}[\psi_i(Y)\psi_j(Y)] = 0$. Then

$$\mathsf{mmse}(W|Y) = \sum_{i=1}^{|\Phi|} \mathsf{mmse}(\phi_i(Y)|X)\rho_i^2. \tag{34}$$

We call the product $\mathsf{mmse}(\phi_i(Y)|X)\rho_i^2$ the error-correlation product. The secret variable W cannot be estimated with low MMSE from Y if and only if the functions ϕ_i that are strongly correlated with W (i.e. large ρ_i^2) cannot be estimated reliably. Consequently, if ρ_i is large and ϕ_i is relevant for the utility provided by the data collector, privacy cannot be achieved without a significant loss of utility: $\mathsf{mmse}(\phi_i(X)|Y)$ is necessarily large if $\mathsf{mmse}(W|Y)$ is large. Conversely, in order to hide W, it is sufficient to hide the functions $\phi_i(X)$ that are strongly correlated with $\phi(X)$. This no-free-lunch result is intuitive, since one would expect that privacy cannot be achieved if utility is based on data that is strongly correlated with the private variables. The results presented here prove that this is indeed the case.

We present next a general description of a two-phase secure communication scheme for the threat model described in Section 1.5, presented in terms of the list-source code constructions derived using linear codes. Note that this scheme can be easily extended to any list-source code by using the corresponding encoding/decoding functions instead of multiplication by parity check

matrices.

7.2 A Secure Communication Scheme Based on List-Source Codes

We assume that Alice and Bob have access to a symmetric-key encryption/decryption scheme (Enc', Dec') that is used with the shared secret key K and is sufficiently secure against the adversary. This scheme can be, for example, a one-time pad. The encryption/decryption procedure is performed as follows, and will be used as components of the overall encryption scheme (Enc, Dec) described below.

Scheme 2. Input: The source encoded sequence $\mathbf{x} \in \mathbb{F}_q^n$, parity check matrix \mathbf{H} of a linear code in \mathbb{F}_q^n , a full-rank $k \times n$ matrix \mathbf{D} such that rank $([\mathbf{H}^T \ \mathbf{D}^T]) = n$, and encryption/decryption functions (Enc', Dec'). We assume both Alice and Bob share a secret key K.

Encryption (Enc):

Phase I (pre-caching): Alice generates $\boldsymbol{\sigma} = \mathbf{H}\mathbf{x}$ and sends to Bob.²

Phase II (send encrypted data): Alice generates $\mathbf{e} = \mathsf{Enc}'(\mathbf{Dx}, K)$ and sends to Bob.

Decryption (Dec): Bob calculates Dec'(e, K) = Dx and recovers x from σ and Dx.

Assuming that (Enc', Dec') is secure, the information-theoretic security of Scheme 2 reduces to the security of the underlying list-source code (i.e. Scheme 1). In practice, the encryption/decryption functions (Enc', Dec') may depend on a secret or public/private key, as long as it provide sufficient security for the desired application. In addition, assuming that the source sequence is uniform and i.i.d. in F_q^n , we can use MDS codes to make strong security guarantees, as described in the next section. In this case, an adversary that observes σ cannot infer any information about any set of k symbols of the original message.

Note that this scheme has a *tunable* level of secrecy: The amount of data sent in phase I and phase II can be appropriately selected to match the properties of the encryption scheme available, the size of the key length, and the desired level of secrecy. Furthermore, when the encryption procedure has a higher computational cost than the list-source encoding/decoding operations, list-source codes can be used to reduce the total number of operations required by allowing encryption of a smaller portion of the message (phase II).

The protocol outline presented in Scheme 2 is useful in different practical scenarios, which are discussed in the following sections. Most of the advantages of the suggested scheme stem from the fact that list-source codes are key-independent, allowing content to be distributed when a key distribution infrastructure is not yet established, and providing an additional level of security if keys are compromised before phase II in Scheme 2.

 $^{^{2}}$ Here, Alice can use message authentication codes and public key encryption to augment security. Furthermore, the list-source coding scheme can be used as an additional layer of security with information-theoretic guarantees in symmetric-key ciphers. Since we are interested in the information-theoretic security properties of the scheme, we will not go into further details. We do recognize that in order to use this scheme in practice additional steps are needed to meet modern cryptographic standards.

7.3 Content pre-caching

As hinted earlier, list-source codes provide a secure mechanism for content pre-caching when a key infrastructure has not yet been established. A large fraction of the data can be list-source coded and securely transmitted before the termination of the key distribution protocol. This is particularly significant in large networks with hundreds of mobile nodes, where key management protocols can require a significant amount of time to complete [46]. Scheme 2 circumvents the communication delays incurred by key compromise detection, revocation and redistribution by allowing data to be efficiently distributed concurrently with the key distribution protocol, while maintaining a level of security determined by the underlying list-source code.

7.4 Application to key distribution protocols

List-source codes can also provide additional robustness to key compromise. If the secret key is compromised before phase II of Scheme 2, the data will still be as secure as the underlying list-source code. Even if a (computationally unbounded) adversary has perfect knowledge of the key, until the last part of the data is transmitted the best he can do is reduce the number of possible inputs to an exponentially large list. In contrast, if a stream cipher based on a pseudo-random number generator were used and the initial seed was leaked to an adversary, all the data transmitted up to the point where the compromise was detected would be vulnerable. The use of list-source codes provide an additional, information-theoretic level of security to the data up to the point where the last fraction of the message is transmitted. This also allows decisions as to which receivers will be allowed to decrypt the data can be delayed until the very end of the transmission, providing more time for detection of unauthorized receivers and allowing a larger flexibility in key distribution.

In addition, if the level of security provided by the list-source code is considered sufficient and the key is compromised before phase II, the key can be redistributed *without the need of retransmitting the entire data*. As soon as the keys are reestablished, the transmitter simply encrypts the remaining part of the data in phase II with the new key.

7.5 Additional layer of security

We also highlight that list-source codes can be used to provide an additional layer of security to the underlying encryption scheme. The message can be list-source coded after encryption and transmitted in two phases, as in Scheme 2. As argued in the previous point, this provides additional robustness against key compromise, in particular when a compromised key can reveal a large amount of information about an incomplete message (e.g. stream ciphers). Consequently, list-source codes are a simple, practical way of augmenting the security of current encryption schemes.

One example application is to combine list-source codes with stream ciphers. The source-coded message can be initially encrypted using a pseudorandom number generator (PRG) initialized with a randomly selected seed, and then list-source coded. The initial random seed would be part of the encrypted message sent in the final transmission phase. This setup has the advantage of augmenting

the security of the underlying stream cipher, and provides randomization to the list-source coded message. In particular, if the LSC is based on MDS codes and assuming that the distribution of the plaintext is nearly uniform, strong information-theoretic symbol secrecy guarantees can be made about the transmitted data, as discussed in Section 2. Even if the underlying PRG is compromised, the message would still be secure.

7.6 Tunable level of secrecy

List-source codes provide a tunable level of secrecy, i.e. the amount of security provided by the scheme can be adjusted according to the application of interest. This can be done by appropriately selecting the size of the list (L) of the underlying code, which determines the amount of uncertainty an adversary will have regarding the input message. In the proposed implementation using linear codes, this corresponds to choosing the size of the parity check matrix **H**, or, analogously, the parameters of the underlying error-correcting code. In terms of Scheme 2, a larger (respectively smaller) value of L will lead to a smaller (larger) list-source coded message in phase I and a larger (smaller) encryption burden in phase II.

8 Conclusions

We conclude the paper with a summary of our contributions. We introduce the concept of LSCs, which are codes that compress a source below its entropy rate. We derived fundamental bounds for the rate list region, and provided code constructions that achieve these bounds. List-source codes are a useful tool for understanding how to perform encryption when the (random) key length is smaller than the message entropy. When the key is small, we can reduce an adversary's uncertainty to a near-uniformly distributed list of possible source sequences with an exponential (in terms of the key length) number of elements by using list-source codes. We also demonstrated how list-source codes can be implemented using standard linear codes.

Furthermore, we presented a new information-theoretic metric of secrecy, namely ϵ -symbol secrecy, which characterizes the amount of information leaked about specific symbols of the source given an encoded version of the message. We derived fundamental bounds for ϵ -symbol secrecy, and showed how these bounds can be achieved using MDS codes when the source is uniformly distributed.

We also introduced results for bounding the probability that an adversary correctly guesses a predicate of the plaintext in terms of the symbol secrecy achieved by the underlying encryption scheme. These results are based on Lemma 3, which, in turn, was used to derive bounds on the information leakage of a security system that does not achieve perfect secrecy. These bounds provide insight on how to design symmetric-key encryption schemes that hide specific functions of the data, where uncertainty is captured in terms of minimum-mean squared error. These results also shed light on the fundamental privacy-utility tradeoff in privacy systems.

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Appendix A Proof of Lemma 3

For fixed $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ where $a_i > 0$ and $b_i \ge 0$, let $z_P : \mathbb{R}^n \to \mathbb{R}$ and $z_D : \mathbb{R}^n \to \mathbb{R}$ be given by

$$z_P(\mathbf{y}) \triangleq \mathbf{a}^T \mathbf{y},$$

$$z_D(\mathbf{u}) \triangleq \mathbf{a}^T \mathbf{b} + \mathbf{u}^T \mathbf{b} + \|\mathbf{u}\|_2.$$

Furthermore, we define $\mathcal{A}(\mathbf{a}) \triangleq \{\mathbf{u} \in \mathbb{R}^n | \mathbf{u} \ge \mathbf{a}\}$ and $\mathcal{B}(\mathbf{b}) \triangleq \{\mathbf{y} \in \mathbb{R}^n \mid ||\mathbf{y}||_2 \le 1, \mathbf{y} \le \mathbf{b}\}.$

The optimal value $z_n(\mathbf{a}, \mathbf{b})$ is given by the following pair of primal-dual convex programs:

$$z_n(\mathbf{a}, \mathbf{b}) = \max_{\mathbf{y} \in \mathcal{B}(\mathbf{b})} z_P(\mathbf{y}) = \min_{\mathbf{u} \in \mathcal{A}(\mathbf{a})} z_D(\mathbf{u}).$$

Assume, without loss of generality, that $b_1/a_1 \le b_2/a_2 \le \ldots \le b_n/a_n$, and let k^* be defined in (13).

Let $c_j \triangleq \sqrt{\frac{(1-\sum_{i=1}^j b_i^2)}{\|\mathbf{a}\|_2^2 - \sum_{i=1}^j a_i^2}}$. Note that since $\sum_{i=1}^{k^*} b_i^2 < 1$, we have $c_{k^*} > 0$. In addition, let

$$\mathbf{y}^* = (b_1, \dots, b_{k^*}, a_{k^*+1}c_{k^*}, \dots, a_nc_{k^*})$$

and

$$\mathbf{u}^* = (-b_1/c_{k^*}, \dots, -b_{k^*}/c_{k^*}, -a_{k^*+1}, \dots, -a_n).$$

From the definition of k^* , $\mathbf{y}^* \in \mathcal{B}(\mathbf{b})$ and $\mathbf{u}^* \in \mathcal{A}(\mathbf{a})$. Furthermore,

$$z_{P}(\mathbf{y}^{*}) = \mathbf{a}^{T} \mathbf{y}^{*}$$

$$= \sum_{i=1}^{k^{*}} a_{i} b_{i} + \sum_{i=k^{*}+1}^{n} c_{k^{*}} a_{i}^{2}$$

$$= \sum_{i=1}^{k^{*}} a_{i} b_{i} + \sqrt{\left(\left\| \mathbf{a} \right\|_{2}^{2} - \sum_{i=1}^{k^{*}} a_{i}^{2} \right) \left(1 - \sum_{i=1}^{k^{*}} b_{i}^{2} \right)}, \qquad (35)$$

and

$$z_D(\mathbf{u}^*) = \mathbf{a}^T \mathbf{b} + \mathbf{u}^{*T} \mathbf{b} + \|\mathbf{u}^*\|_2$$

$$\begin{split} &= \sum_{i=1}^{k^*} \left(a_i b_i - \frac{b_i^2}{c_{k^*}} \right) \\ &+ c_{k^*}^{-1} \sqrt{\sum_{i=1}^{k^*} b_i^2 + c_{k^*}^2 \left(\|\mathbf{a}\|_2^2 - \sum_{i=1}^{k^*} a_i^2 \right)} \\ &= \sum_{i=1}^{k^*} a_i b_i + c_{k^*}^{-1} \left(1 - \sum_{i=1}^{k^*} b_i^2 \right) \\ &= \sum_{i=1}^{k^*} a_i b_i + \sqrt{\left(\left\| \mathbf{a} \right\|_2^2 - \sum_{i=1}^{k^*} a_i^2 \right) \left(1 - \sum_{i=1}^{k^*} b_i^2 \right)} \\ &= z_P(\mathbf{y}^*). \end{split}$$

Since both the primal and the dual achieve the same value at \mathbf{y}^* and \mathbf{u}^* , respectively, it follows that the value $z_P(\mathbf{y}^*)$ given in (35) is optimal.