BOUNDARY SMOOTHNESS OF ANALYTIC FUNCTIONS

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Dedicated to Lawrence Zalcman on the occasion of his 70th birthday

Abstract. We consider the behaviour of holomorphic functions on a bounded open subset of the plane, satisfying a Lipschitz condition with exponent α, with $0 < \alpha < 1$, in the vicinity of an exceptional boundary point where all such functions exhibit some kind of smoothness. Specifically, we consider the relation between the abstract idea of a bounded point derivation on the algebra of such functions and the classical complex derivative evaluated as a limit of difference quotients. We obtain a result which applies, for example, when the open set admits an interior cone at the special boundary point.

1. Introduction

Let U be a nonempty open subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be holomorphic on U. Suppose $0 < \alpha < 1$, and f satisfies a Hölder, or Lipschitz condition with exponent α on U: i.e. there exists $\kappa > 0$ such that

$$|f(z) - f(w)| \leq \kappa |z - w|^\alpha, \ \forall z, w \in U. \tag{1}$$

Then f has a unique continuous extension to $Y = \mathrm{clos}(U)$. This extension also satisfies the Lipschitz condition with exponent α on Y, with the same constant κ.

Let b belong to the boundary of U. It may happen that all such f are in some sense smoother at b than a typical Hölder-continuous complex-valued function. That is, the additional assumption of analyticity on U may force additional smoothness at b.

The strongest possible smoothness that might occur would be that all such f are actually holomorphic on a certain neighbourhood of b.

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In that case, b is usually referred to as a *removable singularity for* Lipα *holomorphic functions on* U. This phenomenon was first investigated by Dolzhenko, who showed [1] that b is a removable singularity of this type if and only if there exists $r > 0$ such that $\mathbb{B}(b, r) \sim U$ has zero $(1+\alpha)$-dimensional Hausforff measure, where $\mathbb{B}(b, r)$ denotes the closed disk having center b and radius r. (It seems appropriate to mention here that author's interest in removable singularities, and in this whole area, was first aroused by Larry Zalcman's Monthly paper [9].)

It may also happen that more limited smoothness occurs at a boundary point b that is not removable. In [4] Lord and the author considered the notion of a *continuous point derivation* at a boundary point, and gave a necessary and sufficient condition for the existence of such a nonzero derivation. The concept of continuous point derivation comes from the theory of commutative Banach algebras. If A is a commutative Banach algebra with character space (maximal ideal space) $M(A)$, and $\phi \in M(A)$, then a continuous point derivation on A at ϕ is a continuous linear functional $\partial : A \to \mathbb{C}$ such that the Leibniz rule

$$\partial(fg) = \phi(f)\partial(g) + \partial(f)\phi(g)$$

holds for all $f, g \in A$. In the present case, we considered the algebra $A = A_\alpha(U)$ of all holomorphic functions f on U that belong to the "little Lipschitz class", i.e. are not just Lipα functions, but have the stronger property that for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $z, w \in U$,

$$|z - w| < \delta \implies |f(z) - f(w)| \leq \epsilon |z - w|^\alpha.$$

The norm on A is given by

$$\|f\|_A = \sup_U |f| + \kappa(f),$$

where $\kappa(f)$ is the least value that works in the inequality (1). The characters on this A are precisely the evaluations $f \mapsto f(a)$, for $a \in Y$ (as follows from a result of Sherbert [8]), and we identify the point a with the corresponding character. At points $a \in U$, the map $\partial : f \mapsto f'(a)$ is a nonzero continuous point derivation on A at a. We gave a necessary and sufficient condition in order that there exist a nonzero continuous point derivation on A at a given $b \in \text{bdy}U$. This condition involved a set function known as lower $(1 + \alpha)$-dimensional Hausdorff content, and denoted $M_*^{1+\alpha}$. The existence of the derivation is equivalent to the convergence of a Wiener-type series

$$\sum_{n=0}^{\infty} 4^n M_*^{1+\alpha}(A_n(b) \sim U),$$

where $A_n(b)$ denotes the annulus consisting of those z with

$$\frac{1}{2^{n+1}} \le |z-b| \le \frac{1}{2^n}.$$

For the definition of the content we refer the reader to [4]. We shall not use the content in the present paper, but we note that the above condition is explicit and practical, and allows one to determine by computation whether or not there exists such a point derivation at a given boundary point.

The paper [4] has similar characterizations of the existence of higher-order continuous point derivations on A. It also has results about the "big Lip" algebra of all Lipα holomorphic functions on U. In the latter case, the character space is still Y, but the results are about weak-star continuous derivations — the big Lip algebra always has nonzero continuous point derivations at every point, but only the weak-star continuous ones are of any interest in connection with analytic function theory.

(For the avoidance of confusion, we note that what is here denoted $A_\alpha(U)$ was denoted $a(U)$ in [4], while the notation $A(U)$ was there used for the big Lip version.)

These results are rather abstract, and the purpose of the present paper is to relate them to concrete classical ideas of derivative. We are going to confine attention to the simplest case: the little lip algebra A, and (first-order) continuous point derivations. The question we address is the following:

Suppose the bounded open set U, a boundary point b, and $\alpha \in (0,1)$ are given, and suppose there exists a nonzero continuous point derivation ∂ on $A = A_\alpha(U)$ at b. Can we evaluate ∂f by a formula

$$\partial f = c \lim_{n\uparrow\infty} \frac{f(z_n) - f(b)}{z_n - b},$$

valid for every $f \in A$? (Here, as before, $f(b)$ denotes the value at b of the unique continuous extension of f to bdy$U$.)

We remark that a continuous point derivation at b is uniquely determined by its value $c = \partial z$ at the identity function $z \mapsto z$. This follows using elementary algebra from the fact [4, Lemma 1.1] that the set of functions $f \in A$ that are holomorphic on a neighbourhood of b is a dense subalgebra of A. We say that ∂ is *normalised* if $\partial z = 1$. If ∂ is any nonzero derivation, then ∂/c is normalised.

In the interest of further simplicity, we confine attention to the situation in which the boundary point is nicely accessible from U. We say that U *has an interior cone* at the boundary point b if there is a segment J ending at b and a constant $t > 0$ such that

$$\mathrm{dist}(z, \mathbb{C} \sim U) \geq t|z - b|, \ \forall z \in J.$$

We call such a segment J a *nontangential ray* to b.

We say that a sequence $(z_n)_n$ of points of U *converges non-tangentially to* b, written $z_n \to_{\mathrm{nt}} b$, if there exists a constant $t > 0$ such that

$$\mathrm{dist}(z_n, \mathbb{C} \sim U) \geq t|z_n - b|, \ \forall z \in J.$$

Obviously, if U has an interior cone at b, then any sequence converging to b along a nontangential ray J is converging nontangentially. However, the existence of a sequence converging nontangentially does not imply that b lies on the boundary of a single connected component of U. Without going into details about Hausdorff content, we remark that for a closed ball $M_*^\beta(\mathbb{B}(a,r)) = r^\beta$, that for a line segment J, $M_*^\beta(J) = 0$ if $\beta > 1$, and also that M_*^β is countably subadditive. As a result, it is easy to construct many examples U in which the complement of U is a countable union of closed balls, line segments and the singleton $\{b\}$, and in which A has a continuous point derivation at b. All you have to do is make sure that the sum of the $(1+\alpha)$-th powers of the radii of all the closed balls that meet $A_n(b)$ is no greater than $s_n/4^n$, where $\sum_n s_n < +\infty$.

Our main result is the following:

Theorem 1.1. *Let* $0 < \alpha < 1$*, let* $U \subset \mathbb{C}$ *be a bounded open set, let* $b \in \mathrm{bdy}U$*,* $z_n \in U$*,* $z_n \to_{\mathrm{nt}} b$*. Suppose* $A = A_\alpha(U)$ *admits a nonzero continuous point derivation at* b*. Let* ∂ *be the normalised derivation at* b*. Then for each* $f \in A$*, we have*

$$\frac{f(z_n) - f(b)}{z_n - b} \to \partial f.$$

Corollary 1.2. *Under the same hypotheses, if* U *has an interior cone at* b*, and* J *is a nontangential ray to* b*, then*

$$\partial f = \lim_{z \to b, z \in J} \frac{f(z) - f(b)}{z - b}, \ \forall f \in A.$$

We would expect that these results could be extended to higher order derivations and to weak-star continuous derivations on the big Lip algebra.

In broad outline, the methods we shall use are adapted from those used to prove a similar result about bounded analytic functions in [5]. In that paper, we used duality ideas from functional analysis, the Riesz Representation Theorem, and the Cauchy transform. In order to transfer the methods to the Lipschitz algebra one has to overcome various technical problems.

The methods we use in this paper, using duality and abstractions from functional analysis, are termed nonconstructive, when contrasted with explicit methods that involve direct use of the Cauchy integral formula applied to individual functions. It should also be possible to approach the proof in a constructive spirit. This would involve the explicit use of Hausdorff contents, and it is to some extent a matter of taste (perhaps influenced by familiarity with various techniques) which might be regarded as preferable. We leave the constructive approach for another day.

As will be seen, we extend here the arsenal of techniques for dealing with spaces of holomorphic functions in Lipschitz classes. We expect that these techniques will prove useful in dealing with other problems in the same area, such as the behaviour of functions near a special boundary point which may not be accessible by nontangential approach, and various approximation problems..

Throughout the paper, $0 < \alpha < 1$, $U \subset \mathbb{C}$ is a bounded open set, $Y = \mathrm{clos}(U)$, and $b \in X = \mathrm{bdy}U$.

2. Extensions and Distributions

2.1. We have already remarked that the elements of $A = A_\alpha(U)$ extend uniquely to continuous functions on $Y = \mathrm{clos}(U)$. It is in fact obvious that this extension imbeds A isometrically as a closed subalgebra of $\mathrm{lip}(\alpha, Y)$.

We are interested in point derivations, and point derivations annihilate the constant functions (because $\partial 1 = \partial(1^2) = 2 \cdot 1 \cdot \partial 1$), so we are really more interested in the quotient space $A/\mathbb{C}$ modulo constants, and the 'pure' Lipschitz seminorm $\|f\|' = \kappa(f)$ from (1). We note that the extension to Y also preserves this pure seminorm.

Our first significant point is that the restriction of the extension to the boundary X is also isometric on both the norm and the pure seminorm. That the sup norm is preserved is the classical maximum principle. But we also have:

Lemma 2.1. *Let $f \in \mathrm{Lip}(\alpha, Y)$ be holomorphic on U. Then*

$$\sup\left\{\frac{|f(z) - f(w)|}{|z - w|^\alpha} : z, w \in Y\right\} = \sup\left\{\frac{|f(z) - f(w)|}{|z - w|^\alpha} : z, w \in X\right\}$$

Proof. For each fixed $w_0 \in Y$, Let $\Sigma(w_0)$ be the Riemann surface of $\log(z - w_0)$, and let $p : \Sigma \to \mathbb{C} \sim \{w_0\}$ be the covering map, so that $\log(Z - w_0)$ is a well-defined holomorphic function on $\Sigma(w_0)$, with real part $\log|p(Z)-w_0|$ and imaginary part $\arg(Z-w_0)$. Then $(Z-w_0)^\alpha$, interpreted as

$$\exp(\alpha\log|p(Z)-w_0|+i\alpha\arg(Z-w_0)),$$

is also holomorphic on $\Sigma(w_0)$, and

$$g(Z):=\frac{f(p(Z))-f(w_0)}{(Z-w_0)^\alpha}$$

is a well-defined function on $\Sigma(w_0)\cap p^{-1}(Y)$, with absolute value that depends only on the projection $p(Z)\in\mathbb{C}$. It is holomorphic on $p^{-1}(U)$.

Since $\dfrac{|f(z)-f(w)|}{|z-w|^\alpha}$ is continuous on $Y\times Y$, it attains its supremum, say m, at some point $(z_0,w_0)\in Y\times Y$. If $z_0\notin X$, then we have a contradiction to the maximum principle unless $|f(p(Z))-f(w_0)|$ is identically equal to $m|p(Z)-w_0|^\alpha$ on the connected component of each preimage of z_0 in $p^{-1}(U)$, and hence at some point of $p^{-1}(X)$. Thus we may assume $z_0\in X$. Similarly, we may assume $w_0\in X$. □

As a result, we may regard $A/\mathbb{C}$ as a subspace of $\mathrm{lip}(\alpha,X)/\mathbb{C}$. Thus, by the Hahn-Banach Theorem, each continuous linear functional $T\in A^*$ that annihilates the constants has an isometric extension to the whole of $\mathrm{lip}(\alpha,X)/\mathbb{C}$, hence (by a standard method) may be represented by a Borel-regular measure μ on $X\times X$ having no mass on the diagonal and total mass equal to $\|T\|'$ (the dual norm on A^* to $\|\dot{\|}'$), via a formula

$$Tf=\int_{X\times X}\frac{f(z)-f(w)}{(|z-w|^\alpha}\,d\mu(z,w),\ \forall f\in A.$$

In particular, if we assume that ∂ is a continuous point derivation on A, then it has a representation of this kind.

2.2. **Extensions.** Let lipα denote, for short, the global space $\mathrm{lip}(\alpha,\mathbb{C})$ of bounded lipα functions on $\mathbb{C}$. Each $f\in A$ may be extended (in many different ways) to an element of lipα, without increasing its pure norm $\|f\|'$ or supremum. (In fact it is not difficult to check that if $\omega(r)$ is any concave upper envelope for the modulus of continuity of f, then f may be extended so that its modulus of continuity remains bounded by ω. For instance, this may be seen by applying the method used to prove Kirszbraun's Theorem in [2].) Thus the restriction map to U (or Y, or X) makes A isometric to a quotient space of

$$\tilde{A}=\{f\in\mathrm{lip}\alpha : f\text{ is holomorphic on }U\}.$$

We shall find it convenient to work with globally-defined functions in the sequel.

2.3. **Distributions.** Let $\mathcal{D}$ denote the space of test functions (i.e. C^∞ functions having compact support), and let $\mathcal{D}'$ denote its dual, the Schwartz distribution space. If μ is any complex measure on $X \times X$, having no mass on the diagonal, then we may define a distribution $T_1 \in \mathcal{D}'$ by setting

$$\langle \phi, T_1 \rangle = \int_{X\times X} \frac{\phi(z)-\phi(w)}{|z-w|^\alpha}\, d\mu(z,w),\ \forall \phi \in \mathcal{D}. \tag{2}$$

This distribution T_1 will not, in general, be representable by integration against a locally-integrable funtion or a measure, but will extend continuously to an element of $(\mathrm{lip}\alpha)^*$. It is a bit 'wilder' than a measure. (It may be represented in the form

$$T_1 = \nu_0 + \frac{\partial}{\partial x}\nu_1 + \frac{\partial}{\partial y}\nu_2,$$

where the ν_j are compactly-supported measures, but we shall not use this representation, as it carries less information than the fact that T_1 acts continuously on $\mathrm{lip}\alpha$.) We denote the extension of T_1 to $\mathrm{lip}\alpha$ by the same notation T_1, and write its value as $T_1(f)$ or $\langle f, T_1\rangle$ for any $f \in \mathrm{lip}\alpha$.

Since $\langle \phi, T_1\rangle$ is unaffected if ϕ is altered away from X, it is clear that T_1 has support in X. Thus we can also define $\langle \phi, T_1\rangle$ for any function ϕ defined and C^∞ on a neighbourhood of X to be $\langle \tilde\phi, T_1\rangle$ where $\tilde\phi$ is any element of $\mathcal{E}$ (the space of globally-defined C^∞ functions) that agrees with ϕ near X. For instance, $\left\langle \frac{1}{z-a}, T_1 \right\rangle$ makes sense, for $a \notin X$. Similarly, $\langle f, T_1\rangle$ makes sense whenever f is defined on some neighbourhood of X and satisfies a little-lipα condition there.

2.4. **Cauchy Transforms.** The main idea behind what follows is that although T_1 is wilder than a measure, it is still tame enough to allow us to treat it almost as though it were a measure. Specifically, the Cauchy transform of T_1 (which we are about to define) is representable by integration against a locally-integrable function. This fact was already noted and exploited in [6].

The Cauchy transform of $\phi \in \mathcal{D}$ is its convolution

$$\hat\phi := \phi * \left(\frac{1}{\pi z}\right)$$

with the fundamental solution of $\frac{\partial}{\partial \bar z}$. In other words,

$$\hat\phi(z) = \frac{1}{\pi}\int \frac{\phi(\zeta)}{z-\zeta} dm(\zeta),$$

for all $z \in \mathbb{C}$, where m denotes area measure. This function belongs to the space $\mathcal{E}$, and satisfies

$$\frac{\partial \hat{\phi}}{\partial z} = \phi.$$

For distributions T having compact support, we define

$$\langle \phi, \hat{T} \rangle = -\langle \hat{\phi}, T \rangle, \forall \phi \in \mathcal{D}.$$

If T_1 is given by Equation (2), then consider

$$H(a) = \frac{1}{\pi} \int \frac{z-w}{(z-a)(w-a)|z-w|^\alpha} \, d\mu(z,w),$$

for $a \in \mathbb{C}$. This is well-defined whenever

$$\tilde{H}(a) = \int \frac{|w-z|^{1-\alpha}}{|z-a| \cdot |w-a|} \, d|\mu|(z,w) < \infty,$$

which happens almost everywhere with respect to area measure, and $\tilde{H}$ is locally-integrable, as is seen by an application of Fubini's Theorem. Also $|H(a)| \leq \tilde{H}(a)$ for all such a. Another Fubini calculation yields

$$\langle \phi, \widehat{T_1} \rangle = \int_{\mathbb{C}} \phi \cdot H dm,$$

for all $\phi \in \mathcal{D}$. Thus H represents $\widehat{T_1}$. Based on this, we sometimes write $\widehat{T_1}(a)$ for $H(a)$. Note that

$$\widehat{T_1}(a) = H(a) = \left\langle \frac{1}{\pi(a-z)}, T_1 \right\rangle,$$

whenever $a \notin X$.

Note that these facts do not depend on the relation of T_1 to a derivation, but only on its representability in the form (2) for some measure μ on $X \times X$.

Note also, for future reference, that if T actually represents a normalized point derivation at a point b of X, then $H(a) = \dfrac{1}{\pi(b-a)^2} = \mathrm{o}(1/|a|^2)$ as $a \to \infty$.

3. Estimates

3.1. **The product $g \cdot T_1$.** The dual of any Banach algebra is naturally a module over the algebra. In the present situation, lipα acts on $(\text{lip}\alpha)^*$, so given $g \in \text{lip}\alpha$ and T_1 as in Equation (2), we may define a new element $g \cdot T_1$ of $(\text{lip}\alpha)^*$ by setting

$$\langle \phi, g \cdot T_1 \rangle = \langle g \cdot \phi, T_1 \rangle, \ \forall \phi \in \mathcal{D}.$$

We remark that $\langle 1, g\cdot T_1\rangle = \langle g, T_1\rangle \neq 0$, in general, so we cannot represent gT_1 by a measure as in Equation (2). However, writing

$$\phi(z)g(z) - \phi(w)g(w) = (\phi(z)-\phi(w))\cdot g(z) + \phi(w)\cdot(g(z)-g(w)),$$

a short calculation gives

$$\langle \phi, g\cdot T_1\rangle = \int_{X\times X} \frac{\phi(z)-\phi(w)}{|z-w|^\alpha} d\mu'(z,w) + \int_X \phi(w)\, d\lambda(w),$$

where μ' is the measure on $X\times X$ such that

$$\mu'(E) = \int_E g(z) d\mu(z,w)$$

whenever $E\subset X\times X$ is a Borel set, and λ is the measure on X such that

$$\lambda(E) = \int_{E\times X} \frac{g(z)-g(w)}{|z-w|^\alpha} d\mu(z,w)$$

whenever $E\subset X$ is Borel, i.e. λ is the first-coordinate marginal of the measure

$$\frac{g(z)-g(w)}{|z-w|^\alpha}\cdot \mu(z,w)$$

(a *bounded* multiple of μ). So we may write $g\cdot T_1 = S_1 + S_2$, where $S_1 \perp \mathbb{C}1$ is represented (as in Equation (2)) by the measure $\mu' = g(z)\cdot\mu$ on $X\times X$, and S_2 is represented by the measure λ on X.

Denoting the total variation of a measure μ by $\|\mu\|$, we note for future reference that

$$\|\lambda\| \le \kappa(g)\cdot\|\mu\|.$$

Let us call S_1 *the main part of* $g\cdot T_1$ and S_2 *the residual part of* $g\cdot T_1$.

3.2. **Estimate.** We are aiming for an estimate for the growth of the Cauchy transform of $g\cdot T_1$ as we approach a boundary point nontangentially. The main step is an estimate for the Cauchy transform $\hat{S}_1$ of the main part.

Lemma 3.1. *Fix a measure μ on $X\times X$. Let $b\in X$ and let $g\in \mathrm{lip}\alpha$ have $g(b)=0$. Let S_1 be the distribution given by*

$$\langle \phi, S_1\rangle = \int_{X\times X} \frac{\phi(z)-\phi(w)}{|z-w|^\alpha} g(z) d\mu(z,w),\ \forall \phi\in\mathcal{D}.$$

Fix t with $0<t<1$. Then there is a constant c that depends only on t such that

$$|\hat{S}_1(a)| \le \frac{c\kappa(g)\cdot\|\mu\|}{|a-b|}$$

for all $a\in\mathbb{C}\sim X$ with $\mathrm{dist}(a,X)\ge t|a-b|$.

Proof. We may assume without loss in generality that $\kappa(g) = 1$. Then $|g(z)| \le |z-b|^\alpha$ for all $z \in \mathbb{C}$.

Assume $\operatorname{dist}(a, X) \ge t|a-b|$.

Let $z \in X$. If $|z-b| \le 2|a-b|$, then

$$|z-b| \le \frac{2\operatorname{dist}(a,X)}{t} \le \frac{2|z-a|}{t},$$

whereas if $|z-b| > 2|a-b|$, then

$$|z-b| \le |z-a| + |a-b| < |z-a| + \frac{1}{2}|z-b|,$$

so $|z-b| < 2|z-a|$. Thus in either case $|z-b| \le 2|z-a|/t$. Hence $|g(z)| \le c|z-a|^\alpha$ for all $z \in X$, where c depends only on t. Henceforth we shall use c to denote a constant, which may differ at each occurrence, depending only on t.

Let

$$K(z,w) = \frac{|w-z|^{1-\alpha}}{|z-a| \cdot |w-a|}.$$

Then

$$|\hat{S}_1(a)| \le \int_{X \times X} K(z,w)|g(z)| d|\mu|(z,w).$$

We have $|z-w|^{1-\alpha} \le |z-a|^{1-\alpha} + |w-a|^{1-\alpha}$, so for $z, w \in X$ we have

$$K(z,w) \le \frac{1}{|z-a| \cdot |w-a|^\alpha} + \frac{1}{|z-a|^\alpha \cdot |w-a|},$$

so

$$K(z,w)|g(z)| \le \frac{c}{|z-a|^{1-\alpha} \cdot |w-a|^\alpha} + \frac{c}{|w-a|} \le \frac{c}{\operatorname{dist}(a,X)}.$$

The desired result follows. □

Lemma 3.2. *If T_1 is given by Equation (2), and $0 < t < 1$, then there is a constant c depending only on t such that*

$$|\widehat{g \cdot T_1}(a)| \le \frac{c\kappa(g)\|\mu\|}{\operatorname{dist}(a,X)}$$

whenever $\operatorname{dist}(a, X) \ge t|a-b|$.

Proof. Let S_1 and S_2 be the parts of $g \cdot T_1$ and λ represent S_2, as in the last section. Since λ is a measure supported on X, we have

$$|\hat{\lambda}(a)| \le \frac{\|\lambda\|}{\operatorname{dist}(a,X)} \le \frac{\|\mu\| \cdot \kappa(g)}{\operatorname{dist}(a,X)}$$

whenever $\mathrm{dist}(a,X) \geq t|a-b|$. Combining this with the last lemma, we get

$$|\widehat{g\cdot T_1}(a)| \leq |\hat{S}_1(a)| + |\hat{S}_2(a)| \leq \frac{c\kappa(g)\|\mu\|}{\mathrm{dist}(a,X)},$$

as required. □

3.3. **Estimate for $\hat{T}_1$.** By a similar (slightly simpler) argument we obtain the following.

Lemma 3.3. *Let T_1 be given by Equation* (2). *Then for each t with $0<t<1$ there exists a constant c, depending only on t, such that*

$$|\hat{T}_1(a)| \leq \frac{c\|\mu\|}{\mathrm{dist}(a,X)^{1+\alpha}},$$

whenever $\mathrm{dist}(a,X) \geq t|a-b|$.

4. Proof of Theorem

Suppose $A = A_\alpha(U)$ admits a nonzero continuous point derivation at b, and let ∂ be the normalised derivation at that point. Given a function $f \in A$, we use the same symbol f to denote some global extension in lipα. Note that the extension is uniquely-determined on X, but not outside Y. None of the quantities we will consider depend on which extension is taken.

As we have seen, there is a measure μ on $X \times X$, having no mass on the diagonal, such that

$$\partial f = \int_{X\times X} \frac{f(z)-f(w)}{(|z-w|^\alpha}\, d\mu(z,w),\ \forall f \in A.$$

Let T_1 be the distribution defined by Equation (2). Then T_1 has support in X, and extends continuously to an element of lipα^*.

Let $\mathcal{A}$ denote the set of all $f \in$ lipα that are holomorphic on U and on a neighbourhood of b. As remarked earlier, $\mathcal{A}$ is dense in A.

$\mathcal{E}'$ is a module over $\mathcal{E}$, via the multiplication operation $(\lambda, T) \mapsto \lambda\cdot T$ defined by

$$\langle \phi, \lambda\cdot T\rangle = \langle \lambda\phi, T\rangle$$

for all $\phi, \lambda \in \mathcal{E}$ and $T \in \mathcal{E}'$. So we may define $T_0 = (z-b)\cdot T_1$, where $(z-b)$ denotes the function $z \mapsto (z-b)$.

We calculate that for $f \in \mathcal{A}$ we have

$$\langle f, T_0\rangle = \langle (z-b)f, T_1\rangle = f(b),$$

and hence by continuity this also holds for all $f \in A$, i.e. the distribution T_0 represents evaluation at b on A. Next,

$$\frac{\partial}{\partial \bar z}\left((z-b)\cdot \hat T_1\right) = (z-b)\cdot T_1 = T_0 = \frac{\partial}{\partial \bar z}\hat T_0,$$

so by Weyl's Lemma (cf. [7, p.72] or [3, Theorem 4.4.1, p. 110])

$$\hat T_0 = (z-b)\cdot \hat T_1 + h,$$

where h is an entire function. Also, if $\phi \in \mathcal{D}$ vanishes on a neighbourhood of Y, then

$$\langle \phi, \hat T_0\rangle = \hat\phi(b) = \langle \phi, \frac{1}{\pi(b-z)}\rangle.$$

Thus $\hat T_0(z) = \dfrac{1}{\pi(b-z)}$ off Y. In particular, $\hat T_0(z)$ tends to 0 as $z \to \infty$. Now we also have

$$\hat T_1(z) = \mathrm{O}\left(\frac{1}{|z|^2}\right)$$

as $z \to \infty$, hence $h(z) \to 0$ as $z \to \infty$, whence h is identically zero, and

$$\hat T_0 = (z-b)\hat T_1.$$

Next, define $T = -\pi(z-b)\cdot T_0 = -\pi(z-b)^2\cdot T_1$. Then T annihilates A, and by a similar argument to that above we see that

$$\hat T = -\pi(z-b)^2 \hat T_1 + k$$

for some entire k. Now the fact that T annihilates A forces $\hat T = 0$ off Y, so we get

$$\hat T = 1 - \pi(z-b)^2\cdot \hat T_1.$$

Now suppose $(z_n)_n \subset U$ and $z_n \to_{\mathrm{nt}} b$. By Lemma 3.3,

$$\hat T(z_n) - 1 = -\pi(z_n-b)^2\cdot \hat T_1(z_n) = \mathrm{O}(|z_n-b|^\alpha) \to 0 \tag{3}$$

as $n \uparrow \infty$. In particular, $\hat T(z_n) \to 1$, so we may choose $N_1 \in \mathbb{N}$ such that $|\hat T(z_n)| > \frac{1}{2}$ for $n > N_1$.

Consider any point $a \in U$ with $\hat T(a) \neq 0$, and define

$$R_a = \frac{1}{\pi \hat T(a)(a-z)}\cdot T = \frac{(z-b)^2}{\hat T(a)(z-a)}\cdot T_1,$$

i.e.

$$\langle \phi, R_a\rangle = \frac{1}{\hat T(a)}\left\langle \frac{\phi(z)}{\pi(a-z)}, T\right\rangle,\ \forall \phi \in \mathcal{D}.$$

This is a well-defined distribution since $\phi(z)/(z-a)$ is C^∞ near X, and hence near the support of T. Also R_a is supported on X, and represents

a on A, since for $f \in \mathcal{A}$ we may write $f(z) = f(a) + (z-a)g(z)$ for a $g \in \mathcal{A}$, and get

$$\langle f, R_a \rangle = \frac{1}{\hat{T}(a)} \left\langle \frac{f(a)}{\pi(a-z)} + \frac{g(z)}{\pi}, T \right\rangle = f(a) - \frac{\langle g, T \rangle}{\pi \hat{T}(a)} = f(a).$$

Thus the functional

$$f \mapsto \frac{f(a) - f(b)}{a-b} - \partial f$$

is represented on A by the distribution

$$D_a = \frac{R_a - T_0}{a-b} - T_1.$$

Hence $D_{z_n}(f) \to 0$ as $n \uparrow \infty$, for all $f \in \mathcal{A}$. To prove the theorem, we have to show that this also holds for all f in the closure A of $\mathcal{A}$. To do this, it suffices to show that the functionals D_{z_n} are uniformly bounded on A, for $n \geq N_1$, i.e that

$$|D_{z_n}(f)| \leq c\kappa(f)$$

for some constant $c > 0$, for all $f \in A$ and all $n > N_1$.

Fix an arbitrary $f \in \text{lip}\alpha$, holomorphic on U. Take $g(z) = f(z) - g(b)$, so $D_a(f) = D_a(g)$, $\kappa(f) = \kappa(g)$ and $g(b) = 0$.

Let

$$L_a = \hat{T}(a) R_a = \frac{(z-b)^2}{z-a} \cdot T_1 = \frac{1}{\pi(a-z)} \cdot T.$$

Then $D_a = E_a + F_a$, where

$$\begin{aligned} E_a &= \frac{L_a - T_0}{a-b} - T_1, \\ F_a &= \frac{R_a - L_a}{a-b}. \end{aligned}$$

.

For $a \in U$, we calculate

$$L_a - T_0 = -\frac{T}{\pi(z-a)} + \frac{T}{\pi(z-b)} = \frac{b-a}{\pi(z-a)(z-b)} \cdot T,$$

$$\frac{L_a - T_0}{a-b} = -\frac{T}{\pi(z-a)(z-b)} = \left(\frac{z-b}{z-a} \right) \cdot T_1,$$

$$E_a = \left(\frac{a-b}{z-a} \right) \cdot T_1,$$

so

$$E_a(g) = (a-b) \cdot \left\langle \frac{g(z)}{z-a}, T_1 \right\rangle = (a-b) \cdot \widehat{g \cdot T_1}(a).$$

Thus Lemma 3.2 gives

$$|E_{z_n} g| \le |z_n - b| \cdot \frac{c\kappa(g)\cdot\|\mu\|}{\operatorname{dist}(z_n, X)} \le c\kappa(f),$$

Next,

$$R_a - L_a = \left(\hat{T}(a) - 1\right) T_a = \frac{\hat{T}(a)-1}{\hat{T}(a)} \cdot R_a.$$

Since $g \in A$ and $g(b) = 0$, we have

$$|R_a(g)| = |g(a)| \le \kappa(g)|b-a|^\alpha.$$

Then for $a = z_n$ with $n \ge N_1$, we get

$$|(R_a - T_a)(g)| \le 2c|a-b|^{1-\alpha}\kappa(g)|a-b|^\alpha = c|a-b|,$$

so

$$|F_{z_n}(g)| \le c, \text{ for } n \ge N_1.$$

Thus $D_{z_n}(g)$ is indeed bounded, as required. This concludes the proof.

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