# A Contribution to Metric Diophantine Approximation : the Lebesgue and Hausdorff Theories

A thesis presented by

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*επιχορηγησατε*
*εν τηι πιστει την αρετην*
*εν δε τηι αρετηι την γνωσιν*

2P, I, 5

*In honorem matris patris sororisque.*

# Contents

# Abstract

This thesis is concerned with the theory of Diophantine approximation from the point of view of measure theory. After the prolegomena which conclude with a number of conjectures set to understand better the distribution of rational points on algebraic planar curves, Chapter 1 provides an extension of the celebrated Theorem of Duffin and Schaeffer. This enables one to set a generalized version of the Duffin–Schaeffer conjecture. Chapter 2 deals with the topic of simultaneous approximation on manifolds, more precisely on polynomial curves. The aim is to develop a theory of approximation in the so far unstudied case when such curves are *not* defined by *integer* polynomials. A new concept of so–called “liminf sets” is then introduced in Chapters 3 and 4 in the framework of simultaneous approximation of independent quantities. In short, in this type of problem, one prescribes the set of integers which the denominators of all the possible rational approximants of a given vector have to belong to. Finally, a reasonably complete theory of the approximation of an irrational by rational fractions whose numerators and denominators lie in prescribed arithmetic progressions is developed in chapter 5. This provides the first example of a Khintchine type result in the context of so–called uniform approximation.

# Acknowledgments

Felix qui potuit
rerum cognoscere causas

Virgil, *Georgics*, II, v. 490

My gratitude naturally goes first to Dr. Detta Dickinson for having supervised my Phd over the past three years with constant care and attention. Most of the results presented in this thesis follow from fruitful discussions that I had with her and which helped me to develop ideas put forward. The supervision she provided me with and the right dose of autonomy she gave me when carrying out my research enabled me to discover many aspects of the wonderland which is the theory of Diophantine approximation and its numerous interactions with other fields of mathematics. May she be sincerely thanked.

I feel also obliged to thank Dr. Patrick Cornelius McCarthy[1] for inspiring conversations I had with him. For instance, the work presented in Chapter 5 started with a question he asked me. I am grateful that he accepted to be the internal examiner.

Finally, I would like to thank Professor Victor Beresnevich for accepting to act as the external examiner. As the reader may notice by taking a look at the bibliography, a substantial part of this work is related to and inspired by his contributions to Diophantine approximation.

[1]Giving his name in a very formal way — instead of “Pat” McCarthy — enables me to unveil to the rest world the splendor of his middle name. Let us hope that he forgives me for this act of bravery…

# Publications

The content of this thesis has been published in the form of five papers :

[A1] F. Adiceam, *A note on the Hausdorff dimension of some liminf sets appearing in simultaneous Diophantine approximation,* Mathematika 59(1), 56–64, 2013.

[A2] F. Adiceam, *An extension of a theorem of Duffin and Schaeffer in Diophantine approximation,* Acta Arith. 162(3), 243–254, 2014.

[A3] F. Adiceam, *Vertical shift and simultaneous Diophantine approximation on polynomial curves*, Proc. Edinb. Math. Soc. 58, 1–26, 2015.

[A4] F. Adiceam, *Diophantine approximation and arithmetic progressions*, Int. J. Number Theory 11(2), 451–486, 2015.

[A5] F. Adiceam, *Liminf sets and simultaneous Diophantine approximation*, to appear in J. Théor. Nombres Bordeaux, 19 pages.

# Prolegomena

Throughout, $\Psi\ : \mathbb{R}^+ \to \mathbb{R}^+$ will denote a function which tends to zero at infinity and will be referred to as an *approximating function.*

**Approximation in dimension one.** Some of the main objects of study in Diophantine approximation are the set $W(\Psi)$ of $\Psi$–well approximable numbers and its numerous generalisations. The set is defined as follows :

$$W(\Psi) \ = \ \{x \in [0,1] \ : \ \|qx\| \leq \Psi(q) \text{ for i.m. } q \in \mathbb{N}\}\,.$$

Here and throughout, *i.m.* stands for *infinitely many* and $\|\,.\,\|$ denotes the distance to the nearest integer; that is,

$$\|x\| \ = \ \min_{p\in\mathbb{Z}}|x-p| \tag{1}$$

for any real number $x$. There is no loss generality in restricting the definition of the set $W\,(\Psi)$ to the unit interval as the distance function defined in (1) is clearly invariant by translation by any integer. In what follows, to avoid cumbersome notation, the set $W(q \mapsto q^{-\tau})$, where $\tau > 0$ is a real parameter, will be more conveniently denoted by $W(\tau)$.

For any $\tau \geq 1$, it is relatively easy to show that $W(\tau)$ is non–empty, for instance with the help of the theory of continued fractions. A much stronger result has long been known : Joseph Liouville (1809–1882) proved in [149] that the intersection $\bigcap_{\tau\geq 1} W(\tau)$ is non–empty. An element in this intersection is referred to as a *Liouville number*, the most famous example of which is the Liouville constant $\sum_{n=1}^{\infty} 10^{-n!}$.

A real number $x \in [0,1]$ is *well approximable* (resp. *very well approximable*) if $x \in W\big(q \mapsto (cq)^{-1}\big)$ for all $c \geq 1$ (resp. if $x \in W\big(q \mapsto q^{-1-\epsilon}\big)$ for some $\epsilon > 0$). It is *badly approximable* if it is not well approximable; that is, if

$$\inf_{q\geq 1} q\,\|qx\| > 0. \tag{2}$$

The set **Bad** of badly approximable numbers admits an easy characterisation with the help of the theory of continued fractions : a real number lies in **Bad** if, and only if, its partial quotients are uniformly bounded above (see, e.g., [136]).

One of the most fundamental results to the understanding of the set $W(\Psi)$, and indeed in the theory of Diophantine approximation, is due to Johann Peter Gustav Lejeune Dirichlet (1805–1859).

**Theorem 0.0.1** (Dirichlet, 1842)**.** *For any real number $x$ and any integer $Q \geq 1$, there exists an integer $q \geq 1$ such that*

$$\|qx\| \leq \frac{1}{Q} \quad \textit{and} \quad 1 \leq q \leq Q. \tag{3}$$

Unlike what its simple formulation might lead one to expect, Theorem 0.0.1 has far–reaching consequences in many areas of mathematics. In proving this result in [72], Dirichlet popularized the *Schubfachprinzip*, also known as pigeon–hole principle, or *principe des tiroirs*, or *principio dei casseti*, or *principio de las cajilas*, or *principiul cutiei*, or *skatulya–elv*, or *lokeroperiaate*, or *zasada pudełkowa*[2]. The latter simply states that, if $Q$ objects are placed in $Q-1$ boxes, then one of the boxes must contain at least two objects. It is remarkable that such a simple argument leads to an optimal result uniformly in all real numbers. Indeed, as already observed by Alexandre Iakovlevitch Khintchine (1878–1959) in 1926, if the right–hand side of the first inequality in (3) was to be replaced by $1/(2Q)$, then Theorem 0.0.1 could hold only when $x$ is rational — see [132, 194]. On the other hand, if one considers the same inequality with the right–hand side replaced with $c/Q$ for some fixed $c$ in the interval $(1/2, 1)$, then the real number $x$ has to be badly approximable (see [51, Chap. 1, p.22] and a generalisation of this result in [66]).

It should be clear from the definition of the distance function (1) that for any real number $x$ and any integer $q \geq 1$, there exists an integer $p$ such that $|x - p/q| \leq 1/(2q)$. Denoting by $\Psi_{1/2}$ the constant function equal to $1/2$, this proves that $W\big(\Psi_{1/2}\big) = [0,1]$. The following trivial corollary of Theorem 0.0.1 improves on the latter showing that $W(1) = [0,1]$.

**Corollary 0.0.2.** *For any real number $x$, there exists arbitrarily large values of $q \geq 1$ such that*

$$\|qx\| \leq \frac{1}{q}\cdot \tag{4}$$

In 1891, Adolf Hurwitz (1859–1919) strengthened Corollary 0.0.2 by proving in [120] that the statement remains true with $\big(\sqrt{5}q\big)^{-1}$ in place of $q^{-1}$ in the right–hand side of (4). Furthermore, the constant $\sqrt{5}$ is optimal in the sense that there exist real numbers $x$ which do not satisfy the inequality $\|qx\| \leq \big((\sqrt{5}+\epsilon)q\big)^{-1}$ infinitely often for any given $\epsilon > 0$. This is for instance the case when $x$ is chosen as the golden ratio $\varphi = (1+\sqrt{5})/2$, which is therefore badly approximable. If the constant $\sqrt{5}$ was to be replaced by 3, then the inequality $\|qx\| \leq (3q)^{-1}$ would still hold true infinitely often for all but countably many values of $x$. The study of the optimal constants that can be chosen in the right–hand side of inequality (4) for the result to hold up to countably many exceptions gives rise to the *Lagrange spectrum* — see [62] for further details.

[2] This list is drawn from [51, Chap. 1].

While Hurwitz's theorem implies that the set $W(1) \setminus W(1+\epsilon)$ is non–empty for any $\epsilon > 0$, another notable result in this direction is the celebrated Roth's Theorem [164], which shows that algebraic numbers are not very well approximable.

**Theorem 0.0.3** (Roth, 1955)**.** *Let $\alpha$ be an algebraic number. Then, for any $\epsilon > 0$, the inequality*

$$\|q\alpha\| \leq \frac{1}{q^{1+\epsilon}} \tag{5}$$

*admits only finitely many solutions in integers $q \geq 1$.*

A far–reaching generalisation of Theorem 0.0.3 has been given by Schmidt in the form of his Subspace Theorem [173] (see also [45, 52, 200] for surveys of some of the numerous consequences of this result). However, determining an effective version of Theorem 0.0.3 (that is, a bound for the largest solution $q$ to (5) in terms of $\alpha$ and $\epsilon$) remains a very topical unsolved problem in Diophantine approximation. Another generalisation of Theorem 0.0.3 due to Waldschmidt [193, p.260] seems beyond reach at the moment (see also [21, § 2.1]) :

**Conjecture 0.0.4** (Waldschmidt, 2004)**.** *Assume that $\Psi$ is a monotonic approximating function such that $\sum_{q=1}^{\infty} \Psi(q) < \infty$. Then no irrational algebraic number lies in $W(\Psi)$.*

**Approximation in higher dimensions.** In higher dimensions, the most general version of the set $W(\Psi)$ can be defined as follows : given integers $m, n \geq 1$, let $W(m, n, \Psi)$ be the set of simultaneously $\Psi$–approximable linear forms in $m$ variables in dimension $n$, that is,

$$W(m,n,\Psi) := \{(\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}) \in [0,1]^{mn} \ : \ \forall i = 1, \ldots, n, \ \|\boldsymbol{q}.\boldsymbol{x_i}\| < \Psi(|\boldsymbol{q}|) \ \text{for i.m. } \boldsymbol{q} \in \mathbb{Z}^m\},$$

where $|\,.\,|$ denotes the infinity norm. Here again, there is no loss of generality in restricting the definition of $W(m, n, \Psi)$ to the unit cube $[0, 1]^{mn}$. The set $\mathcal{S}_n(\Psi)$ of *simultaneously $\Psi$–approximable* vectors in dimension $n$ then corresponds to $W(1, n, \Psi)$ and the set $\mathcal{L}_m(\Psi)$ of *dually $\Psi$–approximable* vectors in dimension $m$ to $W(m, 1, \Psi)$. To avoid cumbersome notation, the set $W(m, n, q \mapsto q^{-\tau})$ (resp. $\mathcal{S}_n(q \mapsto q^{-\tau})$, $\mathcal{L}_m(q \mapsto q^{-\tau})$), where $\tau > 0$ is a real parameter, will be more conveniently denoted by $W(m, n, \tau)$ (resp. by $\mathcal{S}_n(\tau)$, $\mathcal{L}_m(\tau)$).

Dirichlet's theorem (Theorem 0.0.1) can be generalized in this context with the help of Minkowski's Linear Forms Theorem.

**Theorem 0.0.5** (Minkowski's Linear Forms Theorem, 1891)**.** *Let $\boldsymbol{X} = (\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}) \in \mathbb{R}^{n^2}$ be a square real matrix and $c_1, \ldots, c_n$ be $n$ positive real–numbers.*

*Then, there exists a non–zero integer vector $\boldsymbol{q} \in \mathbb{Z}^n$ such that*

$$|\boldsymbol{q}.\boldsymbol{x_1}| \leq c_1 \quad \textit{and} \quad |\boldsymbol{q}.\boldsymbol{x_i}| < c_i \ \ (2 \leq i \leq n)$$

*provided that $c_1 \ldots c_n \geq |\det \boldsymbol{X}|$.*

**Corollary 0.0.6.** *Given any real matrix $\boldsymbol{X} := (\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}) \in [0, 1]^{mn}$ and any integer $Q \geq 1$, there exists an integer vector $\boldsymbol{q} \in \mathbb{Z}^m$ such that*

$$\|\boldsymbol{q}.\boldsymbol{x_i}\| \ \leq \ Q^{-m/n} \quad \textit{and} \quad 1 \leq |\boldsymbol{q}| \leq Q$$

*for all* $i = 1, \ldots, n,$.

This corollary trivially implies the following one :

**Corollary 0.0.7.** *Given any real matrix* $\boldsymbol{X} := (\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}) \in [0,1]^{mn}$*, there exist infinitely many non–zero integer vectors* $\boldsymbol{q} \in \mathbb{Z}^m$ *such that*

$$\|\boldsymbol{q}.\boldsymbol{x_i}\| \leq |\boldsymbol{q}|^{-m/n}$$

*for all* $i = 1, \ldots, n,$. *In other words,* $W(m, n, m/n) = [0,1]^{mn}$*, whence in particular* $\mathcal{S}_n(1/n) = [0,1]^n$ *and* $\mathcal{L}_m(m) = [0,1]^m$.

The deduction of Corollary 0.0.6 from Theorem 0.0.5 can for instance be found in [57, Chap. 1, Theorem VI]. However, Corollary 0.0.6 can also be proved with the help of the pigeon–hole principle (i.e. with a box–counting argument) — see, e.g., [108, 174] for further details.

In view of Corollary 0.0.7, the concept of badly approximable numbers may easily be generalized to higher dimensions in the following way : the set $\mathbf{Bad}(m,n)$ of $n$ *badly approximable linear forms in* $m$ *variables* is the set of all those matrices $\boldsymbol{X} := (\boldsymbol{x_1}, \ldots, \boldsymbol{x_n})$ in $[0,1]^{mn}$ such that

$$\inf_{\boldsymbol{q} \in \mathbb{Z}^m \setminus \{0\}} \min_{1 \leq i \leq n} |\boldsymbol{q}|^{m/n} \|\boldsymbol{q}.\boldsymbol{x_i}\| > 0. \tag{6}$$

Note that $\mathbf{Bad}(1,1) = \mathbf{Bad}$ when $n = m = 1$. Furthermore, when $n = m = 1$, Hurwitz Theorem guarantees that, for any real number $x$, the infimum appearing in (2) is always less than $5^{-1/2}$. However, there is no known analogue of Hurwitz Theorem whenever $mn > 1$, even if some upper bounds valid for all $\boldsymbol{X} \in [0,1]^{mn}$ have been established for the supremum over all $i = 1, \ldots, n$ of the infima appearing in (6) — see, e.g., [174, Chap. 2].

## 0.1 Metric properties of the set of Ψ–approximable points

In this section, $\lambda_n$ will denote the $n$–dimensional Lebesgue measure ($n \geq 1$). For the sake of simplicity of notation, set $\lambda := \lambda_1$. A property will be said to hold for *almost all* $\boldsymbol{x} \in \mathbb{R}^n$, or, for short, *almost everywhere*, if the $n$–dimensional Lebesgue measure of the set of points not satisfying this property is null.

### 0.1.1 The Lebesgue Theory

In 1924, Khintchine [130] (see also [40, Chap. 1] and [136]) proved a beautiful zero–one law enjoyed by the one–dimensional set $W(\Psi)$ under the assumption of the monotonicity of the approximating function.

**Theorem 0.1.1.** *The following holds :*

$$\lambda(W(\Psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q) = \infty \text{ and } \Psi \text{ is monotonic.} \end{cases}$$

Theorem 0.1.1 provides very precise results regarding the behavior of almost all real numbers. Thus, for instance,

$$\inf_{q\geq 16}\ (\log q)\,(\log\log q)\,(\log\log\log q)\,q\,\|qx\| \;=\; 0$$

for almost all $x\in\mathbb{R}$ while

$$\inf_{q\geq 16}\ (\log q)\,(\log\log q)\,(\log\log\log q)^{1+\epsilon}\,q\,\|qx\| \;>\; 0$$

for almost all $x\in\mathbb{R}$ and any $\epsilon>0$.

The convergence part of Theorem 0.1.1 follows from a direct application of the Borel–Cantelli Lemma from probability theory (see, e.g., [51, p.13]). Khintchine actually established the divergent part of the result under the assumption of the monotonicity of the function $q\mapsto q\Psi(q)$. The fact that the divergent part of Theorem 0.1.1 holds with the weaker assumption on $\Psi$ appearing therein emerged with the concept of a *regular system* — see [13, 14] for details.

It should be clear on the one hand that Khintchine's Theorem readily implies that the set $\bigcup_{k\geq 1} W\big(q\mapsto q^{-1-1/k}\big)$ of very well approximable points has zero Lebesgue measure. This is rather intuitive : given $x\in[0,1]$, an inequality of the form $|x-p/q|<q^{-2-\epsilon}$ for integers $p,q\geq 1$ and for $\epsilon>0$ should give roughly $(2+\epsilon)N$ digits in the decimal expansion of $x$ when $q$ has $N\geq 1$ digits. However, one has only $2N$ digits at one's disposal when choosing $p$ and $q$. The fact that $\lambda\left(\bigcup_{k\geq 1} W\big(q\mapsto q^{-1-1/k}\big)\right)=0$ confirms that this gain of information is indeed "rarely" possible in the sense that it happens with zero probability.

On the other hand, Khintchine's Theorem also implies that the set of well approximable numbers, which is the intersection $\bigcap_{k\geq 1} W\Big(q\mapsto (kq)^{-1}\Big)$, has full Lebesgue measure. In turn, its complement, namely the set **Bad**, has zero Lebesgue measure. Therefore, a "generic" real number is neither badly nor very well approximable.

The Khintchine–Groshev[3] Theorem generalizes Khintchine's Theorem to higher dimensions. In full generality, it may be stated as follows :

**Theorem 0.1.2.** *Let $m,n\geq 1$ be integers with $mn>1$. Then*

$$\lambda_{mn}\left(W(m,n,\Psi)\right) \;=\; \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{m-1}\psi(q)^n \;<\;\infty \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{m-1}\psi(q)^n \;=\;\infty. \end{cases}$$

Here again, the convergence part is a consequence of the Borel–Cantelli Lemma. As for the divergence part, Groshev [105] initially proved his result under the assumption of the monotonicity of the function $q\mapsto q^m\Psi(q)^n$. The redundancy of this assumption was established by Schmidt [170, Theorem 2] whenever $m\geq 3$ and by Gallagher [98] whenever $n\geq 2$. The remaining case $m=2$ was solved only recently by Beresnevich and Velani [31], leading to the statement given in Theorem 0.1.2.

[3] Very little biographical data can be found about Groshev, except that the initials of his first and middle names are "A.V.", that he worked for Moscow University and that he published papers in the 1930's.

Together with Theorem 0.1.1, Theorem 0.1.2 implies in particular that $\mathbf{Bad}(m,n)$ has zero Lebesgue measure for any integers $m, n \geq 1$. Also, the fact that the sets of approximation $W(m, n, \Psi)$ satisfy a zero–one law is no coincidence. This ergodic property follows from their invariance by translation by integer vectors. Thus, when proving the divergence part of Theorems 0.1.1 and 0.1.2, showing full measure amounts to proving positive measure. Such ergodic properties have been established for a large class of Diophantine sets — see, e.g., [26, 183].

Despite their generality, in the case of convergence, Theorems 0.1.1 and 0.1.2 do not provide enough refinement to discriminate for example between the two zero–Lebesgue measure sets $\mathcal{S}_n(\tau + \epsilon)$ and $\mathcal{S}_n(\tau)$ for any $\epsilon > 0$ with $\tau > 1/n$, although one would expect the former to be "smaller" than the latter. This intuition can be confirmed with the help of the concept of Hausdorff measure and dimension.

### 0.1.2 Hausdorff measures and Hausdorff dimension

The Hausdorff dimension of subsets in the $n$–dimensional Euclidean space $\mathbb{R}^n$ is an aspect of their size that enables one to discriminate between sets of Lebesgue measure zero. This concept was first introduced by Felix Hausdorff (1868–1942) and then developed by Abraham Samoilovitch Besicovitch (1891–1970). It can be defined in the context of any metric space, but such a generalisation will not be needed in what follows. The definitions and properties of this subsection will therefore be given only in the Euclidean framework. For very general accounts on the topic, see [40, 75, 94, 95].

In what follows, $f$ will denote a *dimension function*; that is, a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $f(0) = 0$ which is monotonic in a neighbourhood of 0. For any $U \subset \mathbb{R}^n$ non–empty, let $d(U) := \sup\{\|\boldsymbol{x} - \boldsymbol{y}\|_2 \,:\, \boldsymbol{x}, \boldsymbol{y} \in U\}$ denote its diameter (here, $\|\,.\,\|_2$ is the Euclidean norm). Given a non–empty subset $X \subset \mathbb{R}^n$, a collection of subsets $\{U_i\}_{i\in\mathbb{N}}$ such that $0 < d(U_i) \leq \rho$ for each $i \in \mathbb{N}$ and $X \subset \bigcup_{i\in\mathbb{N}} U_i$ is said to be a $\rho$–cover for $X$.

For any positive $\rho$, define furthermore

$$\mathcal{H}^f_\rho(X) = \inf\left\{\sum_{i=0}^{\infty} f(d(U_i)) \,:\, \{U_i\}_{i\in\mathbb{N}} \text{ is a } \rho\text{-cover of } X\right\}. \tag{7}$$

It is readily checked that the quantity $\mathcal{H}^f_\rho(X)$ increases as $\rho$ decreases and therefore admits a (finite or infinite) limit as $\rho$ tends to 0. This limit, denoted by $\mathcal{H}^f(X)$, is the *Hausdorff $f$–measure* of $X$ :

$$\mathcal{H}^f(X) \;=\; \lim_{\rho\to 0^+} \mathcal{H}^f_\rho(X) \;=\; \sup_{\rho>0} \mathcal{H}^s_\rho(X).$$

In the case where $f(r) = r^s$ for some $s \geq 0$, $\mathcal{H}^f$ is more conveniently denoted by $\mathcal{H}^s$ and is referred to as the *$s$–dimensional Hausdorff measure.* If $s$ is an integer, say $k$, it can be shown (see, e.g., [94]) that the $k$–dimensional Hausdorff measure $\mathcal{H}^k$ is proportional to the $k$–dimensional Lebesgue measure $\lambda_k$. Furthermore, the constant of proportionality is the inverse of the Lebesgue volume $\nu_k$ of the unit ball in dimension $k$. In other words,

$$\mathcal{H}^k \;=\; \nu_k^{-1}\lambda_k.$$

This relation confirms the fact that the concept of Hausdorff measure refines that of Lebesgue measure.

If $U \subset \mathbb{R}^n$ is non–empty, $0 < \rho < 1$ and $0 < s < t$, then $d(U)^t \leq \rho^{t-s} d(U)^s$. It follows from (7) that $\mathcal{H}^t_\rho(X) \leq \rho^{t-s}\mathcal{H}^s_\rho(X)$, so that when $\mathcal{H}^t(X)$ is positive, $\mathcal{H}^s(X)$ is infinite and when $\mathcal{H}^s(X)$ is finite, $\mathcal{H}^t(X)$ vanishes. The Hausdorff dimension $\dim X$ of $X$ is then defined as the unique value of $s$ at which $\mathcal{H}^s(X)$ "jumps" from infinity to zero (see Figure 0.1.2). In other words,

$$\dim X \;=\; \inf\left\{s > 0 \,:\, \mathcal{H}^s(X) = 0\right\}, \tag{8}$$

in such a way that

$$\mathcal{H}^s(X) = \begin{cases} \infty & \text{if } s < \dim X, \\ 0 & \text{if } s > \dim X. \end{cases} \tag{9}$$

Note that the Hausdorff dimension of a set in $\mathbb{R}^n$ always exists.

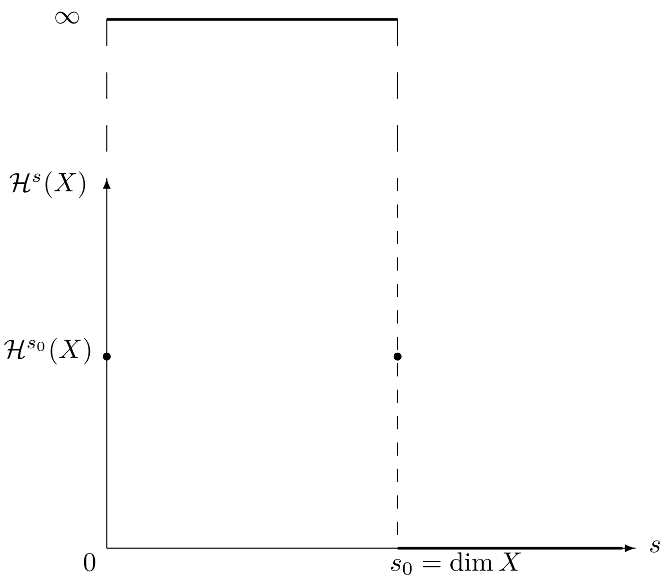



Figure 0.1: The Hausdorff measures of a subset $X \subset \mathbb{R}^n$.

Further details and alternative definitions of Hausdorff measures and dimension can be found in [94], where the following classical properties are also proved :

**Proposition 0.1.3** (Properties of Hausdorff dimension)**.** *Let $n \geq 1$ be an integer and $E$, $F$ and $E_j$ $(j \geq 1)$ be subsets of $\mathbb{R}^n$.*

- *If $E \subset F$, then $\dim E \leq \dim F$.*
- $\dim E \leq n$.
- *The Hausdorff dimension of any countable set is zero and the Hausdorff dimension of any open set in $\mathbb{R}^n$ is $n$.*

- *If* $\lambda_n(E) > 0$*, then* $\dim E = n$ *(the converse is false).*
- *(Countable stability)* $\dim\left(\bigcup_{j=1}^{\infty} E_j\right) = \sup_{j\geq 1} \dim E_j$.

*Also, if* $X$ *and* $Y$ *are two subsets of any Euclidean spaces such that there exists a bi–Lipschitz function* $f$ *mapping* $X$ *onto* $Y$ *(this means that* $f$ *is bijective and* $f$ *and its inverse are Lipschitz continuous), then* $\dim X = \dim Y := s$ *and the* $s$*–dimensional Hausdorff measures of* $X$ *and* $Y$ *are proportional.*

In view of (9), an upper bound for the Hausdorff dimension of $X$ can be obtained by finding a value of $s$ at which $\mathcal{H}^s(X)$ vanishes. When $X$ is a *limsup set*, i.e. when $X$ can be written as

$$X = \limsup_{N\to\infty} X_N = \bigcap_{N=1}^{\infty} \bigcup_{i=N}^{\infty} X_i = \{\boldsymbol{x} \in \mathbb{R}^n \ : \ \boldsymbol{x} \in X_i \text{ for i.m. } i \in \mathbb{N}\}$$

for a sequence of sets $X_i \subset \mathbb{R}^n$, a simple Hausdorff measure counterpart of the convergent case of the Borel–Cantelli Lemma may be used to show that $\mathcal{H}^s(X) = 0$ (see, e.g., [76, Lemma 2.2] for a proof) :

**Lemma 0.1.4.** *With the previous notation, let*

$$X = \{\boldsymbol{x} \in \mathbb{R}^n \ : \ \boldsymbol{x} \in X_i \ \textit{ for i.m. } i \in \mathbb{N}\}.$$

*If, for some* $s > 0$,

$$\sum_{i=1}^{\infty} d(X_i)^s < \infty,$$

*then* $\mathcal{H}^s(X) = 0$ *and therefore* $\dim X \leq s$.

Finding a non–trivial lower bound for the Hausdorff dimension of $X \subset \mathbb{R}^n$ is usually much more difficult. Various tools have been introduced to this end : these include the concept of ubiquity due to Dodson and alii [78] (see also [40, 71, 73]), which reduces in dimension one to the concept of regular systems introduced by Baker and Schmidt [10] (see also [75]) or else the Mass Distribution Principle, which is very convenient when working with Cantor–like sets (see [95, Chap. 1 & 4]).

### 0.1.3 The Hausdorff Theory

Vojtěch Jarník (1897–1970) was the first to apply the concept of Hausdorff dimension to Diophantine sets in order to estimate their "sizes". Thus, he proved in 1928 in [124] with the help of the theory of continued fractions that $\dim \mathbf{Bad} = 1$. The corresponding result for $\mathbf{Bad}(m,n)$ $(m, n \geq 1)$, namely that $\dim \mathbf{Bad}(m,n) = mn$, was proved only in 1969 by Schmidt [172] thanks to his now famous topological game. Since the $mn$–dimensional Hausdorff measure is proportional to the $mn$–dimensional Lebesgue measure, it follows from the fact that $\lambda_{mn}(\mathbf{Bad}(m,n)) = 0$ that

$$\mathcal{H}^s(\mathbf{Bad}(m,n)) = \begin{cases} 0 & \text{if } s \geq mn, \\ \infty & \text{if } 0 \leq s < mn. \end{cases}$$

In 1931, Jarník [126] studied the Hausdorff dimension of the one–dimensional set $\mathcal{S}_1(\tau)$ when $\tau > 1$. Later, in 1934, Besicovitch [44] undertook the same study in higher dimensions. Their result is now known as the "Jarník–Besicovitch Theorem".

**Theorem 0.1.5** (Jarník–Besicovitch, 1931/1934)**.** *For any $n \geq 1$ and any $\tau > 1/n$,*

$$\dim \mathcal{S}_n(\tau) = \frac{n+1}{1+\tau}.$$

Theorem 0.1.5 confirms the intuition that the "size" of the set $\mathcal{S}_n(\tau)$ should be decreasing with $\tau$. For example, $\dim \mathcal{S}_1(2) = 2/3$ while $\dim \mathcal{S}_1(3) = 1/2$. However, Theorem 0.1.5 does not give any information about the Hausdorff measure of the set $\mathcal{S}_n(\tau)$ at the critical value $s = (n+1)/(1+\tau)$. Jarník [125] answered this question in dimension one by establishing the following general result :

**Theorem 0.1.6** (Jarník, 1931)**.** *Given an approximating function $\Psi$ and a dimension function $f$ such that $q \mapsto q^{-1}f(q)$ is a decreasing function tending to infinity as $q$ tends to 0,*

$$\mathcal{H}^f(W(\Psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} qf\left(\frac{\Psi(q)}{q}\right) < \infty \\ \infty & \text{if } \sum_{q=1}^{\infty} qf\left(\frac{\Psi(q)}{q}\right) = \infty. \end{cases}$$

Thus, Theorem 0.1.6 implies that for any $\tau > 1$,

$$\mathcal{H}^{2/(1+\tau)}(W(\tau)) = \infty.$$

But it actually says a lot more. To see this, consider the two approximating functions given by

$$\Psi_1(q) = \frac{1}{q\log^2 q} \qquad \text{and} \qquad \Psi_2(q) = \frac{1}{q\log^3 q}$$

for all $q \geq 2$. It easily follows from Theorem 0.1.6 that

$$\dim(W(\Psi_1)) = \dim(W(\Psi_2)) = 1.$$

Khintchine's Theorem shows furthermore that $W(\Psi_1)$ and $W(\Psi_2)$ are both of Lebesgue measure zero. Let now $f$ denote the dimension function defined for $x \in (0,1)$ as

$$f(x) = -x\log x.$$

Then,

$$\sum_{q=2}^{\infty} qf\left(\frac{\Psi_1(q)}{q}\right) = \sum_{q=2}^{\infty} \frac{2\log(q\log q)}{q\log^2 q} = \infty,$$

whereas

$$\sum_{q=2}^{\infty} qf\left(\frac{\Psi_2(q)}{q}\right) = \sum_{q=2}^{\infty} \frac{2\log\left(q\log^{3/2} q\right)}{q\log^3 q} < \infty.$$

It thus follows from Theorem 0.1.6 that

$$H^f(W(\Psi_1)) \;=\; \infty \qquad \text{and} \qquad H^f(W(\Psi_2)) \;=\; 0;$$

that is, Jarník's Theorem makes it possible to distinguish between sets which share the same Hausdorff dimension $s$ and the same $s$–dimensional Hausdorff measure.

Another feature of Jarník's Theorem worth pointing out is that the case when $H^f$ is proportional to the Lebesgue measure $\lambda$ is excluded by the assumption $q^{-1}f(q) \to \infty$ as $q$ tends to 0. This situation is nevertheless already covered by Khintchine's Theorem.

In the general case of approximations of linear forms, an analogue of the Theorem of Jarník–Besicovitch (Theorem 0.1.5) has been established by Dodson [74]. The statement of the result requires the definition of the *lower order* $\lambda_\Psi$ of $1/\Psi$, where the approximating function $\Psi$ is assumed to be non–vanishing. The quantity $\lambda_\Psi$ is defined as follows :

$$\lambda_\Psi \;=\; \liminf_{q\to\infty}\left(-\frac{\log\Psi(q)}{\log q}\right). \tag{10}$$

Note that $\lambda_\Psi$ is non–negative when $\Psi(q)$ tends to 0 as $q$ tends to infinity.

**Theorem 0.1.7** (Dodson, 1992)**.** *Assume that the approximating function $\Psi$ is decreasing. Then, given integers $m, n \geq 1$,*

$$\dim W(m,n,\Psi) \;=\; \begin{cases} (m-1)n + \frac{m+n}{\lambda_\Psi} & \text{if } \lambda_\Psi \;>\; \frac{m+n}{n}, \\ mn & \text{if } \lambda_\Psi \;\leq\; \frac{m+n}{n}. \end{cases}$$

Defining the *upper order* of the function $1/\Psi$ as a natural analogue to the lower order (10), Theorem 0.1.7 happens to be a particular case of a result due to Dickinson and Velani [71] when the lower and the upper orders of $1/\Psi$ coincide. Indeed, Dickinson and Velani have provided a Jarník–type result (that is, a result similar to Theorem 0.1.6) for systems of linear forms.

**Theorem 0.1.8** (Dickinson & Velani, 1997)**.** *Let $m, n \geq 1$ be integers. Let $f$ be a dimension function such that $q^{-mn}f(q) \to \infty$ as $q \to 0$ and $q \mapsto q^{-mn}f(q)$ is non–increasing. Assume that the approximating function $\Psi$ is such that $q^m\Psi(q)^n \to 0$ as $q \to \infty$ and that $q \mapsto q^m\Psi(q)^n$ is non–increasing. Furthermore, suppose that $q \mapsto q^{m(n+1)}\Psi(q)^{-(m-1)n}f(\Psi(q)/q)$ is non–increasing. Then*

$$\mathcal{H}^f\left(W(m,n,\Psi)\right) \;=\; \begin{cases} 0 & \text{if } \Sigma_\Psi^f \;<\; \infty, \\ \infty & \text{if } \Sigma_\Psi^f \;=\; \infty, \end{cases}$$

*where*

$$\Sigma_\Psi^f \;=\; \sum_{q=1}^{\infty} f\left(\frac{\Psi(q)}{q}\right)\Psi(q)^{-(m-1)n}q^{m(n+1)-1}.$$

Here again, the case when $\mathcal{H}^f$ is proportional to the $mn$–dimensional Lebesgue measure is excluded by the assumption $q^{-mn}f(q) \to \infty$ as $q \to 0$. Furthermore, the condition that $q^m\Psi(q)^n \to 0$ as $q \to \infty$ is natural in the sense that, if it was not met, then the Khintchine–Groshev Theorem would ensure that $W(m,n,\Psi)$ has full measure. Also, it is worth pointing out that, under the assumptions of Theorem 0.1.8, there is *no* dimension function $f$ such that the $f$–Hausdorff measure of $W(m,n,\Psi)$ is positive and finite.

### 0.1.4 The Mass Transference Principle and other Transference Theorems

Given the various forms of approximations (e.g., simultaneous or dual approximations) and the various theories developed to understand them (e.g., the Khintchine–Groshev or the Jarník–Besicovitch theories), it is natural to ask whether information about one kind of approximation or theory can lead to information about another. In full generality, a result of such a type is referred to as a *Transference Theorem.* Two of them are mentioned in what follows for their generality and their importance. Some others can also be found in the literature, e.g. the so–called *Inhomogeneous Transference Principle* [30].

**Transference Theorems for dual approximation.** It is sometimes possible to find solutions to Diophantine inequalities for a given set of linear forms from inequalities satisfied by a related set of linear forms. Results of this kind were first discovered by Khintchine [131, 133] and later developed by Dyson [84] and Jarník [127]. A synthetic and more general result initially due to Mahler [152] is now presented. It can be found in [57, Chap V] — see also [8, Appendix]. First, some notation is introduced.

Let $m, n \geq 1$ be integers. Consider $n$ linear forms in $m$ variables $(L_j)_{1\leq j\leq n}$ defined for all $\boldsymbol{q} = (q_1, \dots, q_m) \in \mathbb{R}^m$ by

$$L_j(\boldsymbol{q}) \;=\; \sum_{i=1}^{m} x_{ji} q_i \quad (1 \leq j \leq n),$$

where $\boldsymbol{X} = (x_{ji})_{1\leq j\leq n, 1\leq i\leq m}$ is a real matrix. Denote by $(M_i)_{1\leq i\leq m}$ the corresponding transposed system of $m$ linear forms in $n$ variables, that is, the system of linear forms defined for all $\boldsymbol{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ by

$$M_i(\boldsymbol{u}) \;=\; \sum_{j=1}^{n} x_{ji} u_j \quad (1 \leq i \leq m).$$

**Theorem 0.1.9.** *Assume that there exists a non–zero integer vector $\boldsymbol{q} \in \mathbb{Z}^m$ such that*

$$\|L_j(\boldsymbol{q})\| \leq C \quad \textit{and} \quad |\boldsymbol{q}| \leq Q$$

*for all $j = 1, \dots, n$, where $C$ and $Q$ are real constants satisfying the inequalities $0 < C < 1 \leq Q$. Then, there exists a non–zero integer vector $\boldsymbol{u} \in \mathbb{Z}^n$ such that*

$$\|M_i(\boldsymbol{u})\| \leq D \quad \textit{and} \quad |\boldsymbol{u}| \leq U$$

*for all $i = 1, \dots, m$, where*

$$D = (l-1)Q^{-(n-1)/(l-1)}C^{n/(l-1)}, \quad U = (l-1)Q^{m/(l-1)}C^{-(m-1)/(l-1)}$$

*and*

$$l = m + n.$$

An easy consequence of Theorem 0.1.9 is that an $m \times n$ matrix $\boldsymbol{X}$ with coefficients in $[0,1]$ lies in $\mathbf{Bad}(m,n)$ if, and only if, its transpose $\boldsymbol{X}^T$ lies in $\mathbf{Bad}(n,m)$ (see [57, p.78] for details).

**The Mass Transference Principle.** Another duality in Metric Diophantine Approximation deals with the Khintchine–Groshev theory on the one hand and the Jarník–Besicovitch theory on the other. In details, the former is concerned with metrical statements relating the size of Diophantine sets to their Lebesgue measures while the latter relates them to their Hausdorff measures and dimensions. A fundamental result to the understanding of this duality is the recent *Mass Transference Principle* due to Beresnevich and Velani [27]. Before stating it, an additional piece of notation is introduced : given a dimension function $f$ and a ball $B(\boldsymbol{x}, r)$ with radius $r > 0$ in $\mathbb{R}^n$ centered at $\boldsymbol{x} \in \mathbb{R}^n$, let $B^f$ denote its dilate by $f$; that is,

$$B^f := B\left(\boldsymbol{x}, f(r)^{1/n}\right).$$

In the case when $f(r) = r^s$ for a given $s > 0$, write $B^s$ instead of $B^f$.

**Theorem 0.1.10** (Mass Transference Principle, 2006)**.** *Let $(B_k)_{k\geq 0}$ be a sequence of balls in $\mathbb{R}^n$. Let $f$ be a dimension function such that $r \mapsto r^{-n}f(r)$ is monotonic and suppose that for any ball $B$ in $\mathbb{R}^n$,*

$$\mathcal{H}^n\left(B \cap \limsup_{k\to\infty} B_k^f\right) = \mathcal{H}^n(B).$$

*Then, for any ball $B$ in $\mathbb{R}^n$,*

$$\mathcal{H}^f\left(B \cap \limsup_{k\to\infty} B_k^n\right) = \mathcal{H}^f(B).$$

A surprising consequence of Theorem 0.1.10 is that Jarník's Theorem (Theorem 0.1.6) is implied by Khintchine's Theorem (Theorem 0.1.1), which in turn is a consequence of Dirichlet's Theorem (Theorem 0.0.1). This means in particular that the Lebesgue theory of limsup sets in Diophantine approximation underpins the Hausdorff theory, which is counterintuitive inasmuch as Hausdorff measures refine Lebesgue measure.

Another version of the Mass Transference Principle has been given in [28] to take into account the case when one considers sequences of hyperplanes in place of balls. This enables one to transfer Lebesgue measure statements regarding approximations of systems of linear forms into Hausdorff measure statements. As a consequence, Theorem 0.1.8 by Dickinson and Velani becomes a corollary of the Khintchine–Groshev Theorem (Theorem 0.1.2).

## 0.2 Adding constraints on the approximants

The theory of Diophantine approximation has long been concerned with approximation of given quantities when constraints are put only on the approximants. This is generally a difficult problem which can be considered both from a metrical and a non–metrical point of view. For instance, it has been proved in [153] that for any $\tau < 1/3$ and any irrational $x$, the inequality $\|px\| \leq p^{-\tau}$ has infinitely many solutions in primes $p$. This best known result stands in sharp contrast with its metrical counterpart which asserts that such approximations are possible almost everywhere for any $\tau < 1$.

In full generality, when considering questions in Diophantine approximation with restricted rational approximants, two problems of a different nature emerge : those in which the numerators and the denominators vary in prescribed but independent sets and those in which they vary in dependent sets. A famous example related to the latter case is the Duffin–Schaeffer conjecture (see subsection 0.2.2), where the rational approximants are required to be irreducible. As for the former case, a survey of some general results may be found in [113]. In what follows, the focus will rather be on one particular type of constraints, namely that of *uniform approximation*. In short, in this type of approximation, one prescribes the interval in which the denominators of the rational approximants satisfying the desired Diophantine inequalities lie. This amounts to controlling their growth.

### 0.2.1 Uniform and asymptotic approximations

There is a fundamental difference between Dirichlet type theorems (Theorems 0.0.1 and 0.0.6) and their corollaries (Corollaries 0.0.2 and 0.0.7). Indeed, while the former claim that an inequality involving a rational with denominator bounded by an integer $Q \geq 1$ happens for all $Q \geq 1$, the latter state something much weaker, namely that the inequality under consideration is realized for arbitrary large values of $Q \geq 1$ (with $q = Q$). In other words, the former impose the constraint that the occurrences of a type of approximation should not be too lacunary while the latter just ask for there to be infinitely many solutions to some inequality. Given an irrational $\xi \in \mathbb{R}$, this motivates the introduction of its *asymptotic irrationality exponent* $\omega(\xi)$ and of its *uniform irrationality exponent* $\hat{\omega}(\xi)$ respectively defined as follows :

$$\omega(\xi) := \sup\left\{w > 0 \,:\, \|q\xi\| \leq Q^{-w} \text{ and } 1 \leq q \leq Q \text{ for arbitrarily large values of } Q \geq 1\right\} \tag{11}$$

and

$$\hat{\omega}(\xi) := \sup\left\{w > 0 \,:\, \|q\xi\| \leq Q^{-w} \text{ and } 1 \leq q \leq Q \text{ for all } Q \geq 1 \text{ large enough}\right\}. \tag{12}$$

Thus, an upper bound for the measure of irrationality $\omega(\xi)$ provides a lower bound for $\|q\xi\|$ in terms of the integer $q \geq 1$. In some cases, the value of $\omega(\xi)$ can be explicitly determined. For instance, the Liouville numbers are precisely those irrationals $\xi$ for which $\omega(\xi) = \infty$ and Roth's Theorem (Theorem 0.0.3) amounts to claiming that $\omega(\xi) = 1$ for any $\xi$ algebraic irrational. Furthermore, Khintchine's Theorem implies that $\omega(\xi) = 1$ for almost all reals $\xi$. Also, for any $\tau \geq 1$, the theory of continued fractions provides an easy way to exhibit an element in $W(\tau)$ which is not in $W(\tau + \epsilon)$ for any $\epsilon > 0$. This shows that the spectrum $\{\omega(\xi) \,:\, \xi \in \mathbb{R}\backslash\mathbb{Q}\}$ of $\omega$ is the interval $[1, +\infty]$.

When $\xi$ is irrational,

$$\omega(\xi) \geq \hat{\omega}(\xi) \geq 1.$$

Moreover, since Dirichlet's Theorem cannot be improved uniformly in $\xi \in \mathbb{R}\backslash\mathbb{Q}$ (see the discussion held after Theorem 0.0.1), this shows that $\hat{\omega}(\xi) = 1$ for all $\xi \in \mathbb{R}\backslash\mathbb{Q}$. The relevance of introducing a function constantly equal to one when the argument is irrational appears clearly when considering generalisations of the functions $\omega$ and $\hat{\omega}$ to higher dimensions or degrees.

More precisely, let $n \geq 1$ be an integer and $\xi$ be an irrational. There are at least three natural ways to extend the definitions of the exponents of irrationality (11) and (12) to more general forms of approximation :

- One can first consider approximations of small linear combinations with integer coefficients of the successive powers $1, \xi, \dots, \xi^n$ of $\xi$. With this in mind, define the quantity $\omega_n(\xi)$ as follows :

$$\omega_n(\xi) := \sup\{w > 0 : |P(\xi)| \leq H^{-w} \text{ with } P(X) \in \mathbb{Z}_n[X] \text{ and } 1 \leq H(P) \leq H \text{ for arbitrarily large values of } H \geq 1\}, \quad (13)$$

  where $\mathbb{Z}_n[X]$ denotes the ring of polynomials with integer coefficients and degree less than or equal to $n$ and $H(P)$ is the height of the polynomial $P(X)$ (that is, the maximum of the moduli of its coefficients). The definition of the function $\omega_n$ first originated in Mahler's classification of real numbers — see [150, 151] and [51, Chap. 3].

- A second possibility is to consider approximations of the irrational $\xi$ by algebraic numbers of degree less than $n$. This leads to the introduction of the following quantity :

$$\omega_n^*(\xi) := \sup\{w^* > 0 : |\xi - \alpha| \leq H^{-w^*-1} \text{ with } \alpha \in \mathbb{A}_n \text{ and } 1 \leq H(\alpha) \leq H \text{ for arbitrarily large values of } H \geq 1\}. \quad (14)$$

  Here, $\mathbb{A}_n$ denotes the set of algebraic numbers of degree at most $n$ and $H(\alpha)$ the naive height of $\alpha$ (that is, the maximum of the moduli of its minimal polynomial over $\mathbb{Z}$). The inclusion of $-1$ in the exponent appearing in (14) is standard and is motivated by some heuristics — see, e.g., [54] for further details. The definition of the function $\omega_n^*$ originated in Koksma's classification [143] of real numbers (see also [51, Chap. 3]).

- Another way to generalize definition (11) is to consider simultaneous rational approximations to $\xi, \dots, \xi^n$. Thus, define

$$\lambda_n(\xi) := \sup\left\{\lambda > 0 : \max_{1 \leq k \leq n} \left\|q\xi^k\right\| \leq Q^{-\lambda} \text{ and } 1 \leq q \leq Q \text{ for arbitrarily large values of } Q \geq 1\right\}. \quad (15)$$

  It is also convenient to work with the inverse of $\lambda_n(\xi)$ which, in what follows, will be denoted by

$$\omega_n'(\xi) := \frac{1}{\lambda_n(\xi)}\cdot$$

Each of the quantities $\omega_n(\xi)$, $\omega_n^*(\xi)$ and $\omega_n'(\xi)$ gives rise to the corresponding hat exponent $\hat{\omega}_n(\xi)$, $\hat{\omega}_n^*(\xi)$ and $\hat{\omega}_n'(\xi)$. The latter are defined in the same way as in (13), (14) and (15) with the exception that the inequalities under consideration are required to hold for all $Q \geq 1$

(resp. for all $H \geq 1$) large enough. Further, set $\hat{\omega}'_n(\xi) := 1/\hat{\lambda}_n(\xi)$. It should be clear that the functions $\omega_n(\xi)$ and $\omega^*_n(\xi)$ are always less than their hat versions and that $\hat{\omega}'_n(\xi)$ is always bigger than $\omega'_n(\xi)$.

It is generally difficult to determine the values of the quantities $\omega_n(\xi), \omega^*_n(\xi)$ and $\omega'_n(\xi)$ (resp. of the quantities $\hat{\omega}_n(\xi)$, $\hat{\omega}^*_n(\xi)$ and $\hat{\omega}'_n(\xi)$) for an arbitrary irrational $\xi$ and an integer $n \geq 1$. However, Theorem 0.0.7 readily implies the inequalities

$$\omega_n(\xi) \geq \hat{\omega}_n(\xi) \geq n \quad \text{and} \quad \omega'_n(\xi) \leq \hat{\omega}'_n(\xi) \leq n.$$

It has been a longstanding problem posed by Schmidt (following the work initiated by Wirsing (1931–) in [199]) to determine a similar estimate for the function $\omega^*_n$ :

**Conjecture 0.2.1** (Wirsing, 1961)**.** *Given an integer $n \geq 1$ and a real number $\xi$ which is not algebraic of degree at most $n$, $\omega^*_n(\xi) \geq n$.*

Conjecture 0.2.1 follows immediately from Dirichlet's Theorem (Theorem 0.0.1) when $n = 1$. It has also been settled in the affirmative by Davenport and Schmidt [64] when $n = 2$. The case when $n \geq 3$ remains open.

The spectrum of the sextuple

$$(\omega_n(\xi),\, \omega^*_n(\xi),\, \omega'_n(\xi),\, \hat{\omega}_n(\xi),\, \hat{\omega}^*_n(\xi),\, \hat{\omega}'_n(\xi)) \in \mathbb{R}^6; \tag{16}$$

that is, the range of this vector as $\xi$ takes all irrational values, is far from being completely determined. From a metrical point of view however, the situation is much clearer as it is known (see [54, 194]) that, for almost all $\xi \in \mathbb{R}$,

$$\omega_n(\xi) \;=\; \omega^*_n(\xi) \;=\; \omega'_n(\xi) \;=\; \hat{\omega}_n(\xi) \;=\; \hat{\omega}^*_n(\xi) \;=\; \hat{\omega}'_n(\xi) \;=\; n.$$

Many partial results or particular cases are known towards the determination of the spectrum of (16) and the reader is referred to [54] and to the references therein for an almost exhaustive account on the topic. For instance, it is known that $\hat{\omega}^*_n(\xi) = 1$ when $\xi$ is a Liouville number (Laurent [148]), that $\hat{\omega}^*_n(\xi) \leq 2n - 1$ whenever $\xi \in \mathbb{R}\backslash\mathbb{A}_n$ (cf. [54]) and that $\hat{\omega}'_n(\xi) \geq \lceil n/2 \rceil$ for any irrational $\xi$ (Davenport & Schmidt [65]), where $\lceil\,.\,\rceil$ denotes the ceiling function.

In the same way, the reader is referred to [54, 194] for an almost exhaustive account on the many inequalities relating the components of the vector (16). In most cases, these transference theorems are not known to be sharp. However, and the situation here is similar to that of Conjecture 0.2.1, the case $n = 2$ is much better understood. The following remarkable results, respectively due to Jarník and Damien Roy, illustrate this fact.

**Theorem 0.2.2** (Jarník, 1938)**.** *For any $\xi \in \mathbb{R}\backslash\mathbb{A}_2$,*

$$\hat{\omega}'_2(\xi) \;=\; \frac{\hat{\omega}_2(\xi)}{\hat{\omega}_2(\xi) - 1}\cdot$$

Jarník established Theorem 0.2.2 in [127] when taking an interest in the relations between exponents of approximation related to systems of linear forms — see [127, 128, 129] and the surveys [55, 56]. In 1969, Davenport and Schmidt [65] proved further that the hat exponent $\hat{\omega}'_2(\xi)$ is always bounded above by $(3+\sqrt{5})/2$ for any real $\xi$ which is neither rational nor quadratic. Roy [165, 166] has provided a very nice proof to show that this upper bound is actually optimal :

**Theorem 0.2.3** (Roy, 2003)**.** *There exists a countable family of real numbers $\xi$ which are neither rational nor quadratic such that*

$$\hat{\omega}'_2(\xi) \;=\; \frac{3+\sqrt{5}}{2}\cdot \tag{17}$$

Roy also observed that, as a consequence of the Subspace Theorem, the real numbers satisfying equation (17) must be transcendental.

It should be noted that the so–called *parametric geometry of numbers* very recently introduced by Schmidt and Summerer [175, 176] has given new impetus to the theory of exponents of approximation. Indeed, the tools developed by Schmidt and Summerer enable one to unify many results in this theory in a unique framework and also to prove new ones (for instance, new transference theorems) — see [167] for a very detailed survey on the topic.

### 0.2.2 The Duffin–Schaeffer conjecture

One of the most profound and fundamental conjectures in the Metric Theory of Diophantine Approximation was enunciated by Richard James Duffin (1909–1996) and Albert Charles Schaeffer (1907–1957) in 1941. It is concerned with removing the assumption of the monotonicity of the approximating function in the divergent part of Khintchine's Theorem (Theorem 0.1.1). In their seminal paper [83], Duffin and Schaeffer provided a counterexample to show that this assumption is necessary in full generality. This led them to consider a modified version of the set of $\Psi$–well approximable numbers, namely

$$W'(\Psi) \;=\; \left\{x\in[0,1] \;:\; |qx-p|\leq\Psi(q) \;\text{ for i.m. }\; p\in\mathbb{Z} \;\text{ and }\; q\in\mathbb{N} \;\text{ with }\; \gcd(p,q)=1\right\}.$$

An easy application of the Borel–Cantelli Lemma shows that $W'(\Psi)$ has Lebesgue measure zero when

$$\sum_{q=1}^{\infty}\frac{\varphi(q)}{q}\Psi(q) \;<\; \infty,$$

where $\varphi$ denotes Euler's totient function. The conjecture of Duffin and Schaeffer states that the divergence of this sum is a sufficient condition for the set $W'(\Psi)$ to have full measure :

**Conjecture 0.2.4** (Duffin & Schaeffer, 1941)**.** *Given any approximating function $\Psi$ taking its values in the interval* $(0,1/2)$*,*

$$\lambda\left(W'(\Psi)\right) \;=\; 1 \quad \textit{if} \quad \sum_{q=1}^{\infty}\frac{\varphi(q)}{q}\Psi(q) \;=\; \infty. \tag{18}$$

There is a natural probabilistic interpretation of Conjecture 0.2.4. To state it, denote by $\mathcal{E}_q$ the intersection with the unit interval of the union over all $p$ of intervals of the form

$$\left(\frac{p}{q}-\frac{\Psi(q)}{q}, \frac{p}{q}+\frac{\Psi(q)}{q}\right) \qquad \text{with} \qquad \gcd(p,q)=1. \tag{19}$$

Let $(X_q)_{q\geq 1}$ be a family of random variables such that, for all $q \geq 1$, $X_q$ equals 1 (and 0 otherwise) if a random variable $U$ uniformly distributed in $[0,1]$ falls in $\mathcal{E}_q$. Thus, it is easily seen that $\mathbb{E}(X_q) = \lambda\left(\mathcal{E}_q\right) \asymp \Psi(q)\varphi(q)/q$ for all $q \geq 1$, where $\asymp$ relates two quantities whose ratio and its inverse are both bounded. Claiming that the divergence of the sum of the probabilities $\mathbb{E}(X_q)$ should ensure that any real number falls with probability one in the set $W'(\Psi)$ comes down to postulating some pairwise independence relation (or, at least, some pairwise non–correlation relation) for the family of random variables $(X_q)_{q\geq 1}$. Note in particular that there would be no reason to believe that the random variables in this family should be pairwise independent, and indeed no reason to state Conjecture 0.2.4, if the distribution of the invertible elements in $\mathbb{Z}/q\mathbb{Z}$ was not believed to distribute "uniformly enough" modulo $q$ for arbitrarily large values of $q$.

If the family of random variables $(X_q)_{q\geq 1}$ was to be pairwise independent, then, for integers $r \neq s$, the relation $\mathbb{E}(X_r X_s) = \mathbb{E}(X_r)\mathbb{E}(X_s)$ would yield the relation $\lambda(\mathcal{E}_r \cap \mathcal{E}_s) \asymp \lambda(\mathcal{E}_r)\lambda(\mathcal{E}_s)$. It is actually enough to prove a weaker form of the latter relation for the Duffin–Schaeffer Conjecture to hold, namely that

$$\lambda\left(\mathcal{E}_r \cap \mathcal{E}_s\right) \ll \Psi(r)\Psi(s)\frac{\varphi(r)}{r}\frac{\varphi(s)}{s} \tag{20}$$

for non–zero integers $r \neq s$, where $\ll$ denotes the usual Vinogradov symbol. Indeed, such an estimate together with Gallagher's zero–one law [97] for the set $W'(\Psi)$ (see also [113, §2.2]) would readily imply Conjecture 0.2.4 from a well–known result initially due to Erdős and Chang [61] which gives a partial converse to the Borel–Cantelli Lemma (see [51, p.125] for a proof) :

**Lemma 0.2.5.** *Let* $(\mathcal{E}_q)_{q\geq 0}$ *be a sequence of* $\lambda$*–measurable sets in* $[0,1]$ *such that the sum* $\sum_{q\geq 0}\lambda\left(\mathcal{E}_q\right)$ *diverges.*

*Then,*

$$\lambda\left(\limsup_{q\to\infty}\mathcal{E}_q\right) \geq \limsup_{q\to\infty}\left(\frac{\left(\sum_{k=1}^{q}\lambda\left(\mathcal{E}_k\right)\right)^2}{\sum_{1\leq r,s\leq q}\lambda\left(\mathcal{E}_r\cap\mathcal{E}_s\right)}\right).$$

Unfortunately, the relation of quasi–independence (20) does not hold in general for arbitrary distinct integers $r$ and $s$ as explained in more detail in [25, §2]. In view of Lemma 0.2.5, the hope is that it should hold on average over $r$ and $s$.

In their seminal paper, Duffin and Schaeffer [83, Lemma II] used the elementary bound

$$\lambda\left(\mathcal{E}_r \cap \mathcal{E}_s\right) \leq 4\Psi(r)\Psi(s) \tag{21}$$

valid for $r \neq s$ to prove a weaker version of their conjecture :

**Theorem 0.2.6** (Duffin & Schaeffer, 1941)**.** *Conjecture 0.2.4 holds under the additional assumption that*

$$\limsup_{q\to\infty}\left(\frac{\sum_{k=1}^{q}\Psi(k)\frac{\varphi(k)}{k}}{\sum_{k=1}^{q}\Psi(k)}\right) > 0.$$

Later, Pollington and Vaughan [160] refined the estimate (21) to establish a higher dimensional analogue of Conjecture 0.2.4 stated by Vladimir Gennadievich Sprindzŭk (1936–1987) in [183]. Evidence is also given in the latter reference to show that the one–dimensional case might be related to the Riemann Hypothesis.

As it stands, the Duffin–Schaeffer Conjecture has been proved under various additional assumptions on the approximating function $\Psi$. Thus, Erdős [91] established it in the case when $\Psi(q)=0$ or $\epsilon/q$ for all $q\geq 1$ and some $\epsilon>0$. This was generalized by Vaaler [188] to the case when $\Psi(q)=O\left(q^{-1}\right)$. The best known results towards the conjecture involve a single extra–divergence assumption on the approximating function. They are due to Haynes, Pollington and Velani [117] on the one hand and Beresnevich, Harman, Haynes and Velani [25] on the other and respectively read as follows :

**Theorem 0.2.7** (Haynes & alii, 2012)**.** *If $\Psi$ is an approximating function such that*

$$\sum_{q=1}^{\infty}\left(\frac{\Psi(q)}{q}\right)^{1+\epsilon}\varphi(q) = \infty$$

*for some $\epsilon>0$, then $\lambda\left(W'(\Psi)\right)=1$.*

**Theorem 0.2.8** (Beresnevich & alii, 2013)**.** *If there exists $c>0$ such that*

$$\sum_{q=16}^{\infty}\frac{\varphi(q)\Psi(q)}{q\exp\left(c\left(\log\log q\right)\left(\log\log\log q\right)\right)} = \infty,$$

*then $\lambda\left(W'(\Psi)\right)=1$.*

The dimensional analogue of the Duffin–Schaeffer Conjecture is also proved in [117], namely that the set $W'(\Psi)$ has full Hausdorff dimension if the sum in (18) diverges. Previously, it had been established in [27] thanks to the Mass Transference Principle (Theorem 0.1.10) that a Hausdorff measure version of the conjecture is implied by the original one. Also, the authors of [117] have set out a programme to tackle the Duffin–Schaeffer Conjecture by making use of the properties of dimension functions close to the Lebesgue measure. Theorem 0.2.8 should be seen as a preliminary advance obtained in carrying out this programme.

One might wonder about the existence of a sufficient condition that would guarantee, without any assumption of monotonicity or coprimality, that the one–dimensional set $W(\Psi)$ should have full measure. Catlin [59] provided the following version of the Duffin–Schaeffer conjecture in this context :

**Conjecture 0.2.9** (Catlin, 1976)**.** *Given any approximating function* $\Psi$*, the set* $W(\Psi)$ *has full Lebesgue measure as soon as*

$$\sum_{q=1}^{\infty} \Psi(q) \max_{t\geq 1} \frac{\Psi(qt)}{qt} \; = \; \infty. \tag{22}$$

The maximum appearing in equation (22) admits a natural interpretation : it is related to the length of the largest interval of the form (19) when one replaces $p/q$ with any other rational it is equal to.

Catlin claimed that his conjecture was equivalent to the Duffin–Schaeffer conjecture. However, Vaaler [188] found a mistake in Catlin's proof in such a way that it is still unknown whether the two statements are equivalent. It should also be noted that, in the same way as for the Duffin–Schaeffer conjecture, a higher dimensional version of Conjecture 0.2.9 has been established — see [21, 31].

## 0.3 Approximation on manifolds

An important and active field of research in Diophantine approximation is the study of the intersection of the set $W(m, n, \Psi)$ with manifolds. This amounts to considering approximations of dependent quantities, which induces major difficulties in the determination of the measure theoretic structure of the underlying set of approximation. In most cases, the study is restricted to the intersection of a given manifold with either the simultaneous set $\mathcal{S}_n(\Psi)$ or the dual one $\mathcal{L}_m(\Psi)$.

In what follows, unless stated otherwise, the manifolds under consideration will be smooth $d$–dimensional ($d \geq 1$) immersed submanifolds of Euclidean space $\mathbb{R}^n$ ($n \geq 1$) and will be taken without boundary. Since the subject of interest is the measure theoretic properties of such manifolds, it is possible to simplify further the problem by working locally. In turn, this means that one may consider throughout and without loss of generality a smooth *embedded* manifold $\mathcal{M}$ arising from a parametrisation map

$$\boldsymbol{f} : \boldsymbol{x} = (x_1, \dots, x_d) \in \boldsymbol{U} \;\mapsto\; \boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \dots, f_n(\boldsymbol{x})) \in \mathbb{R}^n, \tag{23}$$

where $\boldsymbol{U}$ is an open subset of $\mathbb{R}^d$; that is, $\mathcal{M} = \boldsymbol{f}(\boldsymbol{U})$ and the partial derivatives of $\boldsymbol{f}$ exist at all orders. The manifold is moreover *analytic* (resp. of class $C^k$ for some $k \geq 0$) if the coordinate functions $f_1, \dots, f_n$ are analytic (resp. of class $C^k$). Even if it means shrinking further the open set $\boldsymbol{U}$, it may also be assumed without loss of generality that the manifold $\mathcal{M}$ is bounded.

For convenience, write $\mathcal{S}_n(\Psi, \mathcal{M}) := \mathcal{S}_n(\Psi) \cap \mathcal{M}$ and $\mathcal{L}_n(\Psi, \mathcal{M}) := \mathcal{L}_n(\Psi) \cap \mathcal{M}$. In the same vein, define in the natural way the sets $\mathcal{S}_n(\tau, \mathcal{M})$ and $\mathcal{L}_n(\tau, \mathcal{M})$. It should be clear that if the dimension $d$ of the manifold is strictly less than the dimension $n$ of the ambient space, the Lebesgue measures of the sets $\mathcal{S}_n(\Psi, \mathcal{M})$ and $\mathcal{L}_n(\Psi, \mathcal{M})$ will always be null irrespective of the approximating function $\Psi$. It is therefore natural to work with the induced Lebesgue measure on $\mathcal{M}$. The latter, which will be denoted by $\lambda_{\mathcal{M}}$, is defined as follows : for any subset $A \subset \mathcal{M}$,

$$\lambda_{\mathcal{M}}(A) \; := \; \int_{\mathcal{M}} \chi_A(\xi) dV.$$

Here, $\chi_A$ is the characteristic function of the set $A$ and $dV$ is the volume element for the induced Riemannian metric on $\mathcal{M}$ which can be expressed in terms of the *first fundamental form* associated to $\mathcal{M}$ — see [40, §1.4] for details. For instance, in the case of a planar curve $\Gamma$ parametrized by

$$\Gamma \;=\; \{\boldsymbol{x}(t) = (x_1(t), x_2(t)) \;:\; t \in I\} \tag{24}$$

on a real interval $I$ (here, $d = 1$ and $n = 2$), the induced Lebesgue measure of an arc joining the points $\boldsymbol{x}(t_0)$ and $\boldsymbol{x}(t_1)$ is nothing but the usual arc–length of $\Gamma$ between these two points.

A fundamental geometric property of the manifold $\mathcal{M}$ which plays an important role in problems of Diophantine approximation is its curvature $\kappa$. The latter is a function which associates to each point on the manifold a real number that quantifies the "flatness" of $\mathcal{M}$ around that point. The general definition of the curvature is involved and depends on the so–called *second fundamental form* and on the tangent and normal spaces at a given point $\xi \in \mathcal{M}$. For an exhaustive account on the topic, the reader is referred to [142, Chap. IX]. For a more elementary one, see [40, §1.4]. The situation is however simpler for a planar curve $\Gamma$ as in (24). Indeed, the idea of curvature renders in this case the concept of the derivative of the angle between the curve and the tangent at a point. More precisely, the function $\kappa$ is then given, for all $t \in I$, by the usual equation

$$\kappa(t) \;=\; x_1'(t)x_2''(t) - x_1''(t)x_2'(t). \tag{25}$$

It is easily verified that if the curvature vanishes constantly on an interval, then the curve $\Gamma$ reduces to a straight line on this interval. Also, as illustrated by formula (25), in the more general situation of a smooth curve in dimension $n \geq 1$ (with $d = 1$), the curvature is closely related to the determinant of the Wronskian matrix $\left(\mathrm{d}^i f_j(t)/\mathrm{d}t^i\right)_{1 \leq i,j \leq n}$.

### 0.3.1 Non–degenerate manifolds, extremal manifolds

**Non–degeneracy.** Generally speaking, the Diophantine properties of manifolds strongly depend on whether the curvature vanishes or not. It is also convenient to work with a more general concept of "non–flatness" known as non–degeneracy. The latter renders the idea that the manifold, in a neighborhood of a given point, is "bent enough" not to be near any affine subspace. In detail, $\mathcal{M}$ is *non–degenerate* at $\boldsymbol{x} \in \boldsymbol{U}$ if the partial derivatives of $\boldsymbol{f}$ at orders up to some integer $l \geq 1$ span $\mathbb{R}^n$. It is non–degenerate if it is non–degenerate at almost every $\boldsymbol{x} \in \boldsymbol{U}$. Note that in the general case, the function $\boldsymbol{f}$ must be at least $\mathcal{C}^2$ in order to ensure that $\mathcal{M}$ is non–degenerate. It is not difficult to see that any connected and analytic real manifold which is not contained in a hyperplane of $\mathbb{R}^n$ is non–degenerate. With the parametrisation (23), this amounts to claiming that the functions $1, f_1, \ldots, f_n$ are linearly independent over $\mathbb{R}$ (see [16, 141] for details).

**Extremality.** The concept of extremality is useful when trying to extend the classical results of Khintchine and of Groshev (Theorems 0.1.1 and 0.1.2) to the simultaneous and the dual sets $\mathcal{S}_n\left(\Psi, \mathcal{M}\right)$ and $\mathcal{L}_n\left(\Psi, \mathcal{M}\right)$. In order to define it, first notice that the transference theorem stated in Theorem 0.1.9 easily implies that

$$\bigcup_{\tau > \frac{1}{n}} \mathcal{S}_n(\tau, \mathcal{M}) = \bigcup_{\sigma > n} \mathcal{L}_n(\sigma, \mathcal{M}).$$

An element in this set is a very well approximable vector lying on $\mathcal{M}$. By similarity with the case of the Euclidean space $\mathbb{R}^n$, the manifold $\mathcal{M}$ is said to be *extremal* if the set of very well approximable vectors lying on it has zero $\lambda_{\mathcal{M}}$–measure.

The concept of extremal manifolds emerged from transcendental number theory. Indeed, Mahler, in connection with his classification of real numbers (see the definition of the exponent $\omega_n$ in subsection 0.2.1), conjectured in 1932 that the Veronese curve in dimension $n \geq 1$, viz. the curve

$$\mathcal{V}_n = \left\{ \boldsymbol{x} = (x, x^2, \ldots, x^n) \;:\; x \in \mathbb{R} \right\}, \tag{26}$$

is extremal. This was proved in 1965 by Sprindžuk [182] who introduced on this occasion the method of essential and inessential domains. It is clear from Sprindžuk's proof that the same result holds for a manifold parametrized by (23) if $f_1, \ldots, f_n$ are polynomials with rational coefficients and degrees at most $n$. The extremality of many other classes of manifolds has since been established depending on their regularity and their curvature [18, 42, 77, 80, 145, 147, 171] (see also [113, Chap. 9] and [183, 184, 191] for surveys).

In 1975, Roger Baker (1947—) set in [9] the conjecture that the Veronese curve $\mathcal{V}_n$ in dimension $n \geq 1$ is *strongly extremal*; that is, that for almost all $\boldsymbol{x} = (x, x^2, \ldots, x^n) \in \mathcal{V}_n$,

$$\liminf_{q \to \infty} q^{1+\epsilon} \prod_{i=1}^{n} \left\| q x^i \right\| \;>\; 0.$$

This means that there should be almost no very well "multiplicatively approximable" points on $\mathcal{V}_n$. While Bernik [38] proved Baker's conjecture in 1984, Sprindžuk [184] extended it into a very general statement known as the Baker–Sprindžuk conjecture.

**Conjecture 0.3.1** (Baker–Sprindžuk Conjecture, 1980)**.** *Assume that the manifold $\mathcal{M}$ given by the parametrisation* (23) *is analytic and non–degenerate. Then $\mathcal{M}$ is strongly extremal; that is, for $\lambda_{\mathcal{M}}$–almost all $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{M}$,*

$$\liminf_{q \to \infty} q^{1+\epsilon} \prod_{i=1}^{n} \left\| q x_i \right\| \;>\; 0.$$

Conjecture 0.3.1 was emphatically settled in the affirmative by Kleinbock and Margulis in 1998 using the correspondence between multiplicatively very well approximable points on manifolds and unboundedness of orbits of unipotent flows in homogeneous spaces of lattices :

**Theorem 0.3.2** (Kleinbock, Margulis, 1998)**.** *Assume that the manifold $\mathcal{M}$ is $C^k$ for some integer $k \geq 2$ and non–degenerate. Then $\mathcal{M}$ is strongly extremal.*

Since a strongly extremal manifold is clearly extremal, Theorem 0.3.2 includes most of the previously known results on extremality of manifolds. It has also been subsequently generalized to the case of submanifolds in affine subspaces [137], of complex manifolds [138], of $p$–adic approximation [140] and of so–called friendly measures [139]. Also, Beresnevich [15] provided in 2002 a "classical" proof of the extremality of any manifold satisfying the assumptions of Theorem 0.3.2.

**Manifolds of Khintchine or Groshev type for convergence or divergence.** A natural question refining the concept of extremality is to determine whether a Khintchine–Groshev type theory can be developed for the simultaneous (resp. the dual) set $\mathcal{S}_n(\Psi, \mathcal{M})$ (resp. $\mathcal{L}_n(\Psi, \mathcal{M})$) associated to a given extremal manifold $\mathcal{M}$. In this respect, the manifold $\mathcal{M}$ is said to be of *Khintchine type for convergence* if the convergence of the sum

$$\sum_{q=1}^{\infty} \Psi(q)^n \tag{27}$$

implies that $\lambda_{\mathcal{M}}(\mathcal{S}_n(\Psi, \mathcal{M})) = 0$. It is said to be of *Khintchine type for divergence* if the divergence of (27) implies that $\mathcal{S}_n(\Psi, \mathcal{M})$ has full measure under the assumption of the monotonicity of the approximating function $\Psi$. Since the manifold $\mathcal{M}$ has been chosen as being bounded without loss of generality, even if it means renormalizing the induced measure $\lambda_{\mathcal{M}}$, it may be assumed that full measure for a subset $\mathcal{A} \subset \mathcal{M}$ means here that $\lambda_{\mathcal{M}}(\mathcal{A}) = 1$.

In a similar way, $\mathcal{M}$ is of *Groshev type for convergence* if the convergence of the sum

$$\sum_{q=1}^{\infty} q^{n-1}\Psi(q) \tag{28}$$

implies that $\lambda_{\mathcal{M}}(\mathcal{L}_n(\Psi, \mathcal{M})) = 0$. It is said to be of *Groshev type for divergence* if the divergence of (28) implies that $\lambda_{\mathcal{M}}(\mathcal{L}_n(\Psi, \mathcal{M})) = 1$ under the assumption of the monotonicity of $\Psi$. The manifold $\mathcal{M}$ is of *Khintchine–Groshev type* if it is both of Khintchine type and of Groshev type for convergence as well as for divergence. The following conjecture is enunciated in [40, §2.3] :

**Conjecture 0.3.3** (Bernik & Dodson, 1999)**.** *A $C^k$ $(k \geq 2)$ non–degenerate manifold $\mathcal{M}$ is of Khintchine–Groshev type.*

In view of this conjecture, the dual case is much better understood than the simultaneous one. Thus, Beresnevich, Bernik, Kleinbock and Margulis [22] established that any non–degenerate manifold is of Groshev type for divergence. Beresnevich [15] proved the convergence counterpart of this result. Thanks to a new approach involving the geometry of lattices in $\mathbb{R}^n$, Bernik, Kleinbock and Margulis [41] had earlier shown that any smooth non–degenerate manifold is of Groshev type for convergence for a stronger multiplicative version of approximation. On another front, in the case of degeneracy, Ghosh obtained Groshev type results for a fairly large class of affine hyperplanes — see [99, 100, 101].

For simultaneous approximation however, most results are partial and deal with particular cases. Those concerning planar curves (that is, when $d = 1$ and $n = 2$) will be discussed in details in subsection 0.3.3 below. In dimension $d \geq 2$, some manifolds $\mathcal{M}$ are known to be of Khintchine type when they satisfy strong conditions of regularity and geometry (e.g., curvature properties as in [80, 81] or convexity properties as in [40, §2.3]). Also, Bernik [33, 35] has proved that if $\mathcal{M} \subset \mathbb{R}^{nk}$ is defined as the cartesian product of $k$ non–degenerate curves in $\mathbb{R}^n$, then $\mathcal{M}$ is of Khintchine type for convergence if $k \geq n$ and for divergence if $n = 2$ and $k \geq 4$. Most recently, Gorodnik and Shah [104] have established a Khintchine type theorem for varieties defined as non–singular rational quadrics with the further constraint that the rational vectors of approximation have to lie on the variety.

When a manifold is of Khintchine type for divergence, one may further ask about quantitative estimates such as the number of solutions to the Diophantine inequalities under consideration from a metrical point of view. Results of this kind are more difficult and restricted. For instance, Harman studied the problem under the assumption that the approximating function $\Psi$ is such that $\mathcal{S}_n(\Psi) = \mathbb{R}^n$ and Dodson, Rynne and Vickers [81] provided an asymptotic formula for the number of integer solutions to the inequalities defining the set $\mathcal{S}_n(\Psi, \mathcal{M})$ for almost all $\boldsymbol{x} \in \mathcal{M}$. However, the strong curvature conditions imposed in this formula drastically restrict the dimension $d$ of the manifold $\mathcal{M}$. See also [34] and [183, Chap. 2, §12] for other results in the same mould.

The problem of determining the measure theoretic structure of the set $\mathcal{S}_n(\Psi, \mathcal{M})$ is very closely related to that of counting the number of rational points with bounded denominators lying near the manifold $\mathcal{M}$. Indeed, an upper bound accurate enough for such a counting function leads to the convergent half of a Khintchine type result, while for the divergence half an accurate lower bound has to be combined with the proof of the ubiquity of the rational points under consideration (that is, the proof of their uniform distribution in some sense — see [40, Chap. 5] for further details). A volume–based heuristic estimate indicates the result which is to be expected : for a smooth manifold $\mathcal{M}$ of dimension $d$, the number of rational points with denominators bounded by $Q$ lying on a thickening of the manifold by $\epsilon > 0$ should be of the order of magnitude of $\epsilon^{n-d}Q^{n+1}$. While simple counter examples may be found to refute this heuristic (cf. [24, §1.4]), the result has been completely established for $C^3$ planar curves whose curvature is bounded by two positive constants thanks to the work of Huxley [121, 122, 123], of Vaughan and Velani [189] and of Beresnevich, Dickinson and Velani [24]. In higher dimensions, the best known estimate is due to Beresnevich who, in the remarkable paper [16], proved the lower bound of the heuristic estimate for analytic non–degenerate manifolds. As a consequence, he obtained the following very general result, which covers in particular the case of non–degenerate analytic curves in any dimension $n \geq 2$ (with $d = 1$) :

**Theorem 0.3.4** (Beresnevich, 2012)**.** *For any $n \geq 2$, any non–degenerate analytic submanifold of $\mathbb{R}^n$ is of Khintchine type for divergence.*

As for the convergence counterpart of this theorem, it has recently been announced that Beresnevich, Vaughan, Velani and Zorin have been able to establish it when the dimension $d$ of the manifold is strictly larger than $(n+1)/2$.

### 0.3.2 The Generalized Baker–Schmidt Problem

In the case where the manifold $\mathcal{M}$ is extremal, or indeed in the case where it is of Khintchine or Groshev type for convergence, one may try to determine the Hausdorff dimension (or, more generally, Hausdorff measures) of sets of very well approximable vectors lying on it for a given type of approximation. This corresponds to the Generalized Baker–Schmidt Problem, after the pioneering work of Baker and Schmidt [10] who initially only dealt with approximation of linear combinations of the successive powers of a real number and therefore with the Veronese curve (26). Here again, the situation is better understood in the dual than the simultaneous case. The Generalized Baker–Schmidt Problem for planar curves will be discussed in more detail in subsection 0.3.3.

**The dual case.** Regarding the Hausdorff dimension of the set $\mathcal{L}_n(\Psi, \mathcal{M})$, the first results were obtained for particular curves in dimension $n \geq 2$. For instance, Bernik [37, 43] proved that for the Veronese curve (26),

$$\dim \mathcal{L}_n(\tau, \mathcal{V}_n) = \frac{n+1}{\tau+1}$$

for any $\tau \geq n$. A fairly general theory is now available. Thus, Dodson, Rynne and Vickers [77] showed that for any $C^3$ manifold $\mathcal{M}$ with dimension $d \geq 2$ which is 2–curved everywhere except on a set of Hausdorff dimension at most $d-1$,

$$\dim \mathcal{L}_n(\tau, \mathcal{M}) = d - 1 + \frac{n+1}{\tau+1} \tag{29}$$

for any $\tau \geq n$. The condition of 2–curvature is technical in nature and means that at least two of the so–called *principal curvatures* at a given point $\boldsymbol{x} \in \mathcal{M}$ do not vanish for any unit vector in the normal space to the manifold at $\boldsymbol{x}$. For surfaces in $\mathbb{R}^3$ ($d = 2$), the condition reduces to the non–vanishing of the standard Gaussian curvature but it is weaker as soon as the dimension $d$ of the manifold is bigger than 3. For more details, see, e.g., [40, §4.3.1].

The dimensional result (29) has been generalized by Bernik and Dodson [40, Theorem 5.14] to the case of any decreasing approximating function :

**Theorem 0.3.5** (Bernik & Dodson, 1999). *Assume that $\mathcal{M}$ is a $C^3$ manifold embedded in $\mathbb{R}^n$ of dimension $d \geq 2$. Assume furthermore that the set of points at which $\mathcal{M}$ is not 2–curved has dimension at most $d-1$. Then, for any decreasing approximating function $\Psi$,*

$$\dim \mathcal{L}_n(\Psi, \mathcal{M}) = d - 1 + \frac{n+1}{\lambda_\Psi + 1},$$

*where $\lambda_\Psi$ is the lower order of the function $1/\Psi$ as defined in* (10)*.*

Also, the dimension in (29) has been shown to be a lower bound for $\dim \mathcal{L}_n(\tau, \mathcal{M})$ in [68] when $\mathcal{M}$ is extremal and $C^1$. This has been generalized to the case of any dimension function in [23, Theorem 18] in the following form :

**Theorem 0.3.6** (Beresnevich, Dickinson & Velani, 2006). *Let $\mathcal{M}$ denote a non–degenerate $d$–dimensional manifold embedded in the $n$–dimensional Euclidean space $\mathbb{R}^n$. Let $f$ be a dimension function such that the function $q \mapsto q^{-d} f(q)$ tends to infinity as $q$ tends to zero and such that the function $q \mapsto q^{-d} f(q)$ is decreasing. Furthermore, suppose that the function $q \mapsto q^{-(d-1)} f(q)$ is increasing. Then, if the approximating function $\Psi$ is decreasing,*

$$\mathcal{H}^f(\mathcal{L}_n(\Psi, \mathcal{M})) = \infty$$

*provided that*

$$\sum_{q=1}^{\infty} f\left(\frac{\Psi(q)}{q}\right)\Psi(q)^{-(d-1)}q^{n+d-1} \;=\; \infty. \tag{30}$$

Showing that

$$\mathcal{H}^f\left(\mathcal{L}_n\left(\Psi, \mathcal{M}\right)\right) \;=\; 0$$

when the sum appearing in (30) converges remains one of the last major conjectures left open in the metric theory of linear Diophantine approximation on manifolds.

**The simultaneous case.** By contrast, very little is known about the Hausdorff dimension of the set $\mathcal{S}_n\left(\tau, \mathcal{M}\right)$ when the manifold $\mathcal{M}$ is extremal and $\tau$ is strictly bigger than the Dirichlet bound $1/n$. Transference theorems in the manner of Theorem 0.1.9 may be used to obtain lower and upper bounds for $\dim \mathcal{S}_n\left(\tau, \mathcal{M}\right)$ from the much better understood dual case, but the bounds obtained this way remain rather crude. This is not surprising in view of a curious result proved by Rynne [169] which shows that one can find arbitrarily close manifolds with the same regularity but with completely different behaviours with respect to simultaneous approximation.

The precise statement of Rynne's result requires some definitions. Assume that the manifold $\mathcal{M}$ parametrized by (23) is $C^k$ $(k \geq 0)$ on the open set $\boldsymbol{U} \subset \mathbb{R}^d$ and that all the partial derivatives of the component functions $f_1, \dots, f_n$ up to order $k$ are continuous on $\boldsymbol{U}$ and extend continuously on its closure $\overline{\boldsymbol{U}}$. The map $\boldsymbol{f}$, and by extension the manifold $\mathcal{M}$, will then be said to belong to the space $\mathfrak{C}^k\left(\boldsymbol{U}, \mathbb{R}^n\right)$ and the $\mathfrak{C}^k$–norm of $\boldsymbol{f}$ will be defined as the real number

$$\left|\boldsymbol{f}\right|_{\mathfrak{C}^k} \;=\; \sum_{i=1}^{n}\sum_{|\boldsymbol{\beta}|=0}^{k} \left\|\partial_{\boldsymbol{\beta}} f_i\right\|_{\infty}^{\overline{\boldsymbol{U}}}.$$

Here, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$ in the inner sum is a multi–index whose length is the integer $|\boldsymbol{\beta}| = \sum_{l=1}^{d}\beta_l$, $\|f\|_{\infty}^{\overline{\boldsymbol{U}}}$ denotes the supremum norm of a real valued function $f$ defined on $\overline{\boldsymbol{U}}$ and

$$\partial_{\boldsymbol{\beta}} f := \frac{\partial^{|\boldsymbol{\beta}|} f}{\partial x_1^{\beta_1} \dots \partial x_d^{\beta_d}}.$$

The space of functions $\mathfrak{C}^k\left(\boldsymbol{U}, \mathbb{R}^n\right)$ equipped with the norm $|\,.\,|_{\mathfrak{C}^k}$ is then a Banach space (see [102] for details).

**Theorem 0.3.7** (Rynne, 2003)**.** *Let $\mathcal{M} \subset \mathbb{R}^n$ be a $d$–dimensional $C^k$ manifold whose parametrisation map $\boldsymbol{f}$, defined on an open set $\boldsymbol{U} \subset \mathbb{R}^d$, belongs to the space $\mathfrak{C}^k\left(\boldsymbol{U}, \mathbb{R}^n\right)$. Then, for any $\epsilon > 0$, there exist manifolds $\mathcal{M}'$ and $\mathcal{M}''$ such that the $\mathfrak{C}^k$–norm between $\mathcal{M}$ and each of these manifolds is less than $\epsilon > 0$ and such that*

$$\mathcal{S}_n\left(\tau, \mathcal{M}'\right) \;=\; \emptyset \quad \textit{for} \quad \tau > \max\left\{\frac{1}{n}, 2k-1\right\}$$

*and*

$$\dim \mathcal{S}_n\left(\tau, \mathcal{M}''\right) \;\geq\; \frac{1+d}{k(1+\tau)} \;>\; 0 \quad \textit{for} \quad \tau > \max\left\{\frac{1}{n}, \frac{1+d}{kd} - 1\right\}.$$

Theorem 0.3.7 suggests that, unlike the dual case, the properties of the set of simultaneously approximable points lying on a manifold do not only depend on the geometric characteristics of this manifold (e.g., its curvature or its convexity), but also on some of its arithmetical properties. General results holding for a sufficiently large class of manifolds are therefore restricted in this context. One notable exception is a theorem established by Beresnevich [16] when studying the distribution of rational points near manifolds (see also Theorem 0.3.4 above). The latter reads as follows :

**Theorem 0.3.8** (Beresnevich, 2012)**.** *Let $\mathcal{M}$ be a non–degenerate analytic submanifold of dimension $d \geq 1$ in $\mathbb{R}^n$. Let $\Psi$ denote a decreasing function such that $q \mapsto q\Psi(q)^{n-d}$ tends to infinity as $q$ tends to infinity. Then, for any $s \in \left(\frac{(n-d)d}{n-d+1}, d\right)$,*

$$\mathcal{H}^s\left(\mathcal{S}_n\left(\Psi, \mathcal{M}\right)\right) = \infty \quad \textit{whenever} \quad \sum_{q=1}^{\infty} \Psi(q)^{s+n-d} q^{-(s-d)} = \infty.$$

*In particular, if the lower order $\lambda_\Psi$ of $1/\Psi$ as defined in* (10) *lies in the interval $\left(\frac{1}{n}, \frac{1}{n-d}\right)$, then*

$$\dim \mathcal{S}_n\left(\Psi, \mathcal{M}\right) \geq \frac{n+1}{\lambda_\Psi + 1} - n + d.$$

As for the determination of the exact value of $\dim \mathcal{S}_n\left(\tau, \mathcal{M}\right)$, various results have been obtained on a case–by–case basis. Some dealing with planar curves are fairly general but are only valid when $\tau$ is close to the Dirichlet bound $1/n$ (see subsection 0.3.3 below for more details). All the other known results have been obtained in the case where the manifold $\mathcal{M}$ is an algebraic variety defined as the roots of an *integer* polynomial $P$ of degree $d_P$ when $\tau$ is larger than $d_P - 1$. The reason for this is that, under these assumptions, one has only to consider rational points lying *on* the manifold $\mathcal{M}$ when studying the set $\mathcal{S}_n\left(\tau, \mathcal{M}\right)$. This follows indeed straightforwardly from this lemma :

**Lemma 0.3.9.** *Let $P\left(X_1, \dots, X_n\right) \in \mathbb{Z}\left[X_1, \dots, X_n\right]$ be an integer polynomial in $n \geq 1$ variables of degree $d_P \geq 1$. Denote by $\mathcal{M}_P$ the submanifold of $\mathbb{R}^n$ defined as the roots of $P$; that is, $\mathcal{M}_P = \{\boldsymbol{x} \in \mathbb{R}^n \ : \ P(\boldsymbol{x}) = 0\}$. Let $\Psi$ be an approximating function such that $\Psi(q) = o\left(q^{-d_P+1}\right)$ as $q$ tends to infinity. Assume that*

$$\frac{\boldsymbol{p}}{q} := \left(\frac{p_1}{q}, \dots, \frac{p_n}{q}\right) \in \mathbb{Q}^n$$

*is a rational vector realizing a simultaneous $\Psi$–approximation of a point $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathcal{S}_n\left(\Psi, \mathcal{M}_P\right)$; that is, a rational vector such that, for all $i = 1, \dots, n$,*

$$\left| x_i - \frac{p_i}{q} \right| < \frac{\Psi(q)}{q}.$$

*Then, for $q$ large enough,*

$$P\left(\frac{\boldsymbol{p}}{q}\right) = 0;$$

*that is,*

$$\frac{\boldsymbol{p}}{q} \in \mathcal{M}_P.$$

*Proof.* The proof rests on the following trick : let $\boldsymbol{p}/q \in \mathbb{Q}^n$ and $\boldsymbol{x} \in \mathcal{S}_n\left(\Psi, \mathcal{M}_P\right)$ as in the lemma. Then, for $i = 1, \dots, n$, one can write

$$x_i = \frac{p_i}{q} + \theta_i \frac{\Psi(q)}{q},$$

where $\theta_i \in (0,1)$. With some obvious notation, the Taylor expansion of $P$ at $\boldsymbol{p}/q$ yields the equations

$$\begin{aligned} 0 = P(\boldsymbol{x}) &= P\left(\frac{p_1}{q} + \theta_1\frac{\Psi(q)}{q}, \dots, \frac{p_n}{q} + \theta_n\frac{\Psi(q)}{q}\right) \\ &= \sum_{j=0}^{d_P}\left[\frac{1}{j!}\left(\sum_{k=1}^{n}\theta_k\frac{\Psi(q)}{q}\frac{\partial}{\partial x'_k}\right)^j P\left(x'_1, \dots x'_n\right)\right]_{x'_1=\frac{p_1}{q}, \dots, x'_n=\frac{p_n}{q}} \\ &= P\left(\frac{\boldsymbol{p}}{q}\right) + O\left(\frac{\Psi(q)}{q}\right), \end{aligned}$$

where the implicit constant in the remainder may be chosen in such a way that it depends only on $P$ and on a predefined neighborhood of $\boldsymbol{x}$. Multiplying throughout by $q^{d_P}$ gives

$$q^{d_P} P\left(\frac{\boldsymbol{p}}{q}\right) = O\left(q^{d_P - 1}\Psi(q)\right),$$

where the left–hand side is an integer. Due to the growth restriction on $\Psi$, for $q$ large enough, the right–hand side is strictly less than one in absolute value, which implies that the left–hand side must equal zero, hence the lemma. **Q.E.D.**

Lemma 0.3.9 gives a sufficient condition for the set $\mathcal{S}_n\left(\Psi, \mathcal{M}_P\right)$ to be empty when $\Psi(q) = o\left(q^{-d_P+1}\right)$, namely that there be only finitely many solutions $(q, p_1, \dots, p_n)$ to the homogeneous Diophantine equation

$$q^{d_P} P\left(\frac{p_1}{q}, \dots, \frac{p_n}{q}\right) = 0.$$

For instance, since the polynomial $R(X,Y,Z) = X^3 + 2Y^3 + 4Z^3 - 9$ admits no rational root as may be seen by considering the reduction modulo 9 of its homogeneous version, the set $\mathcal{S}_n\left(\tau, \mathcal{M}_R\right)$ is empty as soon as $\tau > 2$.

Using Lemma 0.3.9 as a starting point of the proof, Druţu studied the case of a manifold $\mathfrak{Q}_{\boldsymbol{q}}$ defined by the equation $\boldsymbol{q} = 1$, where $\boldsymbol{q} : \mathbb{R}^n \to \mathbb{R}$ is a non–degenerate rational quadratic form in dimension $n \geq 2$ which is naturally assumed *not* to be negative definite. The comprehensive theory developed in [82] for the set $\mathcal{S}_n\left(\Psi, \mathfrak{Q}_{\boldsymbol{q}}\right)$ implies in particular that

$$\dim \mathcal{S}_n\left(\tau, \mathfrak{Q}_{\boldsymbol{q}}\right) = \frac{n-1}{\tau+1} := s_0 \quad \text{and} \quad \mathcal{H}^{s_0}\left(\mathcal{S}_n\left(\Psi, \mathfrak{Q}_{\boldsymbol{q}}\right)\right) = \infty.$$

More recently, Budarina, Dickinson and Levesley [50] studied properties of simultaneous approximation in dimension $n \geq 2$ on polynomials curves of the form

$$\Lambda := \{(x, P_1(x), \ldots, P_{n-1}(x)) \,:\, x \in \mathbb{R}\},$$

where $P_i(X)$ $(1 \leq i \leq n-1)$ is an integer polynomial of degree $d_i$. Their result, formulated for general dimension functions, shows in particular that

$$\dim \mathcal{S}_n(\Psi, \Lambda) = \frac{2}{d(\tau+1)} := s_1 \quad \text{and} \quad \mathcal{H}^{s_1}(\mathcal{S}_n(\tau, \Lambda)) = \infty$$

for $\tau \geq \max\{1, d-1\}$, where $d := \max_{1\leq i \leq n-1} d_i$. Budarina and Dickinson [49] also considered the graph of the function $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \mapsto x_1^d + \cdots + x_{n-1}^d$ $(d \geq 1, n \geq 2)$, namely the hypersurface

$$\Xi := \left\{\left(x_1, \ldots, x_{n-1}, x_1^d + \cdots + x_{n-1}^d\right) \,:\, x_1, \ldots, x_{n-1} \in \mathbb{R}\right\}.$$

Provided that $\tau + 1 > \max\{n/(n-1), d\}$, $n > 2^d - 1$ and $d \geq 1$, they proved that

$$\dim \mathcal{S}_n(\tau, \Xi) \in \left(\frac{n}{d(\tau+1)}, \frac{n+1-d}{\tau+1}\right).$$

### 0.3.3 The case of planar curves

Diophantine subsets of planar curves have been extensively studied in the literature. This is not only because these are the simplest examples of non–elementary manifolds, but also because a slicing technique introduced by Pyartli [162] shows that planar curves underpin some fundamental aspects of the metric theory of Diophantine approximation for curves in any dimension. The latter may be seen as a first step to the study of higher dimensional manifolds.

It should be noted that, in the case of planar curves, non–zero curvature implies non–degeneracy at a point. Conversely, the set of points at which the curvature vanishes without the curve being non–degenerate is at most countable — see [24, §1.1] for further details. From a metrical point of view, the two concepts are therefore equivalent in this setup.

#### 0.3.3.1 The general theory

A fairly large class of planar curves are known to be of Groshev type. Thus, extending earlier results of extremality due to Schmidt [171] and Baker [11], Bernik, Dickinson and Dodson [39] on the one hand and Beresnevich, Bernik, Dickinson and Dodson [19] on the other have proved that any $C^3$ planar curve with non–zero curvature almost everywhere enjoys this property. In view of the work of Beresnevich, Bernik, Kleinbock and Margulis [22] and of Beresnevich [15], the result may be generalized to the case of any non–degenerate curve. As for the Hausdorff theory of linear approximation on manifolds, Baker [11] proved that for any $C^3$ curve $\Gamma$ with curvature non–zero except on a set of zero Hausdorff dimension, $\dim \mathcal{L}_2(\tau, \Gamma) = 3/(\tau+1)$ for $\tau \geq 2$. This is a particular case of (29).

Regarding simultaneous approximation, the situation is also well understood today. The study began with the work of Bernik [36] who proved that the parabola of equation $y = x^2$ (this is also the Veronese curve (26) when $n = 2$), which is the simplest example of a non–degenerate planar curve, is of Khintchine type for convergence. Beresnevich, Dickinson and Velani [24] generalized this result showing that rational quadrics in dimension 2 are of Khintchine type. They also proved that any $C^3$ non–degenerate planar curve is of Khintchine type for divergence. The convergence half of this result was later established by Vaughan and Velani [189] under weaker assumptions :

**Theorem 0.3.10** (Vaughan & Velani, 2006)**.** *Any $C^2$ non–degenerate planar curve is of Khintchine type for convergence.*

Vaughan and Velani's proof relies on an accurate method of counting rational points lying near planar curves meeting the assumptions of Theorem 0.3.10 and is based on estimates of exponential sums. Note that the assumption that the curve be $C^2$ is the weakest possible when requiring non–degeneracy. In [29], a divergent Khintchine type result is also obtained in the framework of simultaneous approximation on planar curves when each coordinate is approximated by different approximating functions.

General results have also been obtained for the Hausdorff dimension of the set $\mathcal{S}_2(\tau, \Gamma)$ for large classes of curves $\Gamma$ in the plane. However, as Rynne's theorem (Theorem 0.3.7) would suggest, these results are only valid for $\tau$ close to the Dirichlet bound 1/2. Thus, Beresnevich, Dickinson and Velani considered, without loss of generality in view of the Implicit Function Theorem, the case of the graph $\Gamma_g := \{(x, g(x)\,;\, x \in I)\}$ of a $C^3$ function $g\ : I \to \mathbb{R}$ defined on a real interval $I$. The main theorem in [24], stated for general approximating functions, implies in particular that for any given $\tau \in [1/2, 1)$, if

$$\dim \{x \in I\,;\, g''(x) = 0\} \leq (2-\tau)/(1+\tau),$$

then

$$\dim \mathcal{S}_2(\tau, \Gamma_g) \;=\; \frac{2-\tau}{1+\tau} \;:=\; s_2.$$

Moreover, if $\tau \in (1/2, 1)$, then

$$\mathcal{H}^{s_2}(\mathcal{S}_2(\tau, \Gamma_g)) \;=\; \infty.$$

This theorem has been generalized in different ways. On the one hand, Beresnevich and Velani [29] have found an analogue in the case when different approximating functions are used for each coordinate (see also [7]). On the other, Beresnevich and Zorin [32] have recently extended it to a much wider class of functions. The statement of their result requires some preliminary definitions which are taken from their paper.

Let $\mathfrak{F}(I, c_1, c_2)$ be the set of $C^2$ real valued functions defined on a given interval $I$ of positive length such that

$$c_1 \;\leq |f''(x)| \;\leq c_2$$

for all $x \in I$, where $0 < c_1 < c_2$. Clearly, any function in $\mathfrak{F}(I, c_1, c_2)$ is non–degenerate. Denote by $\overline{\mathfrak{F}}(I, c_1, c_2)$ the closure of $\mathfrak{F}(I, c_1, c_2)$ for the $C^0$–topology of uniform convergence on $I$. The curve $\Gamma_g$ is *weakly non–degenerate at* $x_0 \in I$ if there exist constants $c_2 > c_1 > 0$

and a compact subinterval $J \subset I$ centred at $x_0$ such that the restriction of $g$ to $J$ belongs to $\mathfrak{F}(J, c_1, c_2)$. In turn, the curve $\Gamma_g$ and the function $g$ itself are *weakly non–degenerate* if $\Gamma_g$ is weakly non–degenerate almost everywhere on $I$. It should be clear that non–degeneracy implies weak non–degeneracy. The converse, however, is not true : an example given in [32, §2] shows that there exist weakly non–degenerate curves degenerate almost everywhere.

The major part of the proof by Beresnevich and Zorin consists of giving an effective generalisation to the class of weakly non–degenerate curves of the optimal estimates of counting functions of rational points lying near planar curves obtained by Vaughan and Velani [189] and by Beresnevich, Dickinson and Velani [24]. To illustrate the far–reaching nature of this generalisation, the set $\overline{\mathfrak{F}}(I, c_1, c_2)$ is explicitly characterized in [32, §2] in terms of convexity properties satisfied by its elements. In particular, it is shown that $\overline{\mathfrak{F}}(I, c_1, c_2)$ is contained in $C^1(I)$ (the set of continuously differentiable functions on $I$) but not in $C^2(I)$. Estimating the number of rational points lying near $\Gamma_g$ for $f \in \overline{\mathfrak{F}}(I, c_1, c_2)$ leads to the following statement :

**Theorem 0.3.11** (Beresnevich & Zorin, 2010)**.** *Let $\Psi$ be a decreasing approximating function and $g : I \to \mathbb{R}$ be a weakly non–degenerate function. Then :*

- ***a)*** *the curve $\Gamma_g$ is of Khintchine type for divergence.*
- ***b)*** $\mathcal{H}^s\left(\mathcal{S}_2\left(\Psi, \Gamma_g\right)\right) = \infty$ *whenever* $\sum_{q=1}^{\infty} q^{1-s}\Psi(q)^{s+1} = \infty$ *and* $s \in (1/2, 1)$.
- ***c)*** $\dim \mathcal{S}_2\left(\Psi, \Gamma_g\right) = (2-\lambda_\Psi)/(1+\lambda_\Psi) := s_3$ *if the lower order $\lambda_\Psi$ as defined in* (10) *lies in the interval* $(1/2, 1)$ *and if $g$ is weakly non–degenerate everywhere except on a set of dimension less than $s_3$.*

The perturbation approach adopted for the proof of Theorem 0.3.11 allows one to weaken the regularity conditions usually imposed on the curve $\Gamma_g$ in such a way that it is only necessary for the latter to be $C^1$ rather than $C^2$. As a consequence, this enables one to consider a less restrictive framework than non–degeneracy. In particular, the multiplicative version of Theorem 0.3.11 stated in [32, Theorem 4] generalizes Theorem 0.3.2 by Kleinbock and Margulis to this setup.

Theorem 0.3.11 provides a reasonably complete Hausdorff theory for simultaneous approximation on weakly non-degenerate planar curves when the error term does not decay “too fast” and, together with Theorem 0.3.11, it establishes that a $C^2$ curve is of Khintchine type. As for the Lebesgue theory of Diophantine approximation on weakly non-degenerate planar curves, it has very recently been studied by Huang [119].

If one considers now an approximating function decaying at any rate, the Hausdorff theory for the set of simultaneously approximable points lying on a planar curve is very incomplete and is a very active domain of research. One notable exception is the case of the unit circle $\mathbb{S}^1$, where the situation is well understood. Indeed, while Dirichlet’s Theorem in dimension 2 (Corollary 0.0.7) implies that

$$\dim \mathcal{S}_2\left(\tau, \mathbb{S}^1\right) = 1 \quad \text{when} \quad \tau \leq \frac{1}{2},$$

it follows for instance from Theorem 0.3.11 that

$$\dim \mathcal{S}_2\left(\tau, \mathbb{S}^1\right) = \frac{2-\tau}{1+\tau} \quad \text{when} \quad \frac{1}{2} \leq \tau \leq 1.$$

Mel'nichuk [154] studied the distribution of Pythagorean triples to prove that

$$\dim \mathcal{S}_2\left(\tau, \mathbb{S}^1\right) \leq \frac{1}{1+\tau} \quad \text{when} \quad \tau > 1.$$

Dickinson and Dodson [69] completed this study with the help of the concept of ubiquity to establish that

$$\dim \mathcal{S}_2\left(\tau, \mathbb{S}^1\right) = \frac{1}{1+\tau} \quad \text{when} \quad \tau > 1. \tag{31}$$

Thus, the dimension of the set $\mathcal{S}_2\left(\tau, \mathbb{S}^1\right)$ is known for all values of $\tau > 0$. Moreover, complete Hausdorff and Lebesgue theories valid for any decreasing approximating function and a fairly large class of dimension functions are now available for this set thanks to the work of Beresnevich, Dickinson and Velani [23, §12.7.2].

As the situation is much less clear in the case of an arbitrary planar curve, an attempt is made in the next subsection to overcome this lack by setting a number of conjectures concerning a large class of planar curves in a framework where Lemma 0.3.9 applies.

#### 0.3.3.2 A set of conjectures for algebraic planar curves

Let $P(X,Y) \in \mathbb{Z}[X,Y]$ be an integer polynomial of degree $d_P \geq 1$. Denote by $\mathcal{M}_P$ the planar curve defined by $P$, viz.

$$\mathcal{M}_P = \left\{(x_1, x_2) \in \mathbb{R}^2 \;:\; P\left(x_1, x_2\right) = 0\right\}. \tag{32}$$

From an algebraic point of view, $\mathcal{M}_P$ may also be seen as the algebraic variety defined by the equation

$$P(X,Y) = 0. \tag{33}$$

As a consequence of this duality, the set of rational points lying on the planar curve (32) will be denoted by $\mathcal{M}_P\left(\mathbb{Q}\right)$. Furthermore, the Diophantine properties of the set $\mathcal{S}_2\left(\Psi, \mathcal{M}_P\right)$ will be analyzed depending on some algebraic invariants of the variety (33) when $\Psi$ is, unless stated otherwise, a *decreasing* approximating function such that $\Psi(q) = o\left(q^{-d_P+1}\right)$ as $q$ tends to infinity. Under this assumption indeed, in view of Lemma 0.3.9, the problem is reduced to the consideration of the properties of the set $\mathcal{M}_P\left(\mathbb{Q}\right)$. In this respect, it should be noted that the study undertaken henceforth is strongly connected to the recent fury of activity in the area of *intrinsic Diophantine approximation* initiated by Fishman, Kleinbock, Merrill and Simmons [96], Gorodnik and Kadyrov [103] and Gorodnik and Shah [104]. In problems of intrinsic approximation indeed, one is interested in approximation on manifolds by rational points also lying on the manifold.

When the lower order $\lambda_\Psi$ of $1/\Psi$ belongs to the interval $(1/2, 1)$, Theorem 0.3.11 provides a clear picture for the dimension of the set $\mathcal{S}_2(\Psi, \mathcal{M}_P)$. The case when $\lambda_\Psi \in (1, d_p - 1]$ (assuming $d_p \geq 3$) is, on the other hand, not addressed by the various known results and the conjectures set below. More generally, this problem has already been noticed for general planar curves and manifolds in [50, 67].

**Genus of the curve $\mathcal{M}_P$.** The dimensional theory developed for the set $\mathcal{S}_2(\Psi, \mathcal{M}_P)$ will depend on the genus $g(P)$ of the variety (33) considered as a Riemann surface. A succinct account on this invariant is now given. For a more detailed exposition, see, e.g., [196].

The genus $g(P)$ of the variety defined by equation (33) is a positive integer at most equal to

$$\frac{(d_P-1)(d_P-2)}{2}. \tag{34}$$

The genus is exactly equal to this quantity if, and only if, the projective version of the curve is nonsingular over $\mathbb{C}$. This means that, in each of the charts $\{X=1\}$, $\{Y=1\}$ and $\{Z=1\}$, the homogeneous version

$$Q(X, Y, Z) := Z^{d_P} P(X/Z, Y/Z)$$

of $P(X, Y)$ defines a variety with no point with *complex* coordinates at which both $Q$ and its partial derivatives of order one vanish. On the other hand, if such a point exists, it defines a *singular point*. A singular point $S$ of multiplicity $r_S$ such that the projective version of the curve read in the appropriate chart admits $r_S$ *distinct* tangents at $S$ reduces the genus by exactly $r_S(r_S-1)/2$. Such a point is referred to as an *ordinary singular point*. Therefore, if all the singular points $S \in \mathbb{P}^2(\mathbb{C})$ on the projective version of the curve defined by (33) are ordinary, the following equation holds :

$$2g(P) \;=\; (d_P-1)(d_P-2) - \sum_S r_S(r_S-1),$$

where the second sum runs over the (finite) set of singular points. If the curve admits non–ordinary singular points, then the previous formula only gives an upper bound for the genus of the curve and the determination of the actual value of $g(P)$ generally rests on the following fact : if there exists a birational correspondence between two curves, then they have the same genus (recall that birational correspondence between two curves means that the points of one of them can be expressed as a rational function of the other, and conversely). Thus, in the most difficult cases, the computation of $g(P)$ can be done by transforming the curve under consideration into a birationally equivalent curve with only singular points (see [196] for examples and details).

The Diophantine properties of the set of simultaneously very well approximable points lying on $\mathcal{M}_P$ will now be discussed depending on whether $g(P)$ equals 0 or 1 or is bigger than 2.

**Curves of genus at least 2.** Assume first that $g(P) \geq 2$. Falting's Theorem states that, if the variety (33) defines a geometrically irreducible and smooth curve, then $\mathcal{M}_P(\mathbb{Q})$ is finite. It is however possible to dispense with the latter two assumptions (see [47, Chap. 11] for details and for the definition of the concepts at stake) in such a way that Lemma 0.3.9 immediately implies the following statement :

**Theorem 0.3.12.** *Assume that the variety* (33) *defines a curve of genus* $g(P) \geq 2$*. Let* $\Psi$ *be an approximating function satisfying* $\Psi(q) = o\left(q^{-d_P+1}\right)$ *as* $q$ *tends to infinity. Then, the set* $\mathcal{S}_2\left(\Psi, \mathcal{M}_P\right)$ *is finite.*

Note that there is no assumption of monotonicity of the approximating function in Theorem 0.3.12. As an application of this result, consider the family of Fermat curves

$$\mathcal{F}_s \; : X^s + Y^s = 1 \tag{35}$$

parametrized by the integer $s \geq 2$. In view of (34), one has

$$g(\mathcal{F}_s) \; = \; \frac{(s-1)(s-2)}{2}$$

for all $s \geq 2$ (this can also be seen as an easy consequence of the Riemann–Hurwitz formula). Thus, $g(\mathcal{F}_s) \geq 2$ as soon as $s \geq 4$ and Theorem 0.3.12 implies that $\mathcal{S}_2\left(\tau, \mathcal{F}_s\right)$ is finite for any $s \geq 4$ and $\tau > s-1$. This gives an alternative to the solution proposed by Berník and Dodson, who proved the same result in [40, Theorem 4.8] appealing to Wiles' Theorem [198]. Note that, when $s = 3$, the absence of non–trivial rational solutions to equation (35) has been known since Euler [85] and here again Wiles' Theorem may be avoided in order to prove that the set $\mathcal{S}_2\left(\tau, \mathcal{F}_3\right)$ is finite when $\tau > 2$.

**Curves of genus 1.** Assume now that $g(P) = 1$ (in which case $d_P \geq 3$). It may then well be that the set $\mathcal{M}_P(\mathbb{Q})$ is empty. This is for instance the case of the curves defined by the equations

$$X^4 + 2Y^2 = 17 \qquad \text{and} \qquad 3X^3 + 4Y^3 + 5 = 0$$

respectively considered by Reichardt [163] and Selmer [177], who proved that they do not satisfy the Hasse Principle : they admit a global solution in $\mathbb{R}$ and local solutions in $\mathbb{Q}_p$ for every prime $p$, but they admit no rational solution.

However, if the curve defined by $P$ is smooth and if a rational point lies on it, then the chord and tangent method enables one to equip the set $\mathcal{M}_P(\mathbb{Q})$ with a group law. The curve $\mathcal{M}_P$ together with a distinguished rational point on it (referred to as the "point at infinity") then becomes a *model* for an elliptic curve defined over $\mathbb{Q}$. The reader is referred to [180] for a classical reference about the theory of elliptic curves. The main point which needs to be stressed here is that, from an algebraic point of view, elliptic curves are seen as a class of isomorphic algebraic varieties; that is, each elliptic curve may be given by different curves (its models) which are mutually birationally equivalent over $\mathbb{C}$ (this means that the coefficients of the birational transformations under consideration lie in $\mathbb{C}$). From the point of view of Diophantine analysis however, the approximation properties of the points lying on one of the models of an elliptic curve cannot be transferred to another one since they are not necessarily preserved by birational transformations. This shows that one has indeed to state most of the forthcoming results for a *fixed* model of an elliptic curve without the possibility to carry the properties under consideration to other models. However, any result which does not depend on the choice of the model $\mathcal{M}_P$ (and of a rational point lying on it — this choice will have no importance here) will be stated for the corresponding elliptic curve which will be denoted by $\mathcal{E}$.

The celebrated Mordell–Weil Theorem [180, Chap. VIII, §4] states that the set of rational points $\mathcal{E}(\mathbb{Q})$ lying on the algebraic variety $\mathcal{E}$ is a finitely generated abelian group. In other words,

$$\mathcal{E}(\mathbb{Q}) \simeq \mathbb{Z}^r \times \mathcal{E}_{tors}(\mathbb{Q}),$$

where $r \geq 0$ is an integer referred to as the *rank of the elliptic curve* and where $\mathcal{E}_{tors}(\mathbb{Q})$ is the finite subgroup of $\mathcal{E}(\mathbb{Q})$ consisting of all torsion elements. In particular, if $r = 0$, then only finitely many rational points lie on $\mathcal{E}$ and $\mathcal{S}_2(\Psi, \mathcal{M}_P)$ is finite under the assumptions of Theorem 0.3.12. On the other hand, when the rank is non–zero, an accurate estimate of the growth of the number of rational points in $\mathcal{E}(\mathbb{Q})$ has been obtained by André Néron (1922–1985) in [157]. In order to state his result, given a rational point $A \in \mathcal{E}(\mathbb{Q})$, let $A = [x(A) : y(A) : z(A)] \in \mathbb{P}^2(\mathbb{Q})$ denote its coordinates in the predefined projective model for $\mathcal{E}$ induced by $\mathcal{M}_P$. Assume furthermore that $x(A), y(A)$ and $z(A)$ are coprime integers. Define the *absolute height* $H(A)$ of $A$ as follows :

$$H(A) := \max\{|x(A)|, |y(A)|, |z(A)|\}.$$

Let $\mathfrak{N}_P(\mathcal{E}(\mathbb{Q}), B)$ denote the cardinality of the set of rational points lying on the predefined model for $\mathcal{E}$ whose absolute heights are less than $B > 0$.

**Theorem 0.3.13** (Néron, 1965)**.** *Assume that $\mathcal{M}_P$ defines a model for an elliptic curve $\mathcal{E}$ over $\mathbb{Q}$ of rank $r \geq 1$.*

*Then,*

$$\mathfrak{N}_P(\mathcal{E}(\mathbb{Q}), B) \asymp (\log B)^{r/2}$$

*for all $B$ large enough.*

Theorem 0.3.13 thus suggests that the rational points lying on any model for an elliptic curve cannot be "excessively dense". An easy covering argument enables one to deduce from it the following statement :

**Theorem 0.3.14.** *Assume that $\mathcal{M}_P$ defines a model for an elliptic curve. Let $\Psi$ be a decreasing approximating function such that $\Psi(q) = o\left(q^{-d_P+1}\right)$ as $q$ tends to infinity. Then,*

$$\dim \mathcal{S}_2(\Psi, \mathcal{M}_P) = 0.$$

In the case when the rank corresponding to the elliptic curve given by the model $\mathcal{M}_P$ is non–zero, Theorem 0.3.14 follows from Lemmata 0.1.4 and 0.3.9 and Theorem 0.3.13. It should be pointed out that the assumption of the monotonicity of the function $\Psi$ allows one to study, without loss of generality, the metric properties of the set $\mathcal{S}_2(\Psi, \mathcal{M}_P)$ with the additional constraint that the rationals under consideration are irreducible (as in the definition of the counting function $\mathfrak{N}_P(\mathcal{E}(\mathbb{Q}), B)$).

Given that the Hausdorff dimension of the set of very well approximable points $\mathcal{S}_2(\tau, \mathcal{M}_P)$ remains undetermined for $\tau \in (1,2] \subset (1, d_p - 1]$, it seems relevant to pay attention to the set of simultaneously well approximable points lying on $\mathcal{M}_P$. With this in mind, the following theorem due to Baker [12] is of interest :

**Theorem 0.3.15** (Baker, 1976)**.** *Let $f_1,\ f_2,\ \ldots,\ f_n$ be $C^1$ functions defined on an open set $\Omega \subset \mathbb{R}^m$ $(m, n \geq 1)$. Assume that the matrix of partial derivatives*

$$\left(\frac{\partial f_s}{\partial x_j}(\alpha_1, \ldots, \alpha_m)\right)_{1 \leq s \leq n, 1 \leq j \leq m}$$

*is non–zero almost everywhere in $\Omega$.*

*Then, for almost all $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_m) \in \Omega$, the point*

$$(f_1(\boldsymbol{\alpha}), \ldots, f_n(\boldsymbol{\alpha})) \in \mathbb{R}^n$$

*is not badly approximable.*

Thanks to the Implicit Function Theorem, this result implies that the set of simultaneously well approximable vectors lying on $\mathcal{M}_P$, viz. the intersection

$$\bigcap_{c \geq 1} \mathcal{S}_2\left(q \mapsto (cq)^{-1/2}, \mathcal{M}_P\right),$$

has full Lebesgue measure.

The problem is also well understood today for the complement of the set of simultaneously well approximable points on the curve $\mathcal{M}_P$, viz. the set $\mathbf{Bad}(\mathcal{M}_P)$ of simultaneously badly approximable points lying on it. Specifically, a vector $(x_1, x_2) \in \mathcal{M}_P$ is in $\mathbf{Bad}(\mathcal{M}_P)$ if

$$\liminf_{q \to \infty} \max\left\{q^{1/2}\|qx_1\|, q^{1/2}\|qx_2\|\right\} > 0.$$

In a recent work, An, Beresnevich and Velani [4] made use of Schmidt's game to prove that the intersection of the set of simultaneously badly approximable vectors in dimension two with any $C^2$ planar curve which is not a straight line segment has full dimension (see also the main result in [17]). In particular,

$$\dim \mathbf{Bad}(\mathcal{M}_P) = 1.$$

Also, even though the counting function $\mathfrak{N}_P(\mathcal{E}(\mathbb{Q}), B)$ grows relatively slowly, it is well–known that, when the rank of $\mathcal{E}$ is non–zero, the set of rational points lying on any of its models is dense (in the usual Euclidean topology) in the real component containing the point at infinity — see [192] for a proof of this fact. A natural problem associated with Theorem 0.3.15 is then the study of the set of intrinsically well approximable points lying on $\mathcal{M}_P$, which is the intersection

$$\bigcap_{c \geq 1}\left\{(x_1, x_2) \in \mathcal{M}_P\ :\ \max\{|qx_1 - p_1|, |qx_2 - p_2|\} \leq \frac{c}{q^{1/2}} \quad \text{for i.m.} \quad \left(\frac{p_1}{q}, \frac{p_2}{q}\right) \in \mathcal{M}_P(\mathbb{Q})\right\}.$$

**Problem 0.3.16.** *Assume that $\mathcal{M}_P$ defines a model for an elliptic curve with non–zero rank. Does there exist an intrinsically well approximable point lying on $\mathcal{M}_P$? If so, what is the Lebesgue measure of this set?*

It is likely that the solution to this problem will involve some properties of the natural group law with which the set $\mathcal{M}_P(\mathbb{Q})$ may be equipped.

**Curves of genus 0.** Assume now that $g(P) = 0$. Since the Hasse principle applies to all curves of genus 0 (see [196] for a proof), the set $\mathcal{M}_P(\mathbb{Q})$ is non–empty if, and only if, equation (33) admits a solution in $\mathbb{R}$ and in all the local fields $\mathbb{Q}_p$, where $p$ runs through all the prime numbers. The condition about the existence of $\mathbb{Q}_p$–points on the curve concretely means that there is no obstruction to rational solutions to (33) arising from considering this equation modulo various integers. For a given curve, this leads to a finite number of verifications. It is therefore straightforward to check whether $\mathcal{M}_P(\mathbb{Q})$ is empty or not. See [58] and the references therein for details, examples and proofs.

A classical result in the theory of algebraic curves (see, e.g., [196, p.151]) states that a curve is of genus 0 if, and only if, it is unicursal; that is, if, and only if, it admits a rational parametrisation by functions $x(t), y(t) \in \mathbb{C}(t)$. This means that, up to finitely many exceptions, every evaluation of the vector $(x(t), y(t))$ at a real number $t$ determines a point on the curve and, conversely, all but finitely many points on the curve are obtained by the evaluation of the parametrisation vector at some real number. For instance, as is well–known, the unit circle $X^2 + Y^2 = 1$ may be parametrized with the help of the rational map

$$t \mapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right), \tag{36}$$

where $t$ runs through all the real numbers (note that the point $(-1, 0)$ is obtained when "$t = \infty$").

The determination of the minimal field in which the coefficients of the rational functions $x(t), y(t) \in \mathbb{C}(t)$ may be chosen for a given curve of genus 0 is a problem of paramount importance when determining the Hausdorff dimension of the set $\mathcal{S}_2(\Psi, \mathcal{M}_P)$. It was solved a hundred years ago by Hilbert and Hurwitz [118] (see also [178] for a modern approach and a simplified proof) : given a field $\mathbb{K}$ of characteristic zero ($\mathbb{K} = \mathbb{Q}$ in the problem under consideration) and a polynomial $P(X, Y) \in \mathbb{K}[X, Y]$ defining a curve of genus 0, the coefficients of $x(t)$ and $y(t)$ may always be chosen in an algebraic extension $\mathbb{L}$ of $\mathbb{K}$ of degree at most 2. The proof of this result turns out to be very informative as it enables one to distinguish between the cases $\mathbb{L} = \mathbb{K}$ and $[\mathbb{L} : \mathbb{K}] = 2$. Indeed, the construction of an "optimal" parametrisation is obtained with the help of $O(\deg P)$ birational transformations, every such transformation decreasing the degree of the curve by 2. This ultimately leads to either a line in the odd degree case or to an irreducible conic defined over the same field $\mathbb{K}$ as the original curve when the degree is even. In particular, a curve of genus 0 as defined by (33) always admits a rational parametrisation over $\mathbb{Q}$ when the degree of $P$ is odd. For example, the cubic $Y^2 = X^3$ (which has a singularity at the origin and thus is of genus 0) may be parametrized by the function

$$t \in \mathbb{R} \longmapsto (t^2,\, t^3).$$

On the other hand, when the degree of $P$ is even, the reduction process shows that every point on the curve defined by $P$ is in birational correspondence with a point on a conic $\mathcal{C}$ with the exception of a finite number. When one wishes to determine the structure of the set $\mathcal{M}_P(\mathbb{Q})$, this brings the problem down to the well–known case of conics : if a rational point lies on $\mathcal{C}$, then the intersection of $\mathcal{C}$ with a generic line through this rational point determines an additional point on the conic from Bezout's Theorem. Varying the slope of the line leads to a rational parametrisation of $\mathcal{C}$. Furthermore, if the lines have a rational direction–vector, all the rational points on $\mathcal{C}$ are obtained in this way.

All this shows that a rational point lies on a curve of genus zero if, and only if, the curve admits a rational parametrisation by rational functions with coefficients in $\mathbb{Q}$. Since such a function takes rational values when its argument is rational, a curve of genus zero has either no rational point on it or infinitely many of those. In the latter case, all but a finite number of them are given by the evaluation at a rational of a given rational parametrisation of the curve.

It is not difficult to exhibit curves of genus zero with no rational point lying on them (according to the foregoing, the degree of the polynomial defining such curves has to be even) : consider for instance the family of circles

$$\mathbb{S}^1(R)\ : X^2 + Y^2 = R \tag{37}$$

of radius $\sqrt{R}$. They can be parametrized in the form

$$t \mapsto \left(\sqrt{R}\frac{1-t^2}{1+t^2}, \sqrt{R}\frac{2t}{1+t^2}\right)$$

as $t$ runs through the real numbers. When $R = 3$ or $R = 7$, the reduction modulo $R$ of the homogeneous version of equation (37) shows that no rational point lies on the circles $\mathbb{S}^1(3)$ and $\mathbb{S}^1(7)$. In particular, $\mathcal{S}_2\left(\tau, \mathbb{S}^1(3)\right) = \mathcal{S}_2\left(\tau, \mathbb{S}^1(7)\right) = \emptyset$ whenever $\tau > 1$ from Lemma 0.3.9.

These results naturally lead to the statement of the following conjecture which claims that, under some standard assumptions, the metric properties of the set $\mathcal{S}_2(\Psi, \mathcal{M}_P)$ depend only on the embedding of the curve of equation (33) in projective space.

**Conjecture 0.3.17.** *Let $P(X,Y) \in \mathbb{Z}[X,Y]$ be a polynomial of degree $d_P \geq 1$ defining a curve $\mathcal{M}_P$ of genus zero with (infinitely many) rational points lying on it. Let $u(T)$, $v(T)$, $w(T) \in \mathbb{Z}[T]$ be relatively prime polynomials such that the projective version of the curve may be parametrized by*

$$t \longmapsto [u(t), v(t), w(t)].$$

*Then, the Hausdorff dimension of the set $\mathcal{S}_2(\Psi, \mathcal{M}_P)$ depends only on the degrees of $u$, $v$ and $w$ provided that $\Psi$ is a decreasing approximating function such that $\Psi(q) = o\left(q^{-d_P+1}\right)$ as $q$ tends to infinity.*

In the case of the unit circle $\mathbb{S}^1$, all the non–trivial Pythagorean triples are given by the range of the projective version of the vector (36) as the parameter $t$ runs through the rational numbers. In this case, $u(t) = 1 - t^2$, $v(t) = 2t$ and $w(t) = 1 + t^2$ and the actual value of $\dim \mathcal{S}_2\left(\tau, \mathbb{S}^1\right)$ is given by (31) when $\tau > 1$. From this, one could expect that, under the assumptions of Conjecture 0.3.17, the Hausdorff dimension of $\mathcal{S}_2\left(\tau, \mathcal{M}_P\right)$ should be of the form $C/(1+\tau)$ for $\tau > d_P - 1$, where $C > 0$ is a constant depending only on $\deg u$, $\deg v$ and $\deg w$.

## 0.4 Organisation of the chapters

The thesis consists of five chapters :

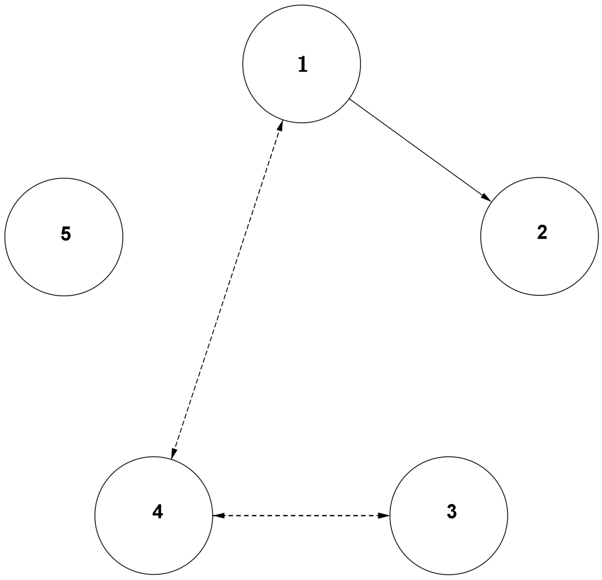
- Chapter 1 deals with the problem of Duffin and Schaeffer (subsection 0.2.2) and provides an extension of their theorem (Theorem 0.2.6) to the case when the numerators and the denominators of the rational approximants are related by a constraint stronger than coprimality. At the end of the chapter, a generalized version of the Duffin–Schaeffer conjecture (Conjecture 0.2.4) is stated in this context.

- Chapter 2 is concerned with the theme of simultaneous approximation on manifolds. The focus will be on the study of the Hausdorff dimension of the set of very well approximable points lying on a polynomial curve which is *not* defined as a polynomial with *integer* coefficients. This will provide the first results in this context and it will be shown that the situation differs immensely from the known results in the case of polynomials with integer coefficients (see subsection 0.3.2 above).

- Chapters 3 and 4 both share a common topic, which seems to be new in the theory of Diophantine approximation. Indeed, while problems of approximation have been so far exclusively concerned with “limsup sets”, the focus in these chapters will be on so–called “liminf sets”. In short, in this type of approximation under constraints (see subsection 0.2 above), one wishes to approach a point in Euclidean space by infinitely many rational vectors whose denominators lie in a prescribed infinite set of integers, but by finitely many rational vectors with denominators in the complement of this set. A Hausdorff theory is developed in Chapters 3 and 4 for simultaneous approximation in this context.

- Chapter 5 is also related to the topic of Diophantine approximation under constraints. A reasonably complete theory of approximation of a real number by rationals whose numerators and denominators lie in prescribed arithmetic progressions is developed here. The results are of a uniform, non–uniform, metrical and non–metrical nature. The main novelty is the proof of a Khintchine type statement in the context of uniform approximation.

The dependencies between the chapters is illustrated in the diagram below, where solid arrows indicate that a result from one chapter is used in another one and where dotted arrows mark a thematic link (in the theories developed or in the ideas involved in the proofs).




Some very active domains of research in Diophantine approximation will not be broached in this thesis. These include the theories of inhomogeneous, weighted and twisted approximation, the theory of multiplicative approximation (in particular the celebrated Littlewood conjecture and its “mixed” versions), the various studies dealing with the properties of the set of badly approximable vectors or with the problems of approximation by algebraic numbers.

# Chapter 1

# An Extension of a Theorem of Duffin and Schaeffer


**Abstract**

Duffin and Schaeffer have generalized the classical theorem of Khintchine in metric Diophantine approximation to the case of any error function under the assumption that all the rational approximants are irreducible. This result is extended to the case where the numerators and the denominators of the rational approximants are related by a congruential constraint stronger than coprimality.


**Notation.** For convenience, the following notation will be used throughout this chapter :

- $\lfloor x \rfloor$ $(x \in \mathbb{R})$ : the integer part of $x$.
- $[\![x, y]\!]$ $(x, y \in \mathbb{R},\ x \leq y)$ : interval of integers, i.e. $[\![x, y]\!] = \{n \in \mathbb{Z} \ : \ x \leq n \leq y\}$.
- $a \ll b$ $(a, b \in \mathbb{R})$ : Vinogradov notation meaning that there exists a constant $c > 0$ such that $a \leq cb$.
- $\mathrm{Card}(X)$ or $|X|$ : the cardinality of a finite set $X$.
- $A^{\times}$ : the set of invertible elements of a ring $A$.
- $\mathcal{P}$ : the set of prime numbers.
- $\pi$ : any prime number.
- $\varphi(n)$ : Euler's totient function.

- $\tau(n)$ : the number of divisors of a positive integer $n$.

- $\omega(n)$ : the number of distinct prime factors dividing an integer $n \geq 2$ ($\omega(1) = 0$).

## 1.1 Introduction

The well–known theorem of Duffin and Schaeffer [83] in metric number theory extends the classical theorem of Khintchine in the following way :

**Theorem 1.1.1** (Duffin & Schaeffer, 1941)**.** *Let $(q_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers and let $(\alpha_k)_{k\geq 1}$ be a sequence of real numbers in $(0, 1/2)$ which satisfies the conditions :*

***(a)*** $\displaystyle\sum_{k=1}^{\infty} \alpha_k = \infty,$

***(b)*** $\displaystyle\sum_{k=1}^{n} \frac{\alpha_k\, \varphi\,(q_k)}{q_k} > c\sum_{k=1}^{n} \alpha_k$ *for infinitely many integers $n \geq 1$ and a real number $c > 0$.*

*Then, for almost all $x \in \mathbb{R}$, there exist infinitely many relatively prime integers $p_k$ and $q_k$ such that*

$$\left| x - \frac{p_k}{q_k} \right| < \frac{\alpha_k}{q_k} \cdot$$

Here as elsewhere, *almost all* must be understood in the sense that the set of exceptions has Lebesgue measure zero.

Several generalisations of Theorem 1.1.1 have been considered : on the one hand, the conjecture of Duffin and Schaeffer asks whether assumption (b) may be weakened by replacing it with the divergence of the series $\sum_{k\geq 1} \alpha_k\, \varphi\,(q_k)\, q_k^{-1}$. Even if the analogue of this problem has been proved in higher dimensions [160] or with some extra assumptions on the sequence $(\alpha_k)_{k\geq 1}$ [117], the full conjecture is still open — see [Prolegomena, subsection 0.2.2] for further details. On the other hand, one may try to see to what extent Theorem 1.1.1 remains true when the numerators $p_k$ and the denominators $q_k$ of the fractional approximations are related by some relationship stronger than coprimality (in a sense to be made precise).

Indeed, many results have been obtained in the metric theory of Diophantine approximation when the denominators of the rational approximants are confined to a prescribed set (see for instance [51, Theorem 5.9]). However, restrictions on the numerators introduce new difficulties which are not always easy to overcome (see [51, p.144] for an account on this fact). In a series of papers [109, 110, 111, 112], G.Harman tackled the problem and gave several results in the case where the denominators and numerators are confined to independent sets of integers. The main theorems proved in this chapter give another approach to this problem and are concerned with the case when numerators and denominators are confined to dependent sets of integers in the sense that they are related, not only by the relation of Diophantine approximation of a given real number, but also by some additional congruential constraints.

Consider first a subsequence $\left(q_k^d\right)_{k\geq 1}$ of the $d^{\text{th}}$ powers of the natural numbers ($d \geq 1$ integer). For any $q \in \mathbb{N}$ denote furthermore by $r_d(q)$ the cardinality of the set of $d^{\text{th}}$ powers in a reduced system of residues modulo $q$ and set for simplicity

$$s_d(q) := \frac{r_d(q)}{q}\cdot \tag{1.1}$$

**Theorem 1.1.2.** *Let $(q_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers and let $(\alpha_k)_{k\geq 1}$ be a sequence of positive real numbers in $(0, 1/2)$. Fix an integer $a \geq 1$ and assume furthermore that :*

***(a)*** $\displaystyle\sum_{k=1}^{\infty} \alpha_k = \infty,$

***(b)*** $\displaystyle\sum_{k=1}^{n} \alpha_k s_d\left(q_k^d\right) > c \sum_{k=1}^{n} \alpha_k$ *for infinitely many integers $n \geq 1$ and a real number $c > 0$,*

***(c)*** $\gcd(q_k, a) = 1$ *for all $k \geq 1$.*

*Then, for almost all $x \in \mathbb{R}$, there exist infinitely many relatively prime integers $p_k$ and $q_k$ such that*

$$\left|x - \frac{p_k}{q_k^d}\right| < \frac{\alpha_k}{q_k^d} \quad \textit{and} \quad p_k \equiv a b_k^d \pmod{q_k} \quad \textit{for some } b_k \in \mathbb{Z} \textit{ relatively prime to } q_k.$$

Theorem 1.1.2 will play a crucial role in Chapter 2 in the study of a problem of simultaneous Diophantine approximation of dependent quantities : given an integer polynomial $P(X)$ and a real number $x$, what is the Hausdorff dimension of the set of real numbers $t$ such that $t$ and $P(t) + x$ are simultaneously $\tau$–well approximable, with $\tau > 0$? It will be proved in Chapter 2 that such an approximation implies an approximation of $x$ by a rational number of the form $p/q^d$, where $d$ is the degree of $P(X)$ and where the integer $p$ satisfies the congruential constraint mentioned in the conclusion of Theorem 1.1.2 with $a$ the leading coefficient of $P(X)$. The emptiness of the set of approximation under consideration is obtained for almost all $x$ as a consequence of the convergent part of the Borel–Cantelli Lemma when $\tau > d + 1$ and Theorem 1.1.2 enables one to prove the optimality of this lower bound.

Theorem 1.1.2 can be generalized in the following way :

**Theorem 1.1.3.** *(Extension of the Theorem of Duffin and Schaeffer). Let $(q_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers and let $(\alpha_k)_{k\geq 1}$ be a sequence of positive real numbers in $(0, 1/2)$. Let $(a_k)_{k\geq 1}$ be a sequence such that for all $k \geq 1$, $a_k \in (\mathbb{Z}/q_k\mathbb{Z})^\times$. For $k \geq 1$, denote by $G_k$ a subgroup of $(\mathbb{Z}/q_k\mathbb{Z})^\times$ and by $a_k G_k$ the coset of $a_k$ in the quotient of $(\mathbb{Z}/q_k\mathbb{Z})^\times$ by $G_k$. Assume furthermore that :*

***(a)*** $\displaystyle\sum_{k=1}^{\infty} \alpha_k = \infty,$

***(b)*** $\displaystyle\sum_{k=1}^{n} \alpha_k \frac{|G_k|}{q_k} > c \sum_{k=1}^{n} \alpha_k$ *for infinitely many integers $n \geq 1$ and a real number $c > 0$,* *(c)* $\dfrac{\varphi(q_k)}{q_k^{1/2-\epsilon}\,|G_k|} \longrightarrow 0$ *as $k$ tends to infinity, for some $\epsilon > 0$.*

*Then, for almost all $x \in \mathbb{R}$, there exist infinitely many relatively prime integers $p_k$ and $q_k$ such that*

$$\left| x - \frac{p_k}{q_k} \right| < \frac{\alpha_k}{q_k} \quad \textit{and} \quad p_k \in a_k G_k. \tag{1.2}$$

**Remark 1.1.4.**

- Given the well–known estimate

$$\varphi(q) \gg \frac{q}{\log\log q},$$

assumption (c) in Theorem 1.1.3 is satisfied as soon as

$$|G_n| \geq q_n^{1/2+\epsilon}$$

for some $\epsilon > 0$ and all $n \geq 1$.

- If $c$ is a real number such that

$$\frac{|G_k|}{q_k} > c > 0 \tag{1.3}$$

for all $k \geq 1$, then (b) holds. However, if, instead of (1.3), one can prove the weaker assertion

$$\sum_{k=1}^{n} \frac{|G_k|}{q_k} > cn \tag{1.4}$$

for some $c > 0$ and all integers $n \geq 1$, then, assuming that the sequence $(\alpha_k)_{k\geq 1}$ is non-increasing, condition (b) still holds true. This may be seen with the help of an Abel transformation in the left–hand side of (b).

It is likely that formula (1.4) can be proved for many sequences $(q_k)_{k\geq 1}$ that do not satisfy (1.3).

Theorem 1.1.2 happens to be a special case of Theorem 1.1.3 when $G_k$ ($k \geq 1$) is taken as the group of $d^{\text{th}}$ powers in a reduced system of residues modulo $q_k$. Nevertheless, it turns out to be more convenient to prove first Theorem 1.1.2 as it will make it possible to introduce some notation much needed in Chapter 2. This chapter is therefore organized as follows : first some lemmata of an arithmetical nature will be recalled (section 1.2). They will be needed to prove Theorem 1.1.2 in section 1.3, where the modifications to make in the proof to establish Theorem 1.1.3 will then be indicated.

## 1.2 Some auxiliary results

In this section, various results which will be needed later are collected.

### 1.2.1 Lemmata on arithmetical functions

For any integer $n \geq 2$, let $\tau(n)$ be the number of divisors of $n$ and let $\omega(n)$ be the number of *distinct* prime factors dividing $n$. If

$$n = \prod_{i=1}^{r} \pi_i^{\alpha_i}$$

is the prime factor decomposition of the integer $n$, recall that

$$\omega(n) = r \quad \text{and} \quad \tau(n) = \prod_{i=1}^{r} (\alpha_i + 1)\,.$$

As is well–known, the average value of $\omega(n)$ is asymptotic to $\log\log n$ when $n$ tends to infinity (cf. [108, §22.11]). However, a stronger statement similar to Lemma 2.3.1 will be needed in the proofs to come. To this end, the definition of the maximal order of an arithmetical function is introduced.

**Definition 1.2.1.** *An arithmetical function $f$ has maximal (resp. minimal) order $g$ if $g$ is a positive nondecreasing arithmetical function such that*

$$\limsup_{n\to\infty} \frac{f(n)}{g(n)} = 1 \quad \left(\textit{resp.} \liminf_{n\to\infty} \frac{f(n)}{g(n)} = 0\right).$$

For instance, it is not difficult to see that the identity function is both a minimal and a maximal order for Euler's totient function.

**Lemma 1.2.2.** *(1) A maximal order for $\omega(n)$ is $\log n/\log\log n$. In particular, for any $\epsilon > 0$ and any positive integer $m$,*

$$\omega(n) = o(\log n) \quad \textit{and} \quad m^{\omega(n)} = o(n^{\epsilon})\,.$$

*(2) For any $\epsilon > 0$, $\tau(n) = o(n^{\epsilon})$.*

*Proof.* Regarding (1), the first claim is implicit in [108, p.355] and the others follow easily from this one. As for (2), see, e.g., [108, §22.13]. **Q.E.D.**

If $n \geq 2$ and $d \geq 1$ are integers, recall that $r_d(n)$ denotes the number of distinct $d^{\text{th}}$ powers in the reduced system of residues modulo $n$ and let $u_d(n)$ be the number of $d^{\text{th}}$ roots of unity modulo $n$; that is,

$$r_d(n) = \text{Card}\left\{m^d \pmod{n} \;:\; m \in (\mathbb{Z}/n\mathbb{Z})^{\times}\right\},$$
$$u_d(n) = \text{Card}\left\{m \in \mathbb{Z}/n\mathbb{Z} \;:\; m^d \equiv 1 \pmod{n}\right\}.$$

Set furthermore $r_d(1) = u_d(1) = 1$.

**Remark 1.2.3.** Let $u(f,n)$ be the number of solutions in $x$ to the congruence equation

$$f(x) := \sum_{k=0}^{d} a_k x^k \equiv 0 \pmod{n}\,,$$

where $f \in \mathbb{Z}[X]$ is a given polynomial of degree $d$. It is well–known that, as a consequence of the Chinese Remainder Theorem, $u(f, n)$ is a multiplicative function of $n$. It follows that $u_d(n)$ is multiplicative with respect to $n$ for any fixed $d$.

The following proposition gives explicit formulae for $r_d(n)$ and $u_d(n)$.

**Proposition 1.2.4.** *The arithmetical functions $r_d(n)$ and $u_d(n)$ are multiplicative when $d$ is fixed. Furthermore, if $n = \pi^k$, where $\pi \in \mathcal{P}$ and $k \geq 1$ is an integer, then the following equations hold :*

$$r_d(n) = \frac{\varphi(\pi^k)}{u_d(\pi^k)} \quad \textit{and} \quad u_d(n) = \begin{cases} \gcd(2d, \varphi(n)) & \textit{if } 2|d,\ \pi = 2 \textit{ and } k \geq 3, \\ \gcd(d, \varphi(n)) & \textit{otherwise,} \end{cases}$$

*where $\varphi$ is Euler's totient function.*

*Proof.* See, e.g., [190]. **Q.E.D.**

### 1.2.2 Dirichlet characters and the Pólya–Vinogradov inequality

Let $G$ be a finite abelian group written multiplicatively and with identity $e$. A *character* $\chi$ over $G$ is a multiplicative homomorphism from $G$ into the multiplicative group of complex numbers. The image of $\chi$ is contained in the group of $|G|^{\text{th}}$ roots of unity.

It is readily seen that the set of characters over $G$ form a group, the dual group of $G$, which will be denoted by $\hat{G}$ in what follows. Its unit $\chi_0$ is the *principal* (or *trivial*) *character* which maps everything in $G$ to the unity.

The following is well–known (see [88, Chapter 7]) :

**Theorem 1.2.5.** *With this notation,*

*i) there are exactly $|G|$ characters over $G$.*

*ii) for any $g \neq e$,*

$$\sum_{\chi \in \hat{G}} \chi(g) = 0.$$

*iii) for any non-principal character $\chi$,*

$$\sum_{g \in G} \chi(g) = 0.$$

If $n > 1$ is an integer, consider the group $G = (\mathbb{Z}/n\mathbb{Z})^{\times}$. A character $\chi$ over $G$ may be extended to all integers by setting $\chi(m) = \chi(m \pmod n)$ if $\gcd(n, m) = 1$ and $\chi(m) = 0$ if $\gcd(n, m) > 1$. Such a function is said to be a *Dirichlet character to the modulus $n$* and will still be denoted by $\chi$ with a slight abuse of notation.

In what follows, an upper bound on the sum of such characters over large intervals will be needed. A fundamental improvement on the trivial estimate given by the triangle inequality is the Pólya–Vinogradov inequality (see [88, Chapter 9] for a proof).

**Theorem 1.2.6** (Pólya & Vinogradov, 1918)**.** *For any non principal Dirichlet characters $\chi$ over $(\mathbb{Z}/n\mathbb{Z})^{\times}$ $(n > 1)$ and any integer $h$, the following inequality holds :*

$$\left|\sum_{k=1}^{h}\chi(k)\right| \leq 2\sqrt{n}\log n.$$

**Remark 1.2.7.** When $\chi$ is a so–called primitive character (which is the case when $n$ is prime), the multiplicative constant 2 in the above may be replaced with 1. This refinement will not be needed.

## 1.3 Irreducible approximations and subgroups of $(\mathbb{Z}/n\mathbb{Z})^{\times}$

The first part of this section will be devoted to the proof of Theorem 1.1.2 : all the tools introduced in the previous section will be used here. In the second subsection, the modifications needed to prove Theorem 1.1.3 are given.

### 1.3.1 Proof of Theorem 1.1.2

All the new notation to be used hereafter is summarized in Figure 1.1.

| Notation | Parameters | Definition |
|---|---|---|
| $\varphi_\mu(n)$ | $n \geq 2$, $\mu > 0$ | $\text{Card}\left\{l \in [\![1, \mu n]\!] \;:\; \gcd(l, n) = 1\right\}$ |
| $G_n$ | $n \geq 2$ integer | Any subgroup of $(\mathbb{Z}/n\mathbb{Z})^{\times}$ |
| $G_n^{(d)}$ | $d \geq 1$ | Group of $d^{th}$ powers in a reduced system of residues modulo a fixed integer $n \geq 2$ |
| $aG_n$ | $a \in (\mathbb{Z}/n\mathbb{Z})^{\times}$, $n \geq 2$ | Coset of $a$ in the quotient of $(\mathbb{Z}/n\mathbb{Z})^{\times}$ by $G_n$, i.e. $aG_n = \{al \;:\; l \in G_n\}$ |
| $\Psi_X(aG_n)$ | $X > 0$ | $\text{Card}\left\{l \in [\![1, X]\!] \;:\; l \in aG_n\right\}$ |
| $d_n(G_n)$ | $n \geq 2$ | Index of $G_n$ in $(\mathbb{Z}/n\mathbb{Z})^{\times}$, i.e. $d_n(G_n) = \varphi(n)/\Psi_n(G_n)$ |

Figure 1.1: Some additional notation

The key–step to the proof of the Theorem of Duffin and Schaeffer (Theorem 1.1.1) is the study of the regularity of the distribution of the numbers less than a given positive integer and relatively prime to this integer. The following is well–known and strengthens the result stated in [83, Lemma III].

**Lemma 1.3.1.** *Let $\mu$ be a positive real number and let $n \geq 2$ be an integer. Let $\varphi_\mu(n)$ denote the number of positive integers which are equal to or less than $\mu n$ and relatively prime to $n$.*

*Then, for any $\epsilon > 0$,*

$$\varphi_\mu(n) = \varphi(n)\left(\mu + O\left(\frac{1}{n^{1-\epsilon}}\right)\right).$$

*Proof.* See, e.g., [155, Theorem 3.1]. **Q.E.D.**

Duffin and Schaeffer provide an error term of the form $O\left(n^{-1/2}\right)$ in Lemma 1.3.1, where the implied constant is absolute. In fact, even such an estimate is too accurate in the sense that their method only requires the error term to tend to zero uniformly in $\mu$. This remark will play a fundamental role in the proof of Theorems 1.1.2 and 1.1.3.

The following theorem deals with the regularity of the distribution of the elements of a given subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$ (where $n \geq 2$) and is the key–step to the generalisation of the result of Duffin and Schaeffer.

**Theorem 1.3.2.** *Let $\mu$ be a positive real number, $n \geq 2$ be an integer and $a \in (\mathbb{Z}/n\mathbb{Z})^\times$. Let $G_n$ be a subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$. Denote by $\Psi_n(G_n)$ the cardinality of $G_n$ (which is also the cardinality of $aG_n$) and by $d_n(G_n)$ the index of $G_n$ in $(\mathbb{Z}/n\mathbb{Z})^\times$; that is,*

$$d_n(G_n) = \frac{\left|(\mathbb{Z}/n\mathbb{Z})^\times\right|}{|G_n|} = \frac{\varphi(n)}{\Psi_n(G_n)}.$$

*Finally, for a real number $\mu > 0$ and an integer $n \geq 1$, let $\Psi_{\mu n}(aG_n)$ denote the number of positive integers $k$ less than or equal to $\mu n$ such that $k \in aG_n$.*

*Then, for any $\epsilon > 0$,*

$$\Psi_{\mu n}(aG_n) = \Psi_n(G_n)\left(\mu + O\left(\frac{d_n(G_n)}{n^{1/2-\epsilon}}\right)\right).$$

*Proof.* The proof makes use of the Dirichlet characters introduced in subsection 1.2.2 and some ideas which probably date back to the works of Erdős and Davenport [63] on character sums.

Let $H_n$ be the quotient group of $(\mathbb{Z}/n\mathbb{Z})^\times$ by $G_n$. Any character $\chi$ over $H_n$ may be extended to $(\mathbb{Z}/n\mathbb{Z})^\times$ by composing with the canonical homomorphism from $(\mathbb{Z}/n\mathbb{Z})^\times$ to $H_n$. Such a character will still be denoted by $\chi$. Let $\hat{G}_{H_n}$ be the set of all characters over $(\mathbb{Z}/n\mathbb{Z})^\times$ arising from a character over $H_n$ : it is readily seen that $\hat{G}_{H_n}$ is a subgroup of the group of characters over $(\mathbb{Z}/n\mathbb{Z})^\times$ of cardinality $|\hat{H}_n|$ (here, the notation of subsection 1.2.2 is kept).

Let $\alpha \in (\mathbb{Z}/n\mathbb{Z})^\times$ be the multiplicative inverse of $a \in (\mathbb{Z}/n\mathbb{Z})^\times$. By Theorem 1.2.5, $|\hat{H}_n| = d_n(G_n)$ and the same theorem implies that

$$\Psi_{\mu n}(aG_n) = \frac{1}{d_n(G_n)} \sum_{k \in [\![1,\mu n]\!]} \sum_{\chi \in \hat{G}_{H_n}} \chi(\alpha k).$$

On inverting the order of summation, two contributions from the sum may be distinguished :

- One comes from the principal character and equals $\operatorname{Card}\left([\![1,\mu n]\!] \cap (\mathbb{Z}/n\mathbb{Z})^\times\right)$. Now, from Lemma 1.3.1,

  $$\operatorname{Card}\left([\![1,\mu n]\!] \cap (\mathbb{Z}/n\mathbb{Z})^\times\right) = \varphi_\mu(n) = \varphi(n)\left(\mu + O\left(\frac{1}{n^{1-\epsilon}}\right)\right)$$

  for any $\epsilon > 0$.

- The other comes from the $(d_n(G_n) - 1)$ non–trivial characters and, by the Pólya–Vinogradov inequality (Theorem 1.2.6), each of them is bounded above in absolute value by $2\sqrt{n}\log n$.

Therefore, for any $\epsilon > 0$,

$$\Psi_{\mu n}(aG_n) = \frac{\varphi(n)}{d_n(G_n)}\left(\mu + O\left(\frac{1}{n^{1-\epsilon}}\right)\right) + \frac{d_n(G_n)-1}{d_n(G_n)}R_n(\mu),$$

where the remainder $R_n(\mu)$ satisfies $|R_n(\mu)| \leq 2\sqrt{n}\log n$. Bearing in mind that $d_n(G_n) = \varphi(n)/\Psi_n(G_n)$ and that $\varphi(n) \geq n/2^{\omega(n)}$ (this follows from the relation $\varphi(n) = n\prod_{\pi|n}\left(1-\pi^{-1}\right)$, where the product runs over primes), Lemma 1.2.2 leads to the inequality

$$\left|\frac{R_n(\mu)}{\varphi(n)}\right| \leq \frac{2\sqrt{n}2^{\omega(n)}\log n}{n} = O\left(\frac{1}{n^{1/2-\epsilon}}\right)$$

for any $\epsilon > 0$. This concludes the proof. **Q.E.D.**

The next result makes the link between Theorem 1.3.2 and Theorem 1.1.2 giving the repartition of the $d^{\text{th}}$ powers in a reduced system of residues modulo an integer. The notation of Theorem 1.3.2 is maintained.

**Corollary 1.3.3.** *Let $n \geq 2$ and $a \geq 1$ be two coprime integers. Denote by $G_n^{(d)}$ the group of $d^{th}$ power residues in a reduced system of residues modulo $n$.*

*Then, for all $\epsilon > 0$,*

$$\Psi_{\mu n}\left(aG_n^{(d)}\right) = \Psi_n\left(G_n^{(d)}\right)\left(\mu + O\left(\frac{1}{n^{1/2-\epsilon}}\right)\right),$$

*where $\Psi_n\left(G_n^{(d)}\right) = r_d(n) = \varphi(n)/u_d(n)$ as defined in Proposition 1.2.4.*

*Proof.* Keeping the notation of Theorem 1.3.2, first notice that $d_n\left(G_n^{(d)}\right) = u_d(n)$. Now, since the arithmetical function $u_d(n)$ is multiplicative (see Remark 1.2.3), Proposition 1.2.4 and Lemma 1.2.2 imply that

$$d_n\left(G_n^{(d)}\right) = u_d(n) \leq (2d)^{\omega(n)} = O\left(n^{\epsilon}\right)$$

for any $\epsilon > 0$. The result then follows from Theorem 1.3.2. **Q.E.D.**

In order to prove Theorem 1.1.2, the following notation will turn out to be convenient.

**Notation.** For any real number $x \in [0, 1/2)$ and any integer $k \geq 1$, let $E_k^x$ denote the collection of intervals of the form

$$\left(\frac{p}{q_k^d} - \frac{x}{q_k^d}, \frac{p}{q_k^d} + \frac{x}{q_k^d}\right)$$

where $0 < p < q_k^d$ is an integer relatively prime to $q_k$ satisfying $p \equiv ab^d \pmod{q_k}$ for an integer $b$ prime to $q_k$ (with the notation of Corollary 1.3.3, this amounts to claiming that $p \in [\![0, q_k^d]\!]$ and $p \in aG_{q_k}^{(d)}$). Here and in what follows, the integer $a$ is fixed and assumed to be relatively prime to $q_k$ for all $k \geq 1$.

For simplicity, set furthermore $E_k := E_k^{\alpha_k}$ for all integers $k \geq 1$.

With the notation of Corollary 1.3.3, $E_k$ is then the set in $(0,1)$ consisting of

$$\Psi_{q_k^d}\left(aG_{q_k}^{(d)}\right)=\Psi_{q_k^d}\left(G_{q_k}^{(d)}\right)=\Psi_{q_k}\left(G_{q_k}^{(d)}\right)q_k^{d-1} \tag{1.5}$$

open intervals which, up to the possible exception of two boundary elements, have length $2\alpha_k/q_k^d$. These intervals are all centered at a rational of the form $p/q_k^d$, where $p$ and $q_k$ are integers satisfying the aforementioned constraints (note that equation 1.5 follows from the fact that $\Psi_{q_k^d}\left(aG_{q_k}^{(d)}\right)$ counts the number of integers $p\in[\![0,q_k^d]\!]$ such that $p\in aG_{q_k}^{(d)}$. Since the integer $a$ is coprime with $q_k$, it should be obvious that $\Psi_{q_k^d}\left(aG_{q_k}^{(d)}\right)=\Psi_{q_k^d}\left(G_{q_k}^{(d)}\right)$).

If $(s,t]$ is some interval in $(0,1)$ (where $s<t$), an estimate of the measure of the set common to $E_k$ and the interval $(s,t]$ is needed. To this end, first note that, for any integer $n\geq 1$ and any real number $\mu>0$, $\Psi_{\mu n^d}\left(aG_{q_k}^{(d)}\right)$ counts the number of positive integers $p$ less than or equal to $\mu n^d$ such that $p\in aG_{q_k}^{(d)}$.

Let $k\geq 1$ be an integer. The number of intervals in $E_k$ whose centers lie in $(s,t]$ is exactly $\Psi_{tq_k^d}\left(aG_{q_k}^{(d)}\right)-\Psi_{sq_k^d}\left(aG_{q_k}^{(d)}\right)$. From this it follows that at least $\Psi_{tq_k^d}\left(aG_{q_k}^{(d)}\right)-\Psi_{sq_k^d}\left(aG_{q_k}^{(d)}\right)-2$ such intervals are entirely contained in $(s,t]$ and at most $\Psi_{tq_k^d}\left(aG_{q_k}^{(d)}\right)-\Psi_{sq_k^d}\left(aG_{q_k}^{(d)}\right)+2$ of them touch $(s,t]$. Thus the measure of the set common to $E_k$ and $(s,t]$ is

$$\frac{2\alpha_k}{q_k^d}\left(\Psi_{tq_k^d}\left(aG_{q_k}^{(d)}\right)-\Psi_{sq_k^d}\left(aG_{q_k}^{(d)}\right)+\theta\right), \tag{1.6}$$

where $|\theta|\leq 2$.

However, since for any $\mu>0$, $\left\lfloor\mu q_k^{d-1}\right\rfloor$ is the greatest integer $m$ satisfying $mq_k\leq\mu q_k^d$, one has

$$\Psi_{\mu q_k^d}\left(aG_{q_k}^{(d)}\right)=\left\lfloor\mu q_k^{d-1}\right\rfloor\Psi_{q_k}\left(G_{q_k}^{(d)}\right)+\mathrm{Card}\left\{p\in[\![\left\lfloor\mu q_k^{d-1}\right\rfloor q_k\,,\,\mu q_k^d]\!]\;:\;p\in aG_{q_k}^{(d)}\right\}.$$

The second term on the right–hand side of this equation is $\Psi_{\nu q_k}\left(aG_{q_k}^{(d)}\right)$, where

$$\nu:=\frac{\mu q_k^d-\left\lfloor\mu q_k^{d-1}\right\rfloor q_k}{q_k}\in[0,1).$$

Therefore, from Corollary 1.3.3,

$$\begin{aligned}\Psi_{\mu q_k^d}\left(aG_{q_k}^{(d)}\right)&=\left\lfloor\mu q_k^{d-1}\right\rfloor\Psi_{q_k}\left(G_{q_k}^{(d)}\right)+\Psi_{\nu q_k}\left(aG_{q_k}^{(d)}\right)\\&=\Psi_{q_k}\left(G_{q_k}^{(d)}\right)\left(\left\lfloor\mu q_k^{d-1}\right\rfloor+\mu q_k^{d-1}-\left\lfloor\mu q_k^{d-1}\right\rfloor+O\left(\frac{1}{q_k^{1/2-\epsilon}}\right)\right)\\&=\Psi_{q_k^d}\left(G_{q_k}^{(d)}\right)\left(\mu+O\left(\frac{1}{q_k^{d-1/2-\epsilon}}\right)\right)\end{aligned}$$

for any $\epsilon>0$.

Substituing this into (1.6) and denoting by $\lambda$ the one–dimensional Lebesgue measure, the measure of the set common to $E_k$ and $(s,t]$ is seen to be

$$\frac{2\alpha_k}{q_k^d}\Psi_{q_k^d}\left(G_{q_k}^{(d)}\right)(t-s+\eta) \;=\; \lambda\left(E_k\right)(t-s)(1+\eta),$$

where $\eta \ll \left(q_k^{d-1/2-\epsilon}(t-s)\right)^{-1}$ for any $\epsilon > 0$.

Thus the following lemma has almost been proved.

**Lemma 1.3.4.** *Let $A$ be a subset of the unit interval $(0,1)$ consisting of a finite number of intervals.*

*Then, there exists a constant $c_A > 0$ which depends only on the set $A$ such that for any integer $k \geq 1$,*

$$\lambda\left(A \cap E_k\right) \leq \lambda\left(A\right)\lambda\left(E_k\right)\left(1 + c_A\,\rho\left(q_k\right)\right),$$

*where*

$$\rho\left(q_k\right) = O\left(\frac{1}{q_k^{d-1/2-\epsilon}}\right)$$

*for any $\epsilon > 0$.*

*Proof.* The lemma has been proved in the case where $A$ is a single interval. The general case follows easily. See [83, Lemma IV]. **Q.E.D.**

All the tools necessary for the proof of Theorem 1.1.2 are now available. In fact, the proof has been reduced to that of the Theorem of Duffin and Schaeffer. The latter will be reproduced for the sake of completeness. Note that a slight variant of this proof has been provided by Sprindžuk in [183, Chapter 1].

The proof of the following lemma, even though it is not difficult, may be found in the original paper by Duffin and Schaeffer. Recall that $E_k := E_k^{\alpha_k}$ (see after the proof of Corollary 1.3.3 for this notation).

**Lemma 1.3.5.** *For any positive and distinct integers $k$ and $l$,*

$$\lambda\left(E_k \cap E_l\right) \leq 4\alpha_k\alpha_l.$$

*Proof.* Cf. [83, lemma II]. **Q.E.D.**

*Proof of Theorem 1.1.2.* Let

$$E := \bigcup_{k=1}^{\infty} E_k.$$

The notation of Corollary 1.3.3 is maintained. If it is shown that $\lambda(E) = 1$, then for almost all $x \in (0,1)$, there will exist at least one pair of integers $(p_k, q_k^q)$ such that

$$\left|\alpha - \frac{p_k}{q_k^d}\right| < \frac{\alpha_k}{q_k^d} \quad \text{with} \quad p_k \in aG_k^{(d)}. \tag{1.7}$$

Assume for a contradiction that $\lambda(E) < 1$. Let $\delta > 0$ and let

$$A := \bigcup_{j=1}^{k_1} E_k,$$

where $k_1$ is chosen large enough so that

$$\lambda(E) - \delta < \lambda(A) \leq \lambda(E). \tag{1.8}$$

Since $A$ consists of a finite union of intervals, it can be inferred from Lemma 1.3.4 that, for any integer $k > k_2$, where $k_2$ is sufficiently large depending only on $A$ and $\delta$,

$$\lambda\left(A \cap E_k\right) \leq \lambda\left(A\right) \lambda\left(E_k\right) \left(1 + \delta\right). \tag{1.9}$$

Let $n$ and $m$ be integers such that $m \geq n > k_1 + k_2$ and let

$$B := \bigcup_{j=n}^{m} E_j.$$

By the inclusion–exclusion principle,

$$\sum_{j=n}^{m} \lambda\left(E_j\right) \;\geq\; \lambda\left(B\right) \;\geq\; \sum_{j=n}^{m} \lambda\left(E_j\right) - \sum_{l=n+1}^{m} \sum_{j=n}^{l-1} \lambda\left(E_j \cap E_l\right),$$

so that from Lemma 1.3.5,

$$\lambda\left(B\right) \;\geq\; \sum_{j=n}^{m} \lambda\left(E_j\right) - 2\left(\sum_{j=n}^{m} \alpha_j\right)^2. \tag{1.10}$$

It follows from (1.9) that

$$\lambda\left(A \cap B\right) \;\leq\; \sum_{j=n}^{m} \lambda\left(A \cap E_j\right) \;\leq\; \lambda\left(A\right) \cdot \left(\sum_{j=n}^{m} \lambda\left(E_j\right)\right) \cdot \left(1 + \delta\right).$$

Since

$$\lambda\left(E\right) \geq \lambda\left(A \cup B\right) = \lambda\left(A\right) + \lambda\left(B\right) - \lambda\left(A \cap B\right),$$

inequality (1.10) implies that

$$\lambda\left(E\right) \;\geq\; \lambda\left(A\right) + \left(\sum_{j=n}^{m} \lambda\left(E_j\right)\right) \cdot \left(1 - \lambda\left(A\right)\left(1 + \delta\right)\right) - 2\left(\sum_{j=n}^{m} \alpha_j\right)^2. \tag{1.11}$$

Let $\delta$ be so small that $\lambda\left(A\right)\left(1 + \delta\right) < 1$. By assumption, there are arbitrarily large integers $n$ and $m$ such that

$$\sum_{j=n}^{m} \alpha_j > 1 \quad \text{and} \quad \sum_{j=n}^{m} \alpha_j s_d\left(q_k^d\right) > \frac{1}{2} c \sum_{j=n}^{m} \alpha_j,$$

where

$$s_d\left(q_k^d\right) = \frac{\Psi_{q_k^d}\left(G_k^{(d)}\right)}{q_k^d}$$

(see the definitions of $s_d(q)$ in (1.1), of $\Psi_{q_k}\left(G_k^{(d)}\right)$ in Corollary 1.3.3 and of $\Psi_{q_k^d}\left(G_k^{(d)}\right)$ in (1.5) for the latter equation).

By substitution in (1.11),

$$\lambda(E) \;\geq\; \lambda(A) + c\cdot(1-\lambda(A)(1+\delta))\cdot\left(\sum_{j=n}^{m}\alpha_j\right) - 2\left(\sum_{j=n}^{m}\alpha_j\right)^2. \tag{1.12}$$

The right–hand side of equation (1.12) is of the form $\lambda(A) + bt - 2t^2$ with $t = \sum_{j=n}^{m}\alpha_j$ and $b = c\cdot(1-\lambda(A)(1+\delta)) \in (0,1)$. The maximum of this second degree polynomial occurs when $t = b/4$. In order to satisfy this last condition, consider the following device : if the length of some interval appearing in the definition $E$ is decreased, the measure of $E$ will obviously not be increased. In view of this, let $z \in (0,1)$ and for any positive integer $k$, let $E_k'$ be the subset of $(0,1)$ consisting of $\Psi_{q_k^d}\left(G_k^{(d)}\right)$ open intervals, each of length $2z\alpha_k/q_k^d$ with centers at $p_k/q_k^d$, where $p_k \in aG_k^{(d)}$. Clearly, $E_k'$ is contained in $E_k$ for any $k \geq 1$. Keeping $A$ the same as before, use in place of $B$ the set

$$B_z := \bigcup_{j=n}^{m} E_j'.$$

Thus (1.12) becomes

$$\lambda(E) \;\geq\; \lambda(A) + c\cdot(1-\lambda(A)(1+\delta))\cdot\left(\sum_{j=n}^{m}z\alpha_j\right) - 2\left(\sum_{j=n}^{m}z\alpha_j\right)^2$$

with $z \in (0,1)$. Choosing $z$ in such a way that $\sum_{j=n}^{m} z\alpha_j = b/4$, one obtains

$$\lambda(E) \geq \lambda(A) + \frac{c^2}{8}\cdot(1-\lambda(A)(1+\delta))^2.$$

On letting $\delta$ tend to zero, it then follows from (1.8) that

$$\lambda(E) \;\geq\; \lambda(E) + \frac{c^2}{8}\cdot(1-\lambda(E))^2,$$

which cannot happen if $\lambda(E) < 1$. This contradiction proves that $\lambda(E) = 1$.

It has thus been shown that for almost all $x \in (0,1)$, inequality (1.7) is satisfied for at least one pair of integers $(p_k, q_k^d)$. To prove that (1.7) is true for arbitrarily many pairs of integers $(p_k, q_k^d)$, let $n \geq 1$ be an integer and let $(\alpha_k')_{k\geq 1}$ be a new sequence defined by

$$\alpha_k' = \begin{cases} 0 & \text{if } k \leq n \\ \alpha_k & \text{if } k > n. \end{cases}$$

Then the sequence $(\alpha'_k)_{k\geq 1}$ satisfies the same conditions as those imposed on $(\alpha_k)_{k\geq 1}$ in Theorem 1.1.2. Consequently, by what has just been established, for almost all $x \in (0,1)$ there is at least one pair of integers $\left(p_k, q_k^d\right)$ satisfying (1.7) with $\alpha'_k$ instead of $\alpha_k$. This amounts to claiming that (1.7) is true for some integer $k > n$. Let then $D_n \subset (0,1)$ be the set of all real numbers $x \in (0,1)$ such that (1.7) is true for at least one pair of integers $\left(p_k, q_k^d\right)$ with $k > n$. Denote by $D$ the set common to all $D_n$, i.e. $D := \cap_{n\geq 1} D_n$. It has been proved that for any $n \geq 1$, $\lambda(D_n) = 1$, hence $\lambda(D) = 1$. Now if $x \in D$, then (1.7) is true infinitely often. This concludes the proof of Theorem 1.1.2. **Q.E.D.**

### 1.3.2 Proof of Theorem 1.1.3

In the course of the proof of Theorem 1.1.2, the main step was the proof of Theorem 1.3.2 and the fact that the subgroup $G_n^{(d)}$ of $(\mathbb{Z}/n\mathbb{Z})^\times$ was sufficiently large in the sense that, for some $\epsilon > 0$,

$$\frac{d_n\left(G_n^{(d)}\right)}{n^{1/2-\epsilon}} \longrightarrow 0$$

as $n$ tends to infinity, with the notation of Corollary 1.3.3. Otherwise, no use whatsoever of any specific property of the group of $d^{\text{th}}$ powers in a reduced system of residues modulo $n$ was made. Consequently, apart from some minor modifications due to the fact that, in Theorem 1.1.2, the denominators of the rational approximants are prescribed to be $d^{\text{th}}$ powers, the same proof as the one provided for Theorem 1.1.2 establishes Theorem 1.1.3.

## 1.4 Notes for the chapter

- Condition (c) in Theorem 1.1.3 is derived from the fact that the Pólya–Vinogradov inequality (Theorem 1.2.6) gives $2\sqrt{n}\log n$ as an upper bound for the absolute value of the sum of the values taken by a non–principal Dirichlet character to the modulus $n$ and the fact that
$$\frac{2\sqrt{n}2^{\omega(n)}\log n}{n} = o\left(\frac{1}{n^{1/2-\epsilon}}\right)$$
for any $\epsilon > 0$ (see the proof of Theorem 1.3.2). Therefore, any improvement on the Pólya–Vinogradov inequality would lead to a condition weaker than (c). However, stated in this form, the exponent $1/2 - \epsilon$ for some $\epsilon > 0$ appearing in condition (c) cannot be improved if a general result is required : indeed, assuming the Riemann Hypothesis for $L$–functions (i.e. the Generalized Riemann Hypothesis), E. Bach [6] has shown that a sharper upper bound for the sum of values of a non–principal Dirichlet character to the modulus $n$ is $2\sqrt{n}\log\log n$. Up to a constant, this is best possible since Paley [158] proved in 1932 that there exist infinitely many quadratic characters $\chi_n$ such that there exists a constant $c > 0$ which satisfies for some $N \in \mathbb{N}^*$ the inequality
$$\left|\sum_{k=1}^{N} \chi_n(k)\right| > c\sqrt{n}\log\log n.$$

Here, a quadratic characters $\chi_n$ refers to a character of the form $\chi_n(k) = \left(\frac{k}{n}\right)$ for some odd integer $n$, where $\left(\frac{\cdot}{n}\right)$ is the Jacobi symbol.

- In view of Theorem 1.1.3, it is tempting to generalize the Duffin–Schaeffer conjecture in the following way :

**Conjecture 1.4.1.** *Let $(q_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers and let $(\alpha_k)_{k\geq 1}$ be a sequence of real numbers in $(0, 1/2)$. Let $(a_k)_{k\geq 1}$ be a sequence such that for all $k \geq 1$, $a_k \in (\mathbb{Z}/q_k\mathbb{Z})^{\times}$. For $k \geq 1$, denote by $G_k$ a subgroup of $(\mathbb{Z}/q_k\mathbb{Z})^{\times}$ and by $a_k G_k$ the coset of $a_k$ in the quotient of $(\mathbb{Z}/q_k\mathbb{Z})^{\times}$ by $G_k$. Assume furthermore that :*

*(1)* $$\sum_{k=1}^{\infty} \alpha_k \frac{|G_k|}{q_k} = \infty,$$

*(2)* $$\frac{\varphi(q_k)}{q_k^{1/2-\epsilon}\,|G_k|} \longrightarrow 0 \quad \text{as } k \text{ tends to infinity, for some } \epsilon > 0.$$

*Then, for almost all $x \in \mathbb{R}$, there exist infinitely many relatively prime integers $p_k$ and $q_k$ such that*

$$\left| x - \frac{p_k}{q_k} \right| < \frac{\alpha_k}{q_k} \quad \text{and} \quad p_k \in a_k G_k.$$

# Chapter 2

# Vertical shifts and very well approximable Points on polynomial Curves


**Abstract**

The Hausdorff dimension of the set of simultaneously $\tau$–well approximable points lying on a curve defined by a polynomial $P(X)+\alpha$, where $P(X) \in \mathbb{Z}[X]$ and $\alpha \in \mathbb{R}$, is studied when $\tau$ is larger than the degree of $P(X)$. This provides the first results related to the determination of the Hausdorff dimension of the set of very well approximable points lying on a curve which is not defined by a polynomial with integer coefficients.

The proofs involve the study of problems in Diophantine approximation in the case where the numerators and the denominators of the rational approximations are related by some congruential constraint.


## 2.1 Introduction

Given a manifold $\mathcal{M} \subset \mathbb{R}^2$ and a real number $\tau > 1$, denote by $\widehat{W}_\tau(\mathcal{M})$ the set of simultaneously $\tau$–well approximable points lying on $\mathcal{M}$, i.e.

$$\widehat{W}_\tau(\mathcal{M}) = \left\{ (x,y) \in \mathcal{M} \ : \ \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \text{ and } \left| y - \frac{r}{q} \right| < \frac{1}{q^\tau} \text{ i.o.} \right\}.$$

Here and in what follows, *i.o.* stands for *infinitely often*; that is, for infinitely many integers $p$, $r$ and $q$ with $q \geq 1$.

Even in the simplest case where $\mathcal{M}$ is prescribed to be a planar curve defined by an equation with integer coefficients, the actual Hausdorff dimension $\dim \widehat{W}_\tau(\mathcal{M})$ of the set $\widehat{W}_\tau(\mathcal{M})$ may exhibit very different behaviours, although the starting point of the computation of the dimension is generally the same : it is first shown that, if a pair of rationals $(p/q, r/q)$ realizes an approximation of $(x, y) \in \mathcal{M}$ at order $\tau$ as in the definition of the set $\widehat{W}_\tau(\mathcal{M})$, then for $\tau$ larger than some constant depending only on the curve, the point $(p/q, r/q)$ has to belong to $\mathcal{M}$ for $q$ large enough. The assumption that $\mathcal{M}$ is a curve defined by some equation with *integer* coefficients is here essential. The following two examples illustrate this fact and complete the discussion held in [Prolegomena, subsection 0.3.2].

Consider first, for any integer $l \geq 2$, the Fermat Curve

$$\mathcal{F}_l := \left\{(x, y) \in \mathbb{R}^2 \ : \ x^l + y^l = 1\right\}.$$

For $\tau > 1$, let $(x, y) \in W_\tau(\mathcal{F}_l)$ and let $(p/q, r/q)$ be a pair of rational numbers such that

$$x = \frac{p}{q} + \frac{\epsilon_x \theta_x}{q^\tau} \quad \text{and} \quad y = \frac{r}{q} + \frac{\epsilon_y \theta_y}{q^\tau}$$

with $\epsilon_x, \epsilon_y \in \{\pm 1\}$ and $\theta_x, \theta_y \in (0, 1)$. In particular, $p = O(q)$ and $r = O(q)$ as $q$ tends to infinity. On rearranging the equation

$$q^l = \left(p + \frac{\epsilon_x \theta_x}{q^{\tau-1}}\right)^l + \left(r + \frac{\epsilon_y \theta_y}{q^{\tau-1}}\right)^l,$$

it is readily seen that

$$\left|q^l - p^l - r^l\right| \leq \frac{C(l, x, y)}{q^{\tau - l}},$$

where $C(l, x, y)$ is a strictly positive constant which depends on $x$, $y$ and $l$, but is independent of $q$. For $\tau > l$ and $q$ large enough, this implies that

$$q^l = p^l + r^l; \tag{2.1}$$

that is, that $(p/q, r/q) \in \mathcal{F}_l$. From Fermat's Last Theorem [198], the latter equation is not solvable in positive integers as soon as $l \geq 3$. Therefore, if $(x, y) \in \widehat{W}_\tau(\mathcal{F}_l)$ $(l \geq 3)$, then $(x, y) \in \{(1, 0); (0, 1)\}$ if $l$ is odd and $(x, y) \in \{(\pm 1, 0); (0, \pm 1)\}$ if $l$ is even. This means that $\widehat{W}_\tau(\mathcal{F}_l)$ contains at most four points if $\tau > l \geq 3$.

In particular, this implies the following result :

**Theorem 2.1.1.** *For $l \geq 3$ and $\tau > l$,*

$$\dim \widehat{W}_\tau(\mathcal{F}_l) = 0.$$

**Remark 2.1.2.** If $l = 2$, equation (2.1) is soluble in infinitely many Pythagorean triples $(p, q, r)$ and the result of Theorem 2.1.1 is no longer true. Indeed, Dickinson and Dodson [69] have proved that

$$\dim \widehat{W}_\tau(\mathcal{F}_2) = \frac{1}{\tau}$$

for $\tau > 2$, which constituted the first reasonably complete non–trivial result for the Hausdorff dimension of the set $\widehat{W}_\tau(\mathcal{M})$ for a smooth manifold $\mathcal{M}$ in $\mathbb{R}^n$ when $\tau$ is larger than the Dirichlet bound $1+1/n$. From their proof, it is also clear that the result holds for any arc contained in $\mathbb{S}^1 = \mathcal{F}_2$.

Consider now the case where the manifold is an integer polynomial curve

$$\Gamma = \{(x, P(x)) \in \mathbb{R}^2 \ : \ x \in \mathbb{R}\}$$

in $\mathbb{R}^2$, where $P(X) \in \mathbb{Z}[X]$ is a polynomial of degree $d \geq 1$. Since Hausdorff dimension is unaffected under locally bi-Lipschitz transformations (see [Prolegomena, Proposition 0.1.3, p.7]), it is not difficult to see that $\widehat{W}_\tau(\Gamma)$ (where $\tau > 0$) has the same Hausdorff dimension as the set

$$W_\tau(P) := \left\{ x \in \mathbb{R} \ : \ \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \text{ and } \left| P(x) - \frac{r}{q} \right| < \frac{1}{q^\tau} \text{ i.o.} \right\}.$$

Working with an appropriate Taylor expansion of $P(X)$, Budarina, Dickinson and Levesley [50] have proved that, for $\tau > d$, the only rational points which need to be taken into account for the computation of the Hausdorff dimension of the set $W_\tau(P)$ actually lie on the polynomial curve under consideration. Their result, which gave impetus to this work, is the following :

**Theorem 2.1.3** (Budarina, Dickinson & Levesley, [50])**.** *For $\tau > \max(d, 2/d)$, the Hausdorff dimension of $W_\tau(P)$ is*

$$\dim W_\tau(P) = \frac{2}{d\tau}\cdot$$

In particular, for any $\tau > 0$, the set $W_\tau(P)$ is always of positive Hausdorff dimension and therefore contains uncountably many points.

The main result of this chapter shows that this no longer holds true in the metric sense as soon as the curve $\Gamma$ is vertically translated by a real number. More precisely, given $\alpha \in \mathbb{R}$, let $W_\tau(P_\alpha)$ denote the set of simultaneously $\tau$–approximable points lying on the polynomial curve $\Gamma_\alpha = \{(x, P(x)+\alpha) \in \mathbb{R}^2 \ : \ x \in \mathbb{R}\}$ in $\mathbb{R}^2$; that is,

$$W_\tau(P_\alpha) = \left\{ x \in \mathbb{R} \ : \ \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \text{ and } \left| P(x) + \alpha - \frac{r}{q} \right| < \frac{1}{q^\tau} \text{ i.o.} \right\}.$$

Then the main result proved hereafter reads as follows :

**Theorem 2.1.4.** *Assume $d \geq 2$. If $\tau > d+1$, then*

$$W_\tau(P_\alpha) = \emptyset$$

*for almost all $\alpha \in \mathbb{R}$.*

Here as elsewhere, *almost all* and *almost everywhere* must be understood in the sense that the set of exceptions has Lebesgue measure zero.

Theorem 2.1.4 improves on a previous result due to Dickinson in [67, Theorem 4], where the weaker lower bound $3d-1$ was proved for $\tau$. The method developed in the proof of Theorem 2.1.4 provides evidence that the bound $d+1$ is in fact optimal. Indeed, it provides an upper bound for the Hausdorff dimension of $W_\tau(P_\alpha)$ valid for almost all $\alpha \in \mathbb{R}$ and for $\tau \in (d, d+1]$ which vanishes when $\tau = d+1$.

**Theorem 2.1.5.** *Assume $d \geq 2$. If $\tau \in (d, d+1]$, then*

$$\dim W_\tau(P_\alpha) \leq \frac{d+1-\tau}{\tau} \tag{2.2}$$

*for almost all $\alpha \in \mathbb{R}$.*

The relevance of the result of Theorem 2.1.5 is also clear when compared with the following one, first proved by Vaughan and Velani in [189] (see also [24]).

**Theorem 2.1.6** (Vaughan & Velani, [189])**.** *Let $f$ be a three times continuously differentiable function defined on an interval $I$ of $\mathbb{R}$ and let $\mathcal{C}_f := \left\{(x, f(x)) \in \mathbb{R}^2 \ : \ x \in I\right\}$. Let $\tau \in [3/2, 2)$ be given. Assume that $\dim\{x \in I \ : \ f''(x) = 0\} \leq (3-\tau)/\tau$. Denote by $W_\tau(f)$ the set of simultaneously $\tau$–well approximable points in $\mathbb{R}^2$ lying on the curve $\mathcal{C}_f$. Then*

$$\dim W_\tau(f) = \frac{3-\tau}{\tau} =: s.$$

*Moreover, if $\tau \in (3/2, 2)$, then the $s$–Hausdorff measure of the set $W_\tau(f)$ is infinite.*

Now if the degree of the polynomial $P(X)$ equals $d=2$, then the upper bound for the Hausdorff dimension of $W_\tau(P_\alpha)$ given by (2.2) for almost all $\alpha \in \mathbb{R}$ and for $\tau$ lying in the interval $(2,3]$ has the same expression as the exact value of $\dim W_\tau(P_\alpha)$ provided by Theorem 2.1.6, which is valid for all $\alpha \in \mathbb{R}$ and for $\tau \in (3/2, 2)$.

Theorems 2.1.4 and 2.1.5 seem to provide the first results related to the study of the Hausdorff dimension of the set of well approximable points lying on a curve which is not defined by a polynomial with integer coefficients. Besides this fact, the method involved in the proofs is also interesting in its own right since it includes the study of problems of Diophantine approximation by rationals whose numerators and denominators are related by some congruential constraint. In this respect, some of the proofs in this chapter will strongly rely on results proved in Chapter 1.

It should also be emphasized that Theorems 2.1.4 and 2.1.5 may easily be generalized to the case of a general decreasing approximating function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ which tends to zero at infinity. To this end, denote by $W_\Psi(P_\alpha)$ the set of $\Psi$–well approximable points lying on the curve defined by the polynomial $P(X)+\alpha$ in such a way that $W_\tau(P_\alpha)$ is the set $W_\Psi(P_\alpha)$ with $\Psi(q) = q^{-\tau}$. Let $\lambda_\Psi$ be the *lower order* of $1/\Psi$, that is,

$$\lambda_\Psi := \liminf_{q \to \infty} \left(-\frac{\log \Psi(q)}{\log q}\right).$$

The lower order $\lambda_\Psi$ indicates the growth of the function $1/\Psi$ in a neighbourhood of infinity. Note that this quantity is non–negative since $\Psi$ tends to zero at infinity. With this notation at one's disposal, the generalisation of Theorems 2.1.4 and 2.1.5 may be stated as follows :

**Corollary 2.1.7.** *Assume $d \geq 2$. If $\lambda_\Psi > d+1$, then*

$$W_\Psi(P_\alpha) = \emptyset$$

*for almost all $\alpha \in \mathbb{R}$. Furthermore, if $\lambda_\Psi \in (d, d+1]$, then*

$$\dim W_\Psi(P_\alpha) \leq \frac{d+1-\lambda_\Psi}{\lambda_\Psi}$$

*for almost all $\alpha \in \mathbb{R}$.*

*Proof.* From the definition of the lower order $\lambda_\Psi$, it is readily verified that, for any $\epsilon > 0$,

$$\Psi(q) \leq q^{-\lambda_\Psi+\epsilon} \text{ for all but finitely many } q \in \mathbb{N}^*.$$

Therefore, for any $\epsilon > 0$,

$$W_\Psi(P_\alpha) \subset W_{\lambda_\Psi-\epsilon}(P_\alpha).$$

The corollary then follows easily from Theorems 2.1.4 and 2.1.5. **Q.E.D.**

The chapter is organized as follows : the problem of simultaneous Diophantine approximation under consideration is first reduced to a problem of Diophantine approximation concerning the quality of approximation of the real number $\alpha$ by rational numbers whose numerators and denominators are related by some congruential constraint (section 2.2). The auxiliary lemmata collected in section 2.3 will be needed in the course of the proofs of Theorem 2.1.4 (section 2.4) and Theorem 2.1.5 (section 2.5). Some remarks on the results and the method developed will conclude the chapter (section 2.6).

For details about Hausdorff dimension and the proof of some of its basic properties which will be used throughout, the reader is referred to [Prolegomena, subsection 0.1.2].

Since the set $W_\tau(P_\alpha)$ is invariant when the real number $\alpha$ is translated by an integer, it will be assumed throughout, without loss of generality, that $\alpha$ lies in the unit interval $[0,1]$. Once and for all, $P(X) \in \mathbb{Z}[X]$ is a fixed polynomial of degree $d \geq 2$ whose leading coefficient will be denoted by $-a_d \in \mathbb{Z}^*$ for convenience.

**Notation.** The following notation will be used throughout this chapter :

- $\lfloor x \rfloor$ (resp. $\lceil x \rceil$), $x \in \mathbb{R}$ : the integer part of $x$ (resp. the smallest integer not less than $x$).
- $(x)_+ := \max\{0, x\}$ $(x \in \mathbb{R})$.
- $f \ll g$ (resp. $f \gg g$) : notation equivalent to $f = O(g)$ (resp. $g = O(f)$).
- $[\![x, y]\!]$ $(x, y \in \mathbb{R},\ x \leq y)$ : interval of integers, i.e. $[\![x, y]\!] = \{n \in \mathbb{Z} \ :\ x \leq n \leq y\}$.

- $\lambda$ : the Lebesgue measure on the real line (or its restriction to the unit interval).
- $\mathrm{Card}(X)$ or $|X|$ : the cardinality of a finite set $X$.
- $A^{\times}$ : the set of invertible elements of a ring $A$.
- $M^* = M \setminus \{0\}$ for any monoid $M$ with identity element 0.
- $\mathcal{P}$ (resp. $\pi$, $\nu_\pi(q)$ for $q \in \mathbb{N}^*$) : the set of primes (resp. any prime, the $\pi$–adic valuation of $q$).
- $\varphi(n)$ : Euler's totient function.
- $\tau(n)$ : the number of divisors of a positive integer $n$.
- $\omega(n)$ : the number of distinct prime factors dividing an integer $n \geq 2$ ($\omega(1) = 0$).
- $\|f\|_\infty^I$ : the supremum norm of a continuous function $f$ over a bounded interval $I \subset \mathbb{R}$, i.e. $\|f\|_\infty^I = \sup_{x \in I} |f(x)|$.
- $G_d(q)$ ($d, q \geq 1$ integers) : the set of $d^{\text{th}}$ powers modulo $q$.
- $aG_d(q) := \{am \ : \ m \in G_d(q)\}$ ($d, q \geq 1$ integers, $a \in \mathbb{Z}/q\mathbb{Z}$).

## 2.2 From the simultaneous case to Diophantine approximation under a constraint

In this section, simultaneous approximation properties of a real number $x$ and of $P(x) + \alpha$ are linked to some properties of Diophantine approximation under a constraint of the real number $\alpha$, and conversely. The aforementioned constraint implies the resolution of a congruence equation involving the polynomial $P(X)$. This section is the key step to the proof of Theorems 2.1.4 and 2.1.5.

### 2.2.1 Reduction of the problem

Let $M$ be an integer and let $W_\tau^M(P_\alpha) = W_\tau(P_\alpha) \cap [M, M+1]$, i.e.

$$W_\tau^M(P_\alpha) = \left\{ x \in [M, M+1] \ : \ \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \text{ and } \left| P(x) + \alpha - \frac{r}{q} \right| < \frac{1}{q^\tau} \text{ i.o.} \right\}.$$

Plainly,

$$W_\tau(P_\alpha) = \bigcup_{M \in \mathbb{Z}} W_\tau^M(P_\alpha).$$

In order to determine the Hausdorff dimension of the set $W_\tau(P_\alpha)$, it is more convenient to first focus on the subsets $W_\tau^M(P_\alpha)$. To this end, the following two lemmata are needed. Recall that $d := \deg P$.

**Lemma 2.2.1.** *Let $\tau > 0$ and $x \in [M, M+1]$ such that there exist rational numbers $p/q$ and $r/q$ satisfying*

$$x - \frac{p}{q} = \frac{\theta_x \epsilon_x}{q^\tau} \quad \textit{and} \quad P(x) + \alpha - \frac{r}{q} = \frac{\theta_y \epsilon_y}{q^\tau},$$

*with* $\theta_x, \theta_y \in (0,1)$ *and* $\epsilon_x, \epsilon_y \in \{\pm 1\}$.

*Then*

$$\left|\alpha - \frac{rq^{d-1} - q^d P\left(\frac{p}{q}\right)}{q^d}\right| < \frac{K_M}{q^\tau},$$

*where* $K_M := 1 + \|P'\|_\infty^{[M,M+1]}$.

*Proof.* The proof is straightforward. First note that

$$\alpha + P(x) - P(x) - \frac{rq^{d-1} - q^d P\left(\frac{p}{q}\right)}{q^d} = \frac{\theta_y \epsilon_y}{q^\tau} - \left(P(x) - P\left(\frac{p}{q}\right)\right).$$

Now, from the Mean Value Theorem, there exists a point $c$ in $\left(x, \frac{p}{q}\right)$ such that

$$P(x) - P\left(\frac{p}{q}\right) = P'(c)\,\frac{\theta_x \epsilon_x}{q^\tau}\cdot$$

Therefore,

$$\alpha - \frac{rq^{d-1} - q^d P\left(\frac{p}{q}\right)}{q^d} = \frac{\theta_y \epsilon_y - P'(c)\,\theta_x \epsilon_x}{q^\tau},$$

which proves the lemma. **Q.E.D.**

The next result provides a partial converse to Lemma 2.2.1. Here again, $K_M := 1 + \|P'\|_\infty^{[M,M+1]}$.

**Lemma 2.2.2.** *Let* $b$ *and* $q \geq 1$ *be integers such that there exists an integer* $p \in [\![Mq, (M+1)q]\!]$ *satisfying*

$$\frac{b}{q^d} + P\left(\frac{p}{q}\right) \in \frac{\mathbb{Z}}{q} := \left\{\frac{a}{q} \;:\; a \in \mathbb{Z}\right\}.$$

*Assume furthermore that*

$$\alpha - \frac{b}{q^d} = \frac{\epsilon \theta K_M}{q^\tau},$$

*where* $\theta \in (0,1)$ *and* $\epsilon \in \{\pm 1\}$.

*Then there exists* $r \in \mathbb{Z}$ *such that for any* $x \in \left(\frac{p}{q} - \frac{1}{q^\tau}, \frac{p}{q} + \frac{1}{q^\tau}\right)$,

$$\left|P(x) + \alpha - \frac{r}{q}\right| < \frac{2K_M}{q^\tau}\cdot$$

*Proof.* Let $r \in \mathbb{Z}$ be such that

$$\frac{b}{q^d} + P\left(\frac{p}{q}\right) = \frac{r}{q}\cdot$$

From the triangle inequality and the Mean Value Theorem,

$$\begin{aligned}\left|P(x) + \alpha - \frac{r}{q}\right| &\leq \left|P(x) - P\left(\frac{p}{q}\right)\right| + \left|\alpha - \frac{b}{q^d}\right| \\ &\leq \frac{\|P'\|_\infty^{[M,M+1]}}{q^\tau} + \frac{\theta K_M}{q^\tau}\cdot\end{aligned}$$

Hence the lemma is established. **Q.E.D.**

For any integer $M$ and any real number $K > 0$, let

$$R_\tau^M(\alpha)[K] := \left\{ x \in [M, M+1] \ : \ \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \text{ and } \left| \alpha - \frac{b}{q^d} \right| < \frac{K}{q^\tau} \right. \\ \left. \text{with } \frac{b}{q^d} + P\left(\frac{p}{q}\right) \in \frac{\mathbb{Z}}{q} \text{ i.o.} \right\} \tag{2.3}$$

and

$$W_\tau^M(P_\alpha)[K] := \left\{ x \in [M, M+1] \ : \ \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \text{ and } \left| P(x) + \alpha - \frac{r}{q} \right| < \frac{K}{q^\tau} \text{ i.o.} \right\}.$$

For the sake of simplicity of notation, omit the square brackets in the above notation when $K = 1$.

With these definitions, Lemmata 2.2.1 and 2.2.2 amount to claiming that, for any integer $M$,

$$W_\tau^M(P_\alpha) \subset R_\tau^M(\alpha)[K_M] \subset W_\tau^M(P_\alpha)[2K_M].$$

It is readily seen that, for any $\epsilon > 0$, the above inclusions imply that

$$W_\tau^M(P_\alpha) \subset R_{\tau-\epsilon}^M(\alpha) \subset W_{\tau-2\epsilon}^M(P_\alpha).$$

Defining

$$R_\tau(\alpha) := \bigcup_{M \in \mathbb{Z}} R_\tau^M(\alpha), \tag{2.4}$$

it follows that, for any $\epsilon > 0$,

$$W_\tau(P_\alpha) \subset R_{\tau-\epsilon}(\alpha) \subset W_{\tau-2\epsilon}(P_\alpha).$$

Thus, the following proposition has been proved :

**Proposition 2.2.3.** *For any* $\tau > 0$, $\dim W_\tau(P_\alpha) \leq \lim_{\epsilon \to 0^+} \dim R_{\tau-\epsilon}(\alpha)$ .

*Furthermore, the equality* $\dim W_\tau(P_\alpha) = \dim R_\tau(\alpha)$ *holds at any point of continuity of the function* $\tau \mapsto \dim W_\tau(P_\alpha)$ .

Since the function $\tau \mapsto \dim W_\tau(P_\alpha)$ is obviously decreasing, it defines a regulated function (that is, it admits at every point both left and right limits). Now, it is well–known that the set of discontinuities of a regulated function is at most countable, from which it follows that, for almost all $\tau > 0$, $\dim W_\tau(P_\alpha) = \dim R_\tau(\alpha)$ .

In fact, much more may be expected. Defining the set $W_\tau(f)$ for any function $f$ in the same way as $W_\tau(P)$, one may indeed state this conjecture :

**Conjecture 2.2.4.** *For any smooth function* $f$ *defined over* $\mathbb{R}$*, the map* $\tau \mapsto \dim W_\tau(f)$ *is continuous.*

Obviously, the statement may be extended both to higher dimensions and by weakening the assumption on the regularity of the function $f$. Note that in the case of simultaneous approximation of independent quantities, the dimension function is known to be continuous in any case (see [168] for the specifics of this assertion). On the other hand, one cannot ask the function $\tau \mapsto \dim W_\tau(f)$ to be differentiable for any positive value of $\tau$ in the general case as shown by the example of the circle $\mathbb{S}^1$. Indeed, combining the bidimensional version of Dirichlet's Theorem in Diophantine approximation, Remark 2.1.2 and Theorem 2.1.2, it is possible to give the value of $\dim W_\tau(\mathbb{S}^1)$ for any $\tau > 0$ :

$$\dim W_\tau(\mathbb{S}^1) = \begin{cases} 1 & \text{if } 0 \leq \tau \leq 3/2 \\ (3-\tau)/\tau & \text{if } 3/2 < \tau \leq 2 \\ 1/\tau & \text{if } \tau > 2. \end{cases}$$

From Remark 2.1.2, this also holds true for any arc contained in $\mathbb{S}^1$. Thus, the function $\tau \mapsto \dim W_\tau(\mathbb{S}^1)$ is piecewise differentiable as a continuous piecewise rational function. It may be expected, as a generalisation of Conjecture 2.2.4, that this behaviour holds true for any function $\tau \mapsto \dim W_\tau(f)$ provided that $f$ is "regular enough".

In what follows, Theorems 2.1.4 and 2.1.5 will be proved for the set $R_\tau(\alpha)$. Since the bounds provided by these theorems are continuous in $\tau$, it suffices to study $\dim R_\tau(\alpha)$ rather than dealing with $\dim R_{\tau-\epsilon}(\alpha)$ before letting $\epsilon$ tend to zero.

### 2.2.2 The congruential constraint

The condition

$$\frac{b}{q^d} + P\left(\frac{p}{q}\right) \in \frac{\mathbb{Z}}{q},$$

with $b$ and $p$ integers and $q$ a positive integer appears in the definition of the set $R_\tau(\alpha)$. Plainly, it amounts to the congruence equation

$$b \equiv -q^d P\left(\frac{p}{q}\right) \pmod{q^{d-1}} \tag{2.5}$$

having a solution. Since the reduction modulo $q$ of (2.5) is

$$b \equiv a_d p^d \pmod{q}$$

(recall that the leading coefficient of $P(X)$ is $-a_d$), it should be obvious that

$$\tilde{R}_\tau(\alpha) \subset R_\tau(\alpha) \subset R^*_\tau(\alpha) \tag{2.6}$$

for any $\tau > 0$, where

$$R^*_\tau(\alpha) := \left\{ x \in \mathbb{R} : \left|x - \frac{p}{q}\right| < \frac{1}{q^\tau} \text{ and } \left|\alpha - \frac{b}{q^d}\right| < \frac{1}{q^\tau} \text{ with } b \equiv a_d p^d \pmod{q} \text{ i.o.} \right\} \tag{2.7}$$

and where the set $\tilde{R}_\tau(\alpha)$ is defined in the same way as $R_\tau(\alpha)$ in (2.3) and (2.4) with the additional constraint $\gcd(q, p d a_d) = 1$ on the rational approximants.

In fact, the upper bound in Theorem 2.1.5 will be established in section 2.5 for the set $R^*_\tau(\alpha)$ whereas Theorem 2.1.4 will follow in an obvious way from the proof in subsection 2.4.1 that the set

$$I^*_\tau(P) := \left\{ \alpha \in (0,1) \ : \ \left| \alpha - \frac{b}{q^d} \right| < \frac{1}{q^\tau} \quad \text{with } b \in a_d G_d(q) \text{ i.o.} \right\} \tag{2.8}$$

has zero Lebesgue measure when $\tau > d+1$ (recall that $G_d(q)$ denotes the set of $d^{\text{th}}$ powers modulo $q$). Furthermore, the bound $d+1$ given by Theorem 2.1.4 cannot be trivially improved if it is shown that $I^*_\tau(P)$ contains a subset which is not of Lebesgue measure zero when $\tau \leq d+1$.

To this end, it will be proved in subsection 2.4.2 that the subset $\tilde{I}_\tau(P) \subset I^*_\tau(P)$ has full measure whenever $\tau \leq d+1$, where

$$\tilde{I}_\tau(P) := \left\{ \alpha \in (0,1) \ : \ \left| \alpha - \frac{b}{q^d} \right| < \frac{1}{q^\tau} \ \text{ with } \ b \in a_d G^\times_d(q) \text{ and } \gcd(q, da_d) = 1 \text{ i.o.} \right\}, \tag{2.9}$$

and where $G^\times_d(q)$ denotes the set of *primitive* $d^{\text{th}}$ powers modulo $q$.

It should be noted that $\tilde{I}_\tau(P)$ is to the set $\tilde{R}_\tau(\alpha)$ as $I^*_\tau(P)$ is to the set $R^*_\tau(\alpha)$ in the following sense : assume that $b \equiv a_d\tilde{p}^d \pmod q$ for some $\tilde{p} \in \mathbb{Z}$ satisfying $\gcd(q, \tilde{p}da_d) = 1$ as in the definition (2.9) of the set $\tilde{I}_\tau(P)$. From the Chinese Remainder Theorem, solving this congruence equation modulo $q$ amounts to solving the same equation modulo $\pi^{\nu_\pi(q)}$ for any prime divisor $\pi$ of $q$. Now, under the assumption that $\gcd(q, \tilde{p}da_d) = 1$, any solution $\tilde{p}$ of $b \equiv a_d\tilde{p}^d \pmod{\pi^{\nu_\pi(q)}}$ may be lifted, thanks to Hensel's lemma, to a unique solution $p$ of the congruence equation (2.5) taken modulo $\pi^{\nu_\pi(q)(d-1)}$ ($d \geq 2$) such that $\pi$ does not divide the product $pda_d$. Therefore, using once again the Chinese Remainder Theorem, a solution in $\tilde{p}$ to $b \equiv a_d\tilde{p}^d \pmod q$ satisfying $\gcd(q, \tilde{p}da_d) = 1$ may be lifted in a unique way to a solution $p$ of equation (2.5) such that $\gcd(q, pda_d) = 1$ as in the definition of the set $\tilde{R}_\tau(\alpha)$.

## 2.3 Growths of arithmetical functions and power residues

In this section, various results which will be needed later are collected. Some are more detailed versions of lemmata seen in [Chapter 1, Section 1.2].

### 2.3.1 Comparative growths of some arithmetical functions

For $n \geq 2$ an integer, let $\tau(n)$ be the number of divisors of $n$ and let $\omega(n)$ be the number of *distinct* prime factors dividing $n$. Some results about the asymptotic behaviour of these two arithmetical function are now recalled.

**Lemma 2.3.1.** *For any $\epsilon > 0$, $\tau(n) = o(n^\epsilon)$ and the average value of $\tau(n)$ is $\log n$, i.e.*

$$\frac{1}{n}\sum_{k=1}^{n} \tau(k) \underset{n\to\infty}{\sim} \log n.$$

*Proof.* See, e.g., [108, Theorems 315 & 320]. **Q.E.D.**

**Lemma 2.3.2.** *For any $\epsilon > 0$ and any positive integer $m$,*

$$\omega(n) = o(\log n) \qquad \textit{and} \qquad m^{\omega(n)} = o(n^{\epsilon}).$$

*Proof.* See [Chap. 1, Lemma 1.2.2, p.45]. **Q.E.D.**

### 2.3.2 Counting the number of power residues in a reduced system of residues

The congruence equations appearing in subsection 2.2.2 in the definition of the sets $\tilde{I}_\tau(P)$ and $I^*_\tau(P)$ on the one hand and $\tilde{R}_\tau(\alpha)$ and $R^*_\tau(\alpha)$ on the other involve power residues modulo an integer $q \geq 1$. The cardinality of such a set is now determined.

Let $n \geq 2$ and $d \geq 2$ be integers. Denote by $r_d(n)$ (resp. by $e_d(n)$) the number of distinct $d^{\text{th}}$ powers in the system of residues modulo $n$ (resp. in the *reduced* system of residues modulo $n$) and by $u_d(n)$ the number of $d^{\text{th}}$ roots of unity modulo $n$; that is,

$$\begin{aligned} r_d(n) &= \text{Card}\left\{m^d \pmod n \ : \ m \in \mathbb{Z}/n\mathbb{Z}\right\}, \\ e_d(n) &= \text{Card}\left\{m^d \pmod n \ : \ m \in (\mathbb{Z}/n\mathbb{Z})^{\times}\right\}, \\ u_d(n) &= \text{Card}\left\{m \in \mathbb{Z}/n\mathbb{Z} \ : \ m^d \equiv 1 \pmod n\right\}. \end{aligned}$$

Set furthermore $r_d(1) = e_d(1) = u_d(1) = 1$.

The following remark, which was already mentioned in Chapter 1, will be used several times in this chapter. It is therefore reproduced hereafter.

**Remark 2.3.3.** If $u(f, n)$ denotes the number of solutions in $x$ of the congruence

$$f(x) := \sum_{k=0}^{d} a_k x^k \equiv 0 \pmod n$$

for a given polynomial $f \in \mathbb{Z}[X]$ of degree $d \geq 1$, it is well–known that, as a consequence of the Chinese Remainder Theorem, $u(f, n)$ is a multiplicative function of $n$. It follows that $u_d(n)$ is multiplicative with respect to $n$ for any fixed $d$.

In fact, the same holds true for $r_d(n)$ and $e_d(n)$ :

**Lemma 2.3.4.** *For any fixed $d$, the functions $r_d(n)$, $e_d(n)$ and $u_d(n)$ are multiplicative with respect to $n$.*

*Proof.* See [144, Lemma 1] for the case of the functions $r_d(n)$ and $e_d(n)$. **Q.E.D.**

Explicit formulae may be given for $r_d(n)$, $e_d(n)$ and $u_d(n)$. Since these arithmetical functions are multiplicative when $d$ is fixed, it suffices to give such formulae in the case where $n$ is a power of a prime.

**Proposition 2.3.5.** *Let $n = \pi^k$ be a power of a prime number ($\pi \in \mathcal{P}$, $k \geq 1$ integer). Then, the following equations hold :*

$$e_d(n) = \frac{\varphi(\pi^k)}{u_d(\pi^k)} \quad \text{and} \quad r_d(n) = \frac{\varphi(\pi^k)}{u_d(\pi^k)} + \frac{\varphi(\pi^{k-d})}{u_d(\pi^{k-d})} + \cdots + \frac{\varphi(\pi^{k-md})}{u_d(\pi^{k-md})} + 1,$$

*where $m$ stands for the largest integer such that $k - md \geq 1$.*

*Furthermore,*

$$u_d(n) = \begin{cases} \gcd(2d, \varphi(n)) & \text{if } 2|d, \ \pi = 2 \text{ and } k \geq 3, \\ \gcd(d, \varphi(n)) & \text{otherwise.} \end{cases}$$

*Proof.* See [144, Lemmata 2 & 3]. **Q.E.D.**

**Remark 2.3.6.** Consider a partition of all numbers in the complete system of residues modulo $\pi^k$ ($\pi \in \mathcal{P}$, $k \geq 1$ integer) into classes with regard to their divisibility by $\pi^s$ and not $\pi^{s+1}$; that is, the numbers of the form $x\pi^s$ with $\gcd(x, \pi) = 1$ belong to the class numbered $s$ (where $0 \leq s \leq k$). As is made clear from the proof of Proposition 2.3.5 in [144], the quantity $\varphi(\pi^{k-sd})/u_d(\pi^{k-sd})$ with $k - sd \geq 1$ counts the number of distinct elements modulo $\pi^k$ obtained when taking the $d^{\text{th}}$ power of the numbers in the $s^{\text{th}}$ class. If $sd \geq k$, then the $d^{\text{th}}$ power of any element in the $s^{\text{th}}$ class is equal to zero modulo $\pi^k$.

Furthermore, the proof of Proposition 2.3.5 also implies that, if $k - sd \geq 1$ and if $b \pmod{\pi^k}$ is the $d^{\text{th}}$ power of an element in the $s^{\text{th}}$ class, then the number of solutions in $x$ to the congruence equation $b \equiv x^d \pmod{\pi^k}$ is exactly $u_d(\pi^{k-sd})$.

## 2.4 The set $W_\tau(P_\alpha)$ when $\tau > d + 1$

Theorem 2.1.4 is now proved and the optimality of the lower bound $d + 1$ appearing in this theorem is also studied.

### 2.4.1 Emptiness of the set for almost all $\alpha \in \mathbb{R}$

In order to establish Theorem 2.1.4, recall that from the discussion held in subsection 2.2.1 and from the inclusions (2.6), it suffices to prove that the set $R^*_\tau(\alpha)$ as defined in (2.7) is empty in the metric sense whenever $\tau > d + 1$. This in turn follows from the fact that, as a consequence of the convergent part of the Borel–Cantelli Lemma, the set $I^*_\tau(P)$ as defined in (2.8) satisfies the same property.

To see this, first notice that, for any $N \geq 1$, a cover of $I^*_\tau(P)$ is given by $\bigcup_{q \geq N} J^*_\tau(q)$, where

$$J^*_\tau(q) := \bigcup_{\substack{0 \leq b \leq q^d - 1 \\ b \in a_d G_d(q)}} \left( \frac{b}{q^d} - \frac{1}{q^\tau}, \frac{b}{q^d} + \frac{1}{q^\tau} \right). \tag{2.10}$$

If $\tau > d$ and $q \geq 1$ is large enough, $J^*_\tau(q)$ is a union of $|a_d G_d(q)| \, q^{d-1}$ non–overlapping intervals, each of length $2q^{-\tau}$; that is,

$$\lambda\left(J_\tau^*(q)\right) = \frac{2\left|a_d G_d(q)\right| q^{d-1}}{q^\tau}, \tag{2.11}$$

where $\lambda$ denotes the Lebesgue measure on the real line. On the other hand, since the ring $a_d\mathbb{Z}/q\mathbb{Z}$ is isomorphic to $\mathbb{Z}/\tilde{q}\mathbb{Z}$, where $\tilde{q} = q/\gcd(q, a_d)$, the following relationships hold true :

$$r_d\left(\tilde{q}\right) := \left|G_d\left(\tilde{q}\right)\right| = \left|a_d G_d(q)\right| \leq \left|G_d(q)\right| =: r_d(q). \tag{2.12}$$

In order to study the convergence of the series $\sum_{q\geq 1} \lambda\left(J_\tau^*(q)\right)$, an upper bound (resp. a lower bound) for $r_d(q)$ (resp. for $r_d(\tilde{q})$) is established. Regarding the upper bound for $r_d(q)$, Lemma 2.3.4 and Proposition 2.3.5 imply that

$$r_d(q) = \prod_{\substack{\pi\in\mathcal{P}\\ \pi|q}} \left(1 + \sum_{s=0}^{m_q(\pi,d)} \frac{\varphi\left(\pi^{\nu_\pi(q)-sd}\right)}{u_d\left(\pi^{\nu_\pi(q)-sd}\right)}\right),$$

where $m_q(\pi, d) := \left\lfloor \frac{\nu_\pi(q)-1}{d} \right\rfloor$. Now, it is easily checked that, for all $s \in [\![0, m_q(\pi, d)]\!]$,

$$\frac{\varphi\left(\pi^{\nu_\pi(q)-sd}\right)}{u_d\left(\pi^{\nu_\pi(q)-sd}\right)} \leq \frac{\varphi\left(\pi^{\nu_\pi(q)}\right)}{u_d\left(\pi^{\nu_\pi(q)}\right)}$$

and hence

$$\begin{aligned} r_d(q) &\leq \prod_{\substack{\pi\in\mathcal{P}\\ \pi|q}} \left[1 + \left(1 + \frac{\nu_\pi(q)-1}{d}\right) \frac{\varphi\left(\pi^{\nu_\pi(q)}\right)}{u_d\left(\pi^{\nu_\pi(q)}\right)}\right] \\ &\leq 2^{\omega(q)} \frac{\varphi(q)}{u_d(q)} \prod_{\substack{\pi\in\mathcal{P}\\ \pi|q}} \left(1 + \nu_\pi(q)\right) \;=\; 2^{\omega(q)} \frac{\varphi(q)}{u_d(q)} \tau(q) \\ &\leq 2^{\omega(q)} \tau(q) q. \end{aligned} \tag{2.13}$$

As for the lower bound for $r_d(\tilde{q})$, first notice that Lemma 2.3.4 and Proposition 2.3.5 lead to the estimate

$$1 \leq u_d(q) \leq (2d)^{\omega(q)} \tag{2.14}$$

valid for all $q \geq 1$. It may then be inferred from that that

$$\begin{aligned} r_d\left(\tilde{q}\right) \geq e_d\left(\tilde{q}\right) = \frac{\varphi\left(\tilde{q}\right)}{u_d\left(\tilde{q}\right)} &= \frac{\tilde{q}}{u_d\left(\tilde{q}\right)} \prod_{\substack{\pi\in\mathcal{P}\\ \pi|\tilde{q}}} \left(1 - \frac{1}{\pi}\right) \\ &\geq \frac{\tilde{q}}{(4d)^{\omega(\tilde{q})}} \\ &\geq \frac{q}{\left|a_d\right| (4d)^{\omega(q)}}, \end{aligned} \tag{2.15}$$

the last inequality following from the definition of $\tilde{q}$.

Finally, the combination of relationships (2.11), (2.12), (2.13) and (2.15) yields the inequalities

$$\sum_{q\geq 1}\frac{1}{(4d)^{\omega(q)}q^{\tau-d}} \ll \sum_{q\geq 1}\lambda\left(J_{\tau}^{*}(q)\right) \ll \sum_{q\geq 1}\frac{2^{\omega(q)}\tau(q)}{q^{\tau-d}}\cdot \tag{2.16}$$

From Lemmata 2.3.1 and 2.3.2, the right–hand side converges for any $\tau > d+1$, hence $\lambda(I_{\tau}^{*}(P)) = 0$ for $\tau > d+1$. This bound is the best possible according to the convergent part of the Borel–Cantelli Lemma since the series $\sum_{q\geq 1}\lambda\left(J_{\tau}^{*}(q)\right)$ diverges for $\tau \leq d+1$. This is indeed implied by (2.16) together with the following general result.

**Lemma 2.4.1.** *Let $n$ be a positive integer and $z$ be a positive real number. Define for any positive real number $s$ the series*

$$L_z(s) := \sum_{\substack{q\geq 1\\ \gcd(q,n)=1}}\frac{z^{\omega(q)}}{q^s}\cdot$$

*Then the series $L_z(s)$ converges if, and only if, $s>1$.*

*Proof.* Let $\chi_n$ be the Dirichlet principal character modulo $n$; that is,

$$\chi_n(q) = \begin{cases} 1 & \text{if } \gcd(n,q)=1\\ 0 & \text{otherwise}\end{cases}$$

for all integers $q\geq 1$.

Then,

$$L_z(s) = \sum_{q\geq 1}\frac{\chi_n(q)z^{\omega(q)}}{q^s}\cdot$$

Since $\chi_n(q)z^{\omega(q)}$ is a multiplicative arithmetical function, $L_z(s)$ admits an Euler product expansion given by

$$L_z(s) = \prod_{\pi\in\mathcal{P}}\left(1+\sum_{k=1}^{\infty}\frac{\chi_n(\pi)z^{\omega(\pi)}}{\pi^{ks}}\right) = \prod_{\substack{\pi\in\mathcal{P}\\ \gcd(\pi,n)=1}}\left(1+\frac{z}{\pi^s-1}\right). \tag{2.17}$$

Since only positive quantities are considered, $L_z(s)$ converges if, and only if, the right–hand side of (2.17) converges. Taking the logarithm on both sides of the latter equation, $L_z(s)$ is seen to converge if, and only if,

$$\sum_{l\in(\mathbb{Z}/n\mathbb{Z})^{\times}}\ \sum_{\substack{\pi\in\mathcal{P}\\ \pi\equiv l \pmod n}}\frac{1}{\pi^s}$$

converges, which is the case if, and only if, for any $l\in(\mathbb{Z}/n\mathbb{Z})^{\times}$,

$$\sum_{\substack{\pi\in\mathcal{P}\\ \pi\equiv l \pmod n}}\frac{1}{\pi^s}$$

converges. By Dirichlet's Theorem on arithmetic progressions (see, e.g., [179, Chapter 6]), for any $l \in (\mathbb{Z}/n\mathbb{Z})^{\times}$,

$$\sum_{\substack{\pi \in \mathcal{P} \\ \pi \equiv l \pmod n}} \frac{1}{\pi^s} \underset{s \to 1^+}{\sim} \frac{1}{\varphi(n)} \log\left(\frac{1}{s-1}\right).$$

This completes the proof. **Q.E.D.**

### 2.4.2 Optimality of the lower bound $d+1$

The divergence of the series $\sum_{q \geq 1} \lambda\left(J_\tau^*(q)\right)$ for $\tau \leq d+1$ does not guarantee that the set $I_\tau^*(P)$ should not be of Lebesgue measure zero, in which case the bound $d+1$ appearing in the statement of Theorem 2.1.4 could be trivially improved. This problem is now tackled by showing, as mentioned in the discussion held in subsection 2.2.2, that the subset $\tilde{I}_\tau(P)$ of $I_\tau^*(P)$ as defined in (2.9) has full measure whenever $\tau \leq d+1$.

To this end, consider the extension of the classical theorem of Duffin and Schaeffer in Diophantine approximation obtained in [Chap. 1, Theorem 1.1.3]. For all integers $q \geq 1$, let

$$s_d(q) := \frac{e_d(q)}{q} \tag{2.18}$$

(see Lemma 2.3.4 and Proposition 2.3.5 for an expression of $e_d(q)$). The following particular case of Theorem 1.1.3 was also proved in Chapter 1 :

**Theorem 2.4.2.** *(Chapter 1, Theorem 1.1.2) Let $(q_k)_{k \geq 1}$ be a strictly increasing sequence of positive integers and let $(\alpha_k)_{k \geq 1}$ be a sequence of positive real numbers in the interval $(0, 1/2)$. Assume that :*

***(a)*** $\displaystyle\sum_{k=1}^{\infty} \alpha_k = \infty,$

***(b)*** $\displaystyle\sum_{k=1}^{n} \alpha_k s_d\left(q_k^d\right) > c \sum_{k=1}^{n} \alpha_k$ *for infinitely many integers $n \geq 1$ and a real number $c > 0$,*

***(c)*** $\gcd(q_k, a_d) = 1$ *for all $k \geq 1$.*

*Then, for almost all $\alpha \in \mathbb{R}$, there exist infinitely many relatively prime integers $b_k$ and $q_k$ such that*

$$\left|\alpha - \frac{b_k}{q_k^d}\right| < \frac{\alpha_k}{q_k^d} \quad \text{and} \quad b_k \in a_d G_d^{\times}(q_k),$$

*where $G_d^{\times}(q_k)$ was defined at the same time as the set $\tilde{I}_\tau(P)$ in (2.9).*

One can deduce from Theorem 2.4.2 a result stronger than the one required to prove that the set $\tilde{I}_\tau(P)$ has full Lebesgue measure when $\tau \leq d+1$ :

**Corollary 2.4.3.** *Let $s \in (0, 1]$ and let $m$ be a positive integer.*

*Then, for almost all $\alpha \in \mathbb{R}$, there exist infinitely many integers $q$ and $b$, $q \geq 1$, satisfying*

***(i)*** $\left|\alpha - \frac{b}{q^d}\right| < \frac{1}{q^{d+s}}$ ,

***(ii)*** $b \in a_d G_d^{\times}(q)$ ,

***(iii)*** $\gcd(q, da_d) = 1$ ,

***(iv)*** $\omega(q) \leq m.$

*In particular,* $\lambda(\tilde{I}_\tau(P)) = \lambda(I^*_\tau(P)) = 1$ *when* $\tau \leq d+1$.

*Proof.* Maintaining the notation of Theorem 2.4.2, choose for the sequence $(q_k)_{k\geq 1}$ the successive elements of the set $\{n \in \mathbb{N}^* \;:\; \gcd(n, da_d) = 1 \;\text{ and }\; \omega(n) \leq m\}$ ordered increasingly and let $(\alpha_k)_{k\geq 1}$ denote the sequence $(1/q_k^s)_{k\geq 1}$.

Then

$$\sum_{k\geq 1} \alpha_k \geq \sum_{\substack{\pi \in \mathcal{P} \\ \pi \nmid da_d}} \frac{1}{\pi^s}$$

and the right–hand side is a divergent series for $s \in (0, 1]$. Furthermore, from (2.14) and (2.18) on the one hand and from the choice of the sequence $(q_k)_{k\geq 1}$ on the other, for any positive integer $k$,

$$s_d(q_k^d) = \frac{\varphi\left(q_k^d\right)}{q_k^d\, u_d\left(q_k^d\right)} \geq \frac{1}{(4d)^m} > 0.$$

Together with Theorem 2.4.2, this completes the proof. **Q.E.D.**

**Remark 2.4.4.** It is not difficult to see that, for almost all $\alpha \in \mathbb{R}$, the sequence of denominators $(q_k)_{k\geq 1}$ in Corollary 2.4.3 may be chosen in such a way that (i), (ii) and (iii) hold and such that the sequence $(\omega(q_k))_{k\geq 1}$ is unbounded. Indeed, define first for any positive integer $m$ the sequence $(n_{m,k})_{k\geq 1}$ as being the sequence of the successive elements of the set

$$\{n \in \mathbb{N}^* \;:\; \gcd(n, da_d) = 1 \;\text{ and }\; \omega(n) = m\}$$

ordered increasingly. Let $(\alpha_{m,k})_{k\geq 1}$ be the sequence $(1/n_{m,k}^s)_{k\geq 1}$, where $s \in (0, 1]$, and let

$$D_m := \{\alpha \in (0, 1] \;:\; (i),\ (ii) \text{ and } (iii) \text{ hold with } \omega(q) = m \text{ i.o}\}.$$

Denote by $(\pi_i)_{i\geq 1}$ the increasing sequence of primes. Since for all $k \geq 1$,

$$s_d(n_{m,k}^d) \geq \frac{1}{(4d)^m} > 0 \;\text{ and }\; \sum_{k\geq 1} \alpha_{m,k} \geq \sum_{\substack{1\leq i_1 < \cdots < i_m \\ \pi_{i_j} \nmid 2da_d}} \frac{1}{(\pi_{i_1} \ldots \pi_{i_m})^s},$$

which is a divergent series, arguing similarly as in the proof of Corollary 2.4.3 yields the equation $\lambda(D_m) = 1$ for any $m \in \mathbb{N}^*$. Then $\lambda\left(\cap_{m\geq 1} D_m\right) = 1$ and the result follows.

## 2.5 Upper bound for the Hausdorff dimension of $W_\tau(P_\alpha)$ when $\tau$ lies in the interval $(d, d+1]$

Theorem 2.1.5 will be proved in this section after the study of the asymptotic behavior of the number of solutions to Diophantine inequalities.

### 2.5.1 Number of solutions to Diophantine inequalities

Given a sequence of intervals $(I_q)_{q\geq 1}$ inside the unit interval and a real number $\alpha$, let $\mathcal{N}(Q,\alpha)$ denote the number of integers $q \leq Q$ such that $q\alpha \in I_q \pmod 1$, that is,

$$\mathcal{N}(Q,\alpha) := \mathrm{Card}\left\{q \in [\![1, Q]\!] \ : \ q\alpha \in I_q \pmod 1\right\}. \tag{2.19}$$

The asymptotic behavior of $\mathcal{N}(Q,\alpha)$ as $Q$ tends to infinity has been studied by Sprindžuk who exploited ideas from the works of W. Schmidt and H. Rademacher on the theory of orthogonal series (see [183] for further details).

**Theorem 2.5.1.** *([183, Theorem 18]) Let $(I_q)_{q\geq 1}$ be a sequence of intervals inside the unit interval $[0,1]$ such that*

$$\sum_{q=1}^{\infty} \lambda(I_q) = \infty.$$

*For any real number $\alpha$, define $\mathcal{N}(Q,\alpha)$ as in (2.19).*

*Then, for almost all $\alpha \in \mathbb{R}$,*

$$\mathcal{N}(Q,\alpha) = \Phi(Q) + O\left(\sqrt{\Psi(Q)}\,(\log\Psi(Q))^{3/2+\kappa}\right),$$

*where*

$$\Phi(Q) := \sum_{q=1}^{Q} \lambda(I_q), \quad \Psi(Q) := \sum_{q=1}^{Q} \lambda(I_q)\,\tau(q)$$

*and $\kappa > 0$ is arbitrary.*

The notation of Theorem 2.5.1 is maintained in the next corollary.

**Corollary 2.5.2.** *Under the assumptions of Theorem 2.5.1, suppose that one of the following conditions holds :*

- ***(i)*** $\Phi(Q) \gg Q^\delta$ *for all $Q > 0$ and for some $\delta > 0$.*
- ***(ii)*** $\lambda(I_q)$ *is decreasing and $\Phi(Q) \gg (\log Q)^{1+\delta}$ for all $Q > 0$ and for some $\delta > 0$.*

*Then*

$$\mathcal{N}(Q,\alpha) \sim \sum_{q=1}^{Q} \lambda(I_q) \qquad \textit{as } Q \to \infty.$$

*Proof.* If condition (i) holds, then the result is a simple consequence of Theorem 2.5.1 and the fact that $\tau(q) \ll q^\epsilon$ for any $\epsilon > 0$ (cf. Lemma 2.3.1).

If condition (ii) holds, since $\sum_{1\leq k\leq q}\tau(k)\ll q\log q$ by Lemma 2.3.1, making an Abel transformation in the expression for $\Psi(Q)$ shows that $\Psi(Q)\ll\Phi(Q)\log Q$. The conclusion follows in this case also. **Q.E.D.**

**Remark 2.5.3.** In the statement of Theorem 2.5.1, no restrictions whatsoever are imposed on the way the intervals $I_q$ vary with $q$. Therefore, the condition $q\alpha\in I_q \pmod 1$ appearing in the definition (2.19) of $\mathcal{N}(Q,\alpha)$ may be regarded as holding for the numbers $q_k$ of an arbitrarily increasing sequence. Then Corollary 2.5.2 is still valid for such a sequence $(q_k)_{k\geq 1}$.

### 2.5.2 Proof of Theorem 2.1.5

In order to prove Theorem 2.1.5, recall that it suffices to establish the upper bound for the Hausdorff dimension of $W_\tau(P_\alpha)$ in the case of the set $R^*_\tau(\alpha)$ as defined in (2.7). Without loss of generality, it may be assumed that $\tau\in(d,d+1)$, the result in the case $\tau=d+1$ following from an obvious passage to the limit. Furthermore, since the set $R^*_\tau(\alpha)$ is invariant when translated by an integer, it suffices to prove Theorem 2.1.5 for the subset $R^*_\tau(\alpha)\cap[0,1]$ which, for the sake of simplicity, shall still be denoted by $R^*_\tau(\alpha)$ in what follows.

The fact that the fractions $p/q$ are not necessarily irreducible in the definition of the set $R^*_\tau(\alpha)$ induces considerable difficulties as one needs to take into account the order of magnitude of the highest common factor between $p$ and $q$ to determine $\dim R^*_\tau(\alpha)$. It is in fact more convenient to work with $\gcd(b,q)$. To this end, define for $\epsilon\in[0,1]$ and $\delta>0$ the set $R^*_\tau(\alpha,\epsilon,\delta)$ as

$$\Big\{x\in[0,1]\,:\,\left|x-\frac{p}{q}\right|<\frac{1}{q^\tau}\text{ and }\left|\alpha-\frac{b}{q^d}\right|<\frac{1}{q^\tau}\text{ i.o.}\\ \text{with } b\equiv a_dp^d \pmod q \text{ and } q^\epsilon\leq\gcd(b,q)<q^{\epsilon+\delta}\Big\}. \tag{2.20}$$

It should be obvious that

$$R^*_\tau(\alpha)=\bigcup_{0\leq\epsilon<\epsilon+\delta\leq 1}R^*_\tau(\alpha,\epsilon,\delta).$$

Let furthermore $I^*_\tau(\alpha,\epsilon,\delta)$ be the set

$$I^*_\tau(\alpha,\epsilon,\delta):=\Big\{\alpha\in(0,1)\,:\,\left|\alpha-\frac{b}{q^d}\right|<\frac{1}{q^\tau}\text{ i.o.}\\ \text{with } b\in a_dG_d(q)\text{ and } q^\epsilon\leq\gcd(b,q)<q^{\epsilon+\delta}\Big\}. \tag{2.21}$$

**Notation.** Given $\epsilon\in[0,1]$ and $\delta>0$, let $\mathcal{N}(Q,\alpha,\epsilon,\delta)$ denote the counting function of the set $I^*_\tau(\alpha,\epsilon,\delta)$, which can be defined more conveniently in this case as follows :

$$\mathcal{N}(Q,\alpha,\epsilon,\delta):=\mathrm{Card}\Big\{q^d\in[\![1,Q]\!]\,:\,\left|q^d\alpha-b\right|<q^{d-\tau}\text{ i.o.}\\ \text{with } b\in a_dG_d(q)\text{ and } q^\epsilon\leq\gcd(b,q)<q^{\epsilon+\delta}\Big\}. \tag{2.22}$$

With these definitions and this notation at one's disposal, one may now state this lemma :

**Lemma 2.5.4.** *Assume that $\tau \in (d, d+1)$. Then the set $R^*_\tau(\alpha, \epsilon, \delta)$ is empty for almost all $\alpha \in [0,1]$ if $\epsilon > d+1-\tau$.*

*Furthermore, if $0 \le \epsilon < \epsilon + \delta < d+1-\tau$, then, for almost all $\alpha \in [0,1]$,*

$$Q^{d+1-\tau-\epsilon-\delta-\mu} \ll \mathcal{N}(Q, \alpha, \epsilon, \delta) \ll Q^{d+1-\tau-\epsilon+\nu},$$

*where $\mu, \nu > 0$ are arbitrarily small.*

*Proof.* To prove the first part of the statement, it suffices to show that the set $I^*_\tau(\alpha, \epsilon, \delta)$ is empty in the metric sense as soon as $\epsilon > d+1-\tau$. With this goal in mind, define

$$B_P(q, \epsilon, \delta) := \left\{ b \pmod q \ : \ b \in a_d G_d(q) \ \text{ and } \ q^\epsilon \le \gcd(b,q) < q^{\epsilon+\delta} \right\}$$

and

$$J^*_\tau(q, \epsilon, \delta) := \bigcup_{\substack{0 \le b \le q^d - 1 \\ b \in B_P(q,\epsilon,\delta)}} \left( \frac{b}{q^d} - \frac{1}{q^\tau}, \frac{b}{q^d} + \frac{1}{q^\tau} \right), \tag{2.23}$$

in such a way that $\bigcup_{q \ge N} J^*_\tau(q, \epsilon, \delta)$ is a cover for $I^*_\tau(\alpha, \epsilon, \delta)$ for any $N \ge 1$.

Since

$$|B_P(q, \epsilon, \delta)| = \sum_{\substack{a \mid q \\ q^\epsilon \le a < q^{\epsilon+\delta}}} \operatorname{Card}\{ b \pmod q \ : \ b \in a_d G_d(q) \ \text{ and } \ \gcd(b,q) = a \},$$

it should be clear that

$$|B_P(q, \epsilon, \delta)| = \sum_{\substack{a \mid q \\ q^\epsilon \le a < q^{\epsilon+\delta}}} \operatorname{Card}\{ b \pmod q \ : \ b \in G_d(q) \text{ and } \gcd(b,q) = a \}$$

if $\gcd(a_d, q) = 1$ and that

$$|B_P(q, \epsilon, \delta)| \le \sum_{\substack{a \mid q \\ q^\epsilon \le a < q^{\epsilon+\delta}}} \operatorname{Card}\{ b \pmod q \ : \ b \in G_d(q) \text{ and } \gcd(b,q) = a \}$$

if $\gcd(a_d, q) > 1$.

Now, if $a$ divides $q$, the ring $a\mathbb{Z}/q\mathbb{Z}$ is isomorphic to $\mathbb{Z}/\tilde{q}\mathbb{Z}$, where $\tilde{q} = q/a$. Therefore, for such an integer $a$,

$$\begin{aligned} \operatorname{Card}\{ b \pmod q \ : \ b \in G_d(q) \text{ and } \gcd(b,q) = a \} \ &= \ \operatorname{Card}\left\{ b \ \left(\operatorname{mod} \frac{q}{a}\right) \ : \ b \in G_d\left(\frac{q}{a}\right) \right\} \\ &:= \ r_d\left(\frac{q}{a}\right) \end{aligned}$$

from the definition of $r_d(n)$ in subsection 2.3.2. Therefore,

$$|B_P(q, \epsilon, \delta)| = \sum_{\substack{a \mid q \\ q^\epsilon \le a < q^{\epsilon+\delta}}} r_d\left(\frac{q}{a}\right) = \sum_{\substack{l \mid q \\ q^{1-\epsilon-\delta} < l \le q^{1-\epsilon}}} r_d(l) \tag{2.24}$$

if $\gcd(a_d, q) = 1$ and

$$|B_P(q, \epsilon, \delta)| \leq \sum_{\substack{a|q \\ q^{\epsilon} \leq a < q^{\epsilon+\delta}}} r_d\left(\frac{q}{a}\right) = \sum_{\substack{l|q \\ q^{1-\epsilon-\delta} < l \leq q^{1-\epsilon}}} r_d(l)$$

if $\gcd(a_d, q) > 1$.

From (2.13) and (2.15), it is readily checked that

$$\frac{q^{1-\epsilon-\delta}}{(4d)^{\omega(q)}} \leq \sum_{\substack{l|q \\ q^{1-\epsilon-\delta} < l \leq q^{1-\epsilon}}} r_d(l) \leq 2^{\omega(q)} \tau(q)^2 q^{1-\epsilon}. \tag{2.25}$$

Thus, combining (2.23), (2.24) and (2.25), it follows that, if $\gcd(a_d, q) = 1$,

$$\frac{2q^{d-\epsilon-\delta}}{(4d)^{\omega(q)} q^{\tau}} \leq \lambda\left(J_\tau^*(q, \epsilon, \delta)\right) = \frac{2\,|B_P(q, \epsilon, \delta)|\, q^{d-1}}{q^{\tau}} \leq \frac{2 \cdot 2^{\omega(q)} \tau(q)^2 q^{d-\epsilon}}{q^{\tau}}\cdot \tag{2.26}$$

On the one hand, Lemmata 2.3.1 and 2.3.2 imply that the right–hand side of (2.26) is the general term of a series which converges whenever $\epsilon > 1 + d - \tau$; hence, from the convergent part of the Borel–Cantelli Lemma, $\lambda\left(I_\tau^*(q, \epsilon, \delta)\right) = 0$ as soon as $\epsilon > 1 + d - \tau$.

On the other hand, Lemmata 2.3.1 and 2.3.2 and inequalities (2.26) also imply that, for any $\mu, \nu > 0$,

$$\begin{aligned} Q^{1+d-\tau-\epsilon-\delta-\mu} &\ll \sum_{\substack{1 \leq q \leq Q \\ \gcd(a_d, q) = 1}} \frac{1}{q^{\tau-d+\epsilon+\delta+\mu}} \ll \sum_{1 \leq q \leq Q} \lambda\left(J_\tau^*(q, \epsilon, \delta)\right) \\ &\ll \sum_{1 \leq q \leq Q} \frac{1}{q^{\tau-d+\epsilon-\nu}} \ll Q^{1+d-\tau-\epsilon+\nu}. \end{aligned}$$

To conclude the proof, it suffices to notice that, if $\mu$ is chosen so small that $1+d-\tau-\epsilon-\delta-\mu > 0$, then, from Corollary 2.5.2,

$$\mathcal{N}(Q, \alpha, \epsilon, \delta) \sim \sum_{1 \leq q \leq Q} \lambda\left(J_\tau^*(q, \epsilon, \delta)\right) \quad \text{as } Q \to \infty$$

almost everywhere. **Q.E.D.**

**Corollary 2.5.5.** *Let $\tau \in (d, d+1)$. Assume that $\epsilon$ and $\delta$ are such that $0 \leq \epsilon < \epsilon+\delta < 1+d-\tau$. Then, for almost all $\alpha \in [0, 1]$,*

$$\dim R_\tau^*(\alpha, \epsilon, \delta) \leq \frac{1 + d - \tau + \delta}{\tau}\cdot$$

*Proof.* By the definition of the set $R_\tau^*(\alpha, \epsilon, \delta)$ in (2.20), its $s$–dimensional Hausdorff measure $\mathcal{H}^s\left(R_\tau^*(\alpha, \epsilon, \delta)\right)$ satisfies the inequality

$$\mathcal{H}^s\left(R_\tau^*(\alpha, \epsilon, \delta)\right) \leq \sum_{q \geq 1} \sum_{b} \sum_{\substack{0 \leq p \leq q-1 \\ b \equiv a_d p^d \pmod{q}}} \frac{2}{q^{\tau s}}, \tag{2.27}$$

where the second sum runs over all the possible integers $b$ such that

$$\left|\alpha - \frac{b}{q^d}\right| < \frac{1}{q^\tau} \quad \text{and} \quad q^\epsilon \leq \gcd(b, q) < q^{\epsilon+\delta}. \tag{2.28}$$

Note that, provided that $q \geq 1$ is large enough and that $\tau > d$, there exists at most one integer $b$ which is a solution to (2.28). So let $(q_n)_{n\geq 1}$ denote the strictly increasing sequence of denominators $q_n$ such that (2.28) is satisfied for some integer $b_n$. From the definition of this sequence, (2.27) may be rewritten as

$$\mathcal{H}^s\left(R_\tau^*(\alpha, \epsilon, \delta)\right) \ll \sum_{n\geq 1} \frac{c_n}{q_n^{\tau s}}, \tag{2.29}$$

where $c_n := \text{Card}\left\{p \pmod{q_n} \ : \ b_n \equiv a_d p^d \pmod{q_n}\right\}.$

In order to determine the value of $c_n$, first note that, from Remark 2.3.3, $c_n$ is multiplicative in $q_n$ (i.e. $c_{nm} = c_n c_m$ whenever $\gcd(q_n, q_m) = 1$). Consider now the equation $b_n \equiv a_d p^d \left(\text{mod } \pi^{\nu_\pi(q_n)}\right)$, where $\pi$ is any prime divisor of $q_n$ :

- If $b_n \equiv 0 \left(\text{mod } \pi^{\nu_\pi(q_n)}\right)$, then the equation $a_d p^d \equiv 0 \left(\text{mod } \pi^{\nu_\pi(q_n)}\right)$ amounts to the following one : $d\nu_\pi(p) + \nu_\pi(a_d) \geq \nu_\pi(q_n)$. It is readily checked that the number of solutions in $p \pmod{q_n}$ to this equation is

$$\pi^{\nu_\pi(q_n)} - \pi^{k_d(n,\pi)}, \quad \text{where} \quad k_d(n, \pi) := \left\lceil \frac{(\nu_\pi(q_n) - \nu_\pi(a_d))_+}{d} \right\rceil.$$

- If $b_n \not\equiv 0 \left(\text{mod } \pi^{\nu_\pi(q_n)}\right)$, then the equation $b_n \equiv a_d p^d \left(\text{mod } \pi^{\nu_\pi(q_n)}\right)$ amounts to

$$p^d \equiv \frac{b_n}{\pi^{\nu_\pi(a_d)}} \left(\frac{a_d}{\pi^{\nu_\pi(a_d)}}\right)^{-1} \left(\text{mod } \pi^{\nu_\pi(q_n) - \nu_\pi(a_d)}\right),$$

  where the division by $\pi^{\nu_\pi(a_d)}$ denotes ordinary integer division while multiplicative inversion is performed in $\mathbb{Z}/\left(\pi^{\nu_\pi(q_n)-\nu_\pi(a_d)}\right)\mathbb{Z}$. Using the terminology introduced in Remark 2.3.6, the class of any solution $p \left(\text{mod } \pi^{\nu_\pi(q_n)-\nu_\pi(a_d)}\right)$ to this equation has to be $(\nu_\pi(b_n) - \nu_\pi(a_d))/d$. Therefore, from Remark 2.3.6, the number of solutions in $p \left(\text{mod } \pi^{\nu_\pi(q_n)-\nu_\pi(a_d)}\right)$ to this equation is

$$u_d\left(\pi^{\nu_\pi(q_n) - \nu_\pi(a_d) - d\frac{\nu_\pi(b_n)-\nu_\pi(a_d)}{d}}\right) = u_d\left(\pi^{\nu_\pi(q_n)-\nu_\pi(b_n)}\right) \leq u_d\left(\pi^{\nu_\pi(q_n)}\right)$$

  (see Proposition 2.3.5 for this last inequality).

All things considered,

$$c_n \leq \prod_{\substack{\pi | q_n \\ b_n \equiv 0 \left(\text{mod } \pi^{\nu_\pi(q_n)}\right)}} \pi^{\nu_\pi(q_n)} \prod_{\substack{\pi | q_n \\ b_n \not\equiv 0 \left(\text{mod } \pi^{\nu_\pi(q_n)}\right)}} u_d\left(\pi^{\nu_\pi(q_n)}\right) \leq \gcd(b_n, q_n)\, u_d\left(q_n\right).$$

Now, from the definition of the set $R_\tau^*(\alpha, \epsilon, \delta)$, it may be assumed that $\gcd(b_n, q_n) \leq q_n^{\epsilon+\delta}$.

Then, upon using (2.14) and Lemma 2.3.2, it is readily seen that (2.29) implies that

$$\mathcal{H}^s\left(R_\tau^*(\alpha,\epsilon,\delta)\right) \ll \sum_{n\geq 1} \frac{1}{q_n^{\tau s-\epsilon-\delta-\gamma}} \tag{2.30}$$

for arbitrarily small $\gamma > 0$. Since $\mathcal{N}\left(q_n,\alpha,\epsilon,\delta\right) = n$ by the definition of the sequence $(q_n)_{n\geq 1}$, Lemma 2.5.4 yields the estimate

$$n^{1/(d+1-\tau-\epsilon+\gamma)} \ll q_n$$

valid for almost all $\alpha \in [0,1]$ and for arbitrarily small $\gamma > 0$. Thus,

$$\mathcal{H}^s\left(R_\tau^*(\alpha,\epsilon,\delta)\right) \ll \sum_{n\geq 1} \frac{1}{n^{(\tau s-\epsilon-\delta-\gamma)/(d+1-\tau-\epsilon+\gamma)}},$$

which is a convergent series for $s \geq (d+1-\tau+\delta+2\gamma)/\tau$; that is,

$$\dim R_\tau^*(\alpha,\epsilon,\delta) \leq \frac{d+1-\tau+\delta+2\gamma}{\tau}.$$

The result follows on letting $\gamma$ tend to zero. **Q.E.D.**

The proofs of Lemma 2.5.4 and Corollary 2.5.5 rely strongly on the fact that, when $\epsilon < 1+d-\tau$, it is always possible to choose $\delta > 0$ and $\mu > 0$ so small that $1+d-\tau-\epsilon-\delta-\mu > 0$. While Lemma 2.5.4 also implies that the set $R_\tau^*(\alpha,\epsilon,\delta)$ is empty in the metric sense whenever $\epsilon > 1+d-\tau$, this leaves a gap corresponding to the case when $\epsilon = 1+d-\tau$. This limit case is now studied.

Since $R_\tau^*(\alpha,\epsilon,\delta) = \emptyset$ for almost all $\alpha \in [0,1]$ when $\epsilon > 1+d-\tau$, it should be clear that $R_\tau^*(\alpha,1+d-\tau,\delta) = R_\tau^*(\alpha,1+d-\tau,\mu)$ for any $\delta,\mu > 0$, the equality holding true in the metric sense. Denote by $R_\tau^*(\alpha,1+d-\tau)$ the common set determined by these different values of $\delta > 0$ and $\mu > 0$, i.e.

$$R_\tau^*(\alpha,1+d-\tau) := \bigcap_{\delta>0} R_\tau^*(\alpha,1+d-\tau,\delta).$$

In other words, $R_\tau^*(\alpha,1+d-\tau) = R_\tau^*(\alpha,1+d-\tau,\delta)$ for any $\delta > 0$ and for almost all $\alpha \in [0,1]$. In a similar way, let

$$I_\tau^*(\alpha,1+d-\tau) := \bigcap_{\delta>0} I_\tau^*(\alpha,1+d-\tau,\delta).$$

Thus, $I_\tau^*(\alpha,1+d-\tau)$ is to $R_\tau^*(\alpha,1+d-\tau)$ as $I_\tau^*(\alpha,\epsilon,\delta)$ is to $R_\tau^*(\alpha,\epsilon,\delta)$ when $0 \leq \epsilon < \epsilon+\delta < 1+d-\tau$, these last two sets being defined in (2.20) and (2.21).

**Notation.** The quantity $\mathcal{N}\left(Q,\alpha,1+d-\tau\right)$ will denote the counting function of the set $I_\tau^*(\alpha,1+d-\tau)$ defined in a similar way as in (2.22).

As might be expected, the asymptotic behavior of the function $\mathcal{N}\left(Q,\alpha,1+d-\tau\right)$ is different from that of $\mathcal{N}\left(Q,\alpha,\epsilon,\delta\right)$ when $0 \leq \epsilon < \epsilon+\delta < 1+d-\tau$ :

**Lemma 2.5.6.** *Assume that $\tau \in (d, d+1)$. Then for almost all $\alpha \in \mathbb{R}$,*

$$\mathcal{N}(Q, \alpha, 1+d-\tau) \ \ll\ Q^{\mu},$$

*where $\mu > 0$ is arbitrarily small.*

*Proof.* Let $\delta > 0$. Define $J^*_\tau(q, 1+d-\tau, \delta)$ as in (2.23). Then the upper bound for $\lambda\left(J^*_\tau(q, 1+d-\tau, \delta)\right)$ provided by (2.26) still holds, namely

$$\lambda\left(J^*_\tau(q, 1+d-\tau, \delta)\right) \ \le\ \frac{2 \cdot 2^{\omega(q)}\tau(q)^2}{q}\cdot$$

Therefore, since $2^{\omega(q)} = o\left(q^{\mu}\right)$ and $\tau(q) = o\left(q^{\mu}\right)$ for any $\mu > 0$ from Lemmata 2.3.1 and 2.3.2, for all $Q \ge 1$,

$$\Phi(Q) := \sum_{q=1}^{Q} \lambda\left(J^*_\tau(q, 1+d-\tau, \delta)\right) \ \ll\ \sum_{q=1}^{Q} \frac{1}{q^{1-3\mu}} \ \ll\ Q^{3\mu}$$

and, in a similar way,

$$\Psi(Q) := \sum_{q=1}^{Q} \lambda\left(J^*_\tau(q, 1+d-\tau, \delta)\right)\tau(q) \ \ll\ Q^{4\mu}.$$

Now, $\bigcup_{q \ge N} J^*_\tau(q, 1+d-\tau, \delta)$ is a cover for $I^*_\tau(\alpha, 1+d-\tau, \delta)$ for any $N \ge 1$. Since the latter set is equal to $I^*_\tau(\alpha, 1+d-\tau)$ for almost all $\alpha \in [0,1]$, it follows from Theorem 2.5.1 that, for almost all $\alpha \in [0,1]$,

$$\mathcal{N}(Q, \alpha, 1+d-\tau) \ =\ \Phi(Q) + O\left(\sqrt{\Psi(Q)}\left(\log \Psi(Q)\right)^{3/2+\kappa}\right) \ \ll\ Q^{3\mu},$$

where $\kappa > 0$ has been chosen arbitrarily. **Q.E.D.**

**Corollary 2.5.7.** *Assume that $\tau \in (d, d+1)$. Then, for almost all $\alpha \in [0,1]$,*

$$\dim R^*_\tau(\alpha, 1+d-\tau) \le \frac{1+d-\tau}{\tau}\cdot$$

*Proof.* Let $\delta > 0$. Denote by $(q_n)_{n\ge 1}$ the strictly increasing sequence of denominators $q_n$ such that (2.28) with $\epsilon = 1+d-\tau$ is satisfied for some integer $b_n$. Then inequality (2.30) still holds for the set $R^*_\tau(\alpha, 1+d-\tau, \delta)$, namely

$$\mathcal{H}^s\left(R^*_\tau(\alpha, 1+d-\tau, \delta)\right) \ll \sum_{n\ge 1} \frac{1}{q_n^{\tau s - 1 - d + \tau - \delta - \gamma}}$$

for arbitrarily small $\gamma > 0$.

Since $I^*_\tau(\alpha, 1+d-\tau, \delta) = I^*_\tau(\alpha, 1+d-\tau)$ for almost all $\alpha \in [0,1]$, the counting functions of these two sets have the same asymptotic behaviour, hence, from Lemma 2.5.6,

$$n \ =\ \mathcal{N}(q_n, \alpha, 1+d-\tau) \ \ll\ q_n^{\mu}$$

almost everywhere, with $\mu > 0$ arbitrary. Therefore, for almost all $\alpha \in [0,1]$,

$$\mathcal{H}^s\left(R_\tau^*(\alpha, 1+d-\tau)\right) \ll \sum_{n\geq 1} \frac{1}{n^{(\tau s - 1 - d + \tau - \delta - \gamma)/\mu}},$$

which is a convergent series for $s \geq (1 + d - \tau + \delta + \gamma + \mu)/\tau$; that is,

$$\dim R_\tau^*(\alpha, 1+d-\tau) \leq \frac{d + 1 - \tau + \delta + \gamma + \mu}{\tau}.$$

The result follows on letting $\gamma$, $\delta$ and $\mu$ tend to zero. **Q.E.D.**

*Completion of the proof of Theorem 2.1.5.* In order to prove that $\dim R_\tau^*(\alpha) \leq (d+1-\tau)/\tau$ for almost all $\alpha \in [0,1]$ when $\tau \in (d, d+1)$, recall first that, from Lemma 2.5.4, the equation

$$R_\tau^*(\alpha) = \left( \bigcup_{0 \leq \epsilon < \epsilon + \delta < 1 + d - \tau} R_\tau^*(\alpha, \epsilon, \delta) \right) \cup R_\tau^*(\alpha, 1 + d - \tau)$$

holds almost everywhere. Corollary 2.5.7 also implies that it suffices to prove that

$$\dim \left( \bigcup_{0 \leq \epsilon < \epsilon + \delta < 1 + d - \tau} R_\tau^*(\alpha, \epsilon, \delta) \right) \leq \frac{1 + d - \tau}{\tau}$$

for almost all $\alpha \in [0,1]$ when $\tau$ lies in the interval $(d, d+1)$.

To this end, consider a strictly increasing sequence $(\beta_p)_{p\geq 0}$ of real numbers from the interval $(0, 1 + d - \tau)$ tending to $1 + d - \tau$ as $p$ tends to infinity. It should then be obvious that

$$\bigcup_{0 \leq \epsilon < \epsilon + \delta < 1 + d - \tau} R_\tau^*(\alpha, \epsilon, \delta) = \bigcup_{p \geq 0} R_\tau^*(\alpha)[p], \quad \text{where} \quad R_\tau^*(\alpha)[p] := \bigcup_{0 \leq \epsilon < \epsilon + \delta \leq \beta_p} R_\tau^*(\alpha, \epsilon, \delta).$$

Given $p \in \mathbb{N}$, let $(\epsilon_p(k))_{0 \leq k \leq n}$ be the finite sequence subdividing the interval $[0, \beta_p]$ into $n \geq 1$ intervals of equal length $\delta_p(n)$ and satisfying the equations

$$\epsilon_p(0) = 0 \quad \text{and} \quad \epsilon_p(n) = \beta_p = \epsilon_p(n-1) + \delta_p(n).$$

Then,

$$R_\tau^*(\alpha)[p] = \bigcup_{k=0}^{n-1} R_\tau^*\left(\alpha, \epsilon_p(k), \delta_p(n)\right).$$

Thus, from Corollary 2.5.5,

$$\dim R_\tau^*(\alpha)[p] = \sup_{0 \leq k \leq n-1} \dim R_\tau^*\left(\alpha, \epsilon_p(k), \delta_p(n)\right) \leq \frac{d + 1 - \tau + \delta_p(n)}{\tau},$$

which holds for almost all $\alpha \in [0,1]$ and for any $n \geq 1$. On letting $n$ tend to infinity, $\delta_p(n)$ tends to zero and it follows that, outside a set of zero Lebesgue measure,

$$\dim R_\tau^*(\alpha)[p] \leq \frac{1 + d - \tau}{\tau}$$

when $\tau \in (d, d+1)$. Since the set $R^*_\tau(\alpha)$ is the countable union of $R^*_\tau(\alpha, 1+d-\tau)$ and of $R^*_\tau(\alpha)[p]$ $(p \in \mathbb{N})$ for almost all $\alpha \in [0,1]$, this completes the proof. **Q.E.D.**

## 2.6 Notes for the chapter

- The upper bound for the Hausdorff dimension of the set $W_\tau(P_\alpha)$ stated in Theorem 2.1.5 is easily seen to be non–optimal as soon as $\tau < d-1$ as it is superseded by the Hausdorff dimension of the set of $\tau$–well approximable numbers given by the Theorem of Jarník and Besicovitch (cf. [Prolegomena, Theorem 0.1.5, p.9]) : if $W_\tau$ denotes the latter set, then $W_\tau(P_\alpha) \subset W_\tau$ for any $\tau > 0$ and $\dim W_\tau = 2/\tau$ whenever $\tau > 2$.

  Now if $\tau \in [d-1, d]$, the study of the case $d = 3$ also tends to provide evidence that the upper bound $(1+d-\tau)/\tau$ is still not relevant in the general case. Indeed, when $d = 3$, on letting $\tau$ tend to 2 from above (resp. from below) in Theorem 2.1.5 (resp. in Theorem 2.1.6), the upper bound thus found for $\lim_{\tau \to 2^+} \dim W_\tau(P_\alpha)$ is clearly seen not to be sharp.

  More generally, the determination of the actual Hausdorff dimension of the set of $\tau$–well approximable points lying on a polynomial curve when $\tau$ is larger than 2 and less than the degree of the polynomial remains an open question (see also [50] for another mention of this problem).

- As mentioned in the introduction of this chapter, the upper bound for $\dim W_\tau(P_\alpha)$ given by Theorem 2.1.5 is more than likely the actual value for the Hausdorff dimension of $W_\tau(P_\alpha)$ for almost all $\alpha \in [0,1]$ when $\tau$ lies in the interval $(d, d+1]$. To also obtain $(d+1-\tau)/\tau$ as a lower bound for $\dim W_\tau(P_\alpha)$, it would be sufficient to prove such a result for the set $\tilde{R}_\tau(\alpha)$ as defined in (2.6). However, this would require the study of the distribution of solutions to congruence equations and a quantitative result on the uniformity of such a distribution. For arbitrary polynomials, this appears to be out of reach at the moment.

- The set of exceptions (with respect to $\alpha$) left by Theorem 2.1.4 actually contains uncountably many points. Indeed, let $\tau > d+1$ be given and let $(x, y)$ be a pair of real numbers simultaneously $\tau$–well approximable — this set is uncountable as its Hausdorff dimension is $3/\tau$ from the Theorem of Jarník and Besicovitch (cf. [Prolegomena, Theorem 0.1.5, p.9]). Then, setting $\alpha = y - P(x)$, it is readily seen that $x$ lies in $W_\tau(P_\alpha)$ since $x$ and $P(x) + \alpha$ are simultaneously $\tau$–well approximable[1].

- A natural continuation to the study undertaken in this chapter would be to answer the following question : what can be said, for $\tau > 0$ "large enough", about the Hausdorff dimension of the set

  $$W_\tau\left(P^{[\beta]}\right) := \left\{ x \in \mathbb{R} \; : \; \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \text{ and } \left| P(x+\beta) - \frac{r}{q} \right| < \frac{1}{q^\tau} \text{ i.o.} \right\},$$

  where $\beta \in \mathbb{R}$ and $P(X) \in \mathbb{Z}[X]$?

[1] This concluding remark is entirely due to Dr. Detta Dickinson.

Some preliminary considerations seem to indicate that the problem boils down to considering approximations to a linear form whose coefficients come from the Taylor expansion of $P(X)$ at $\beta$.

Since the result of Rynne mentioned in [Prolegomena, Theorem 0.3.7, p.25] rules out any likely general result regarding the Hausdorff dimension of the set of very well approximable points lying on a polynomial curve, the aim is to try to understand what a *generic* result could be. Specifically, the aim is to recover a generic behaviour for the set of approximation under consideration for almost all $\beta \in \mathbb{R}$ and thus to obtain in combination with the main theorems in this chapter a result regarding all polynomials that can be obtained by vertical and horizontal shifts from an integer polynomial.

# Chapter 3

# Liminf Sets in simultaneous Diophantine Approximation : some particular Cases


**Abstract**

Let $\mathcal{Q}$ be an infinite set of positive integers. Denote by $W_{\tau,n}(\mathcal{Q})$ the set of points in dimension $n \geq 1$ simultaneously $\tau$–approximable by infinitely many rationals with denominators in $\mathcal{Q}$. A non–trivial lower bound for the Hausdorff dimension of the liminf set $W^*_{\tau,n}(\mathcal{Q}) = W_{\tau,n}(\mathbb{N}) \backslash W_{\tau,n}(\mathcal{Q})$ is established when $n \geq 2$ and $\tau > 1 + 1/(n-1)$ in the case where the complement set of $\mathcal{Q}$ satisfies some divisibility properties. The determination of the actual value of this Hausdorff dimension as well as the one–dimensional analogue of the problem are also discussed. Furthermore, the dimensional properties of a $p$–adic version of the set $W^*_{\tau,n}(\mathcal{Q})$ are also studied.


## 3.1 Introduction

Let $n \geq 1$ be an integer and $\tau > 1$ be a real number. Given an infinite set of positive integers $\mathcal{Q}$, denote by $W_{\tau,n}(\mathcal{Q})$ the set of points in dimension $n \geq 1$ approximable at order $\tau$ by infinitely many rationals with denominators in $\mathcal{Q}$, i.e. the limsup set

$$W_{\tau,n}(\mathcal{Q}) := \left\{ \boldsymbol{x} \in \mathbb{R}^n \ : \ |\boldsymbol{x} - \boldsymbol{p}/q| < q^{-\tau} \ \text{ for i.m. } (\boldsymbol{p}, q) \in \mathbb{Z}^n \times \mathcal{Q} \right\}. \tag{3.1}$$

Here and throughout, *i.m.* stands for *infinitely many*, $|\boldsymbol{x}|$ is the usual supremum norm of a vector $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{p}/q$ is shorthand notation for the rational vector $(p_1/q, \ldots, p_n/q)$, where $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$.

Jarník [126] and Besicovitch [44] proved independently that the Hausdorff dimension $\dim W_{\tau,n}(\mathbb{N})$ of the set $W_{\tau,n}(\mathbb{N})$ is

$$\dim W_{\tau,n}(\mathbb{N}) = \frac{n+1}{\tau} \tag{3.2}$$

as soon as $\tau > 1 + 1/n$. Subsequently, Borosh and Fraenkel generalized this result in [48] to the case of any infinite subset $\mathcal{Q} \subset \mathbb{N}$ by showing that

$$\dim W_{\tau,n}(\mathcal{Q}) = \frac{n+\nu(\mathcal{Q})}{\tau} \tag{3.3}$$

when $\tau > 1 + \nu(\mathcal{Q})/n$, where $\nu(\mathcal{Q})$ is the exponent of convergence of $\mathcal{Q}$ defined by

$$\nu(\mathcal{Q}) := \inf \left\{ \nu > 0 \;:\; \sum_{q \in \mathcal{Q}} q^{-\nu} < \infty \right\} \in [0, 1]. \tag{3.4}$$

For recent developments and related results, see [113] (in particular Chapters 6 and 10), [70, 168] and [Prolegomena, section 0.1.3].

On the other hand, the corresponding liminf set

$$W^*_{\tau,n}(\mathcal{Q}) := W_{\tau,n}(\mathbb{N}) \setminus W_{\tau,n}\left(\mathbb{N} \backslash \mathcal{Q}\right) = W_{\tau,n}(\mathcal{Q}) \setminus W_{\tau,n}\left(\mathbb{N} \backslash \mathcal{Q}\right) \tag{3.5}$$

has received much less attention. Explicitly, this is the set of all those vectors $\boldsymbol{x}$ in $\mathbb{R}^n$ which admit infinitely many approximations at order $\tau$ as in (3.1) by rational vectors $(\boldsymbol{p}, q)$ whose denominators $q$ lie in $\mathcal{Q}$, but only finitely many approximations by rational vectors whose denominators do not lie in $\mathcal{Q}$. To the best of our knowledge, the actual Hausdorff dimension of the liminf set $W^*_{\tau,n}(\mathcal{Q})$ has not been studied yet. This seems to be a difficult problem which will be discussed in this chapter in the case where $\mathcal{Q}$ is a set defined by divisibility properties. Most of the results of this chapter will be complemented by the general result of Chapter 4.

## 3.2 $\mathbb{N}^* \backslash \mathcal{Q}$–free sets

In this section, $\mathcal{Q}$ will denote an infinite set of positive integers satisfying the following property : for any integer $q \in \mathbb{N}^*$,

$$(q \in Q) \;\Leftrightarrow\; (\forall v \notin Q,\; v \nmid q)\,. \tag{3.6}$$

Such a set will be referred to as an $\mathbb{N}^* \backslash \mathcal{Q}$*–free set* following the definition of a $B$–free set introduced by Erdős [90] [1]. Examples of sets satisfying this property are the square–free integers (or, more generally, the $k$–free integers with $k \geq 2$ — conventionally, 1 is considered

[1]However, unlike Erdős, the convergence of the series $\sum_{q \in \mathbb{N}^* \backslash \mathcal{Q}} q^{-1}$ is not required here. Note that a set $\mathcal{Q}$ which is $\mathbb{N}^* \backslash \mathcal{Q}$–free must contain 1.

here as a $k$–free integer for any $k \geq 2$) or the set of integers coprime to a given natural number $m \neq 1$.

The focus will be on the liminf set $W^*_{\tau,n}(\mathbb{N}^* \backslash \mathcal{Q})$ which, for the sake of simplicity, will be denoted by $\widehat{W}^*_{\tau,n}(\mathcal{Q})$ in what follows. Thus,

$$\widehat{W}^*_{\tau,n}(\mathcal{Q}) := \left\{ \boldsymbol{x} \in \mathbb{R}^n \ : \ |\boldsymbol{x} - \boldsymbol{p}/q| < q^{-\tau} \quad \text{for i.m. } q \notin \mathcal{Q} \text{ and f.m. } q \in \mathcal{Q} \right\}, \tag{3.7}$$

where *f.m.* stands for *finitely many.* For the definition of $\widehat{W}^*_{\tau,n}(\mathcal{Q})$ to make sense, it is natural to impose the further condition that $\mathcal{Q} \subsetneq \mathbb{N}^*$, which will be assumed throughout. It is indeed easily seen that such a condition ensures that the complement set $\mathbb{N}^* \backslash \mathcal{Q}$ is infinite.

### 3.2.1 A lower bound in any dimension

An elementary property of Hausdorff dimension (see [Prolegomena, Proposition 0.1.3, p.7]) leads to the relationship

$$\dim W_{\tau,n}\left(\mathbb{N}\right) = \max\left\{ \dim W_{\tau,n}(\mathcal{Q}),\, \dim \widehat{W}^*_{\tau,n}(\mathcal{Q}) \right\}. \tag{3.8}$$

It follows from (3.2) and (3.3) that $\dim \widehat{W}^*_{\tau,n}(\mathcal{Q}) = (n+1)/\tau$ as soon as $\nu\left(\mathcal{Q}\right) < 1$ and $\tau > 1 + 1/n$. In the case $\nu\left(\mathcal{Q}\right) = 1$ however, the situation is much less well understood and the determination of the actual Hausdorff dimension of the set $\widehat{W}^*_{\tau,n}(\mathcal{Q})$ for any value of $\tau > 0$ and $n \geq 1$ may involve some deep results on the distribution of $B$–free numbers which are still in the state of conjecture, as for instance those mentioned by Erdős in [90]. It is nevertheless always possible to find a non–trivial lower bound for $\dim \widehat{W}^*_{\tau,n}(\mathcal{Q})$ when $n \geq 2$.

**Theorem 3.2.1.** *Let $\mathcal{Q}$ be an infinite $\mathbb{N}^* \backslash \mathcal{Q}$–free set. Assume that $n \geq 1$. Then*

$$\dim \widehat{W}^*_{\tau,n}(\mathcal{Q}) \begin{cases} = (n+1)/\tau & \text{if } \nu\left(\mathcal{Q}\right) < 1,\ \tau > 1 + 1/n \ \text{ and } \ n \geq 1, \\ \in [n/\tau, (n+1)/\tau] & \text{if } \nu\left(\mathcal{Q}\right) = 1,\ \tau > 1 + 1/(n-1) \ \text{ and } \ n \geq 2. \end{cases}$$

**Remark 3.2.2.** If $\mathcal{Q}$ is any $\mathbb{N}^* \backslash \mathcal{Q}$–free set, define its *support* $\operatorname{Supp}\left(\mathcal{Q}\right)$ as the set of all primes dividing at least one element in $\mathcal{Q}$. It is readily seen that, if the support of $\mathcal{Q}$ is finite, then $\nu\left(\mathcal{Q}\right) = 0$.

On the other hand, if $\mathcal{S}$ is an infinite set of positive integers such that the series $\sum_{s \in \mathcal{S}} s^{-\mu}$ diverges for some $\mu > 0$, one may construct by the diagonal process a subset $\mathcal{S}'$ of $\mathcal{S}$ such that the series $\sum_{s \in \mathcal{S}'} s^{-\mu}$ diverges and the series $\sum_{s \in \mathcal{S}'} s^{-\mu-\epsilon}$ converges for any $\epsilon > 0$ (that is, a subset $\mathcal{S}'$ such that $\nu(S') = \mu$). Since the series of the reciprocals of the primes is divergent, this shows that, for any $\alpha \in (0, 1]$, there exists an $\mathbb{N}^* \backslash \mathcal{Q}$–free set $\mathcal{Q}$ such that $\nu\left(\mathcal{Q}\right) = \alpha$.

It should also be noted that the exponent of convergence of an $\mathbb{N}^* \backslash \mathcal{Q}$–free set $\mathcal{Q}$ depends only on its support. Indeed, for any such set, it should be clear that

$$\sum_{\pi \in \operatorname{Supp}(\mathcal{Q})} \frac{1}{\pi^\nu} \ \leq \ \sum_{q \in \mathcal{Q}} \frac{1}{q^\nu} \ \leq \ \sum_{n \in \mathcal{C}(\operatorname{Supp}(\mathcal{Q}))} \frac{1}{n^\nu}, \tag{3.9}$$

where $\nu > 0$ and $\mathcal{C}\left(\operatorname{Supp}\left(\mathcal{Q}\right)\right)$ stands for the set of positive integers all of whose prime factors belong to $\operatorname{Supp}\left(\mathcal{Q}\right)$. The series on the right–hand side of (3.9) admits an Euler product expansion given by

$$\sum_{n\in\mathcal{C}(\mathrm{Supp}(\mathcal{Q}))}\frac{1}{n^{\nu}}=\prod_{\pi\in\mathrm{Supp}(\mathcal{Q})}\left(1+\frac{1}{\pi^{\nu}-1}\right).$$

Taking the logarithm of this last quantity, it is readily seen that the right–hand side of (3.9) converges if, and only if, the left–hand side of (3.9) converges, hence the claim.

From Theorem 3.2.1, as soon as $\tau>1+1/(n-1)$ (where $n\geq 2$), the inequality

$$\dim\widehat{W}^{*}_{\tau,n}\left(\mathcal{Q}\right)\geq\frac{n}{\tau}$$

holds true as it does for the set $W_{\tau,n}\left(\mathcal{Q}\right)$ when $\tau>1+\nu\left(\mathcal{Q}\right)/n$ from (3.3). The result of the theorem also implies that, for any fixed $\tau>1$,

$$\dim\widehat{W}^{*}_{\tau,n}\left(\mathcal{Q}\right)\underset{n\to\infty}{\sim}\dim W_{\tau,n}\left(Q\right).$$

The proof of Theorem 3.2.1 is based on the following lemma, which states that good simultaneous rational approximations to given rationally dependent real numbers must satisfy the same rational dependence relationship as do the given numbers. This is a particular case of [Prologomena, subsection 0.3.2, Lemma 0.3.9]. However, the proof is much simpler in this case.

**Lemma 3.2.3.** *Let $n\geq 1$ be an integer and $\tau>1$ be a real number. Let $\boldsymbol{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$ be such that there exist integers $a_1,\dots,a_n$ and $b$ satisfying*

$$\sum_{i=1}^{n}a_ix_i=b.$$

*Assume furthermore that, for all $i\in[\![1,n]\!]$,*

$$\left|x_i-\frac{p_i}{q}\right|<\frac{1}{q^{\tau}},$$

*where $p_1/q,\dots,p_n/q$ are rational numbers.*

*Then, if $q$ is large enough (depending only on the integers $a_1,\dots,a_n$ and on the real number $\tau$),*

$$\sum_{i=1}^{n}a_i\frac{p_i}{q}=b.$$

*Proof.* The proof is straightforward : notice that

$$\left|qb-\sum_{i=1}^{n}a_ip_i\right|=\left|\sum_{i=1}^{n}a_i(qx_i-p_i)\right|\leq\frac{\sum_{i=1}^{n}|a_i|}{q^{\tau-1}}.$$

Thus, if $q$ is large enough, the left hand side of the above inequality is an integer with absolute value less than one and therefore vanishes. **Q.E.D.**

For any $A = (a_1, \ldots, a_n) \in \mathbb{N}^n \backslash \{0\}$ (where $n \geq 2$) and any $u, v \in \mathbb{N}^*$, denote by $\Gamma_n(A, u, v)$ the rational hyperplane

$$\Gamma_n(A,u,v) := \left\{ \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} a_i x_i = \frac{u}{v} \right\}. \tag{3.10}$$

Even if it means relabeling the axes, it may be assumed, without loss of generality, that $a_n \neq 0$.

For an infinite $\mathbb{N}^* \backslash \mathcal{Q}$–free set $\mathcal{Q}$, choose $v \in \mathbb{N}^* \backslash \mathcal{Q}$ and $u \in \mathbb{N}^*$ coprime to $v$. Then Lemma 3.2.3 implies that

$$\Gamma_n(A,u,v) \cap W_{\tau,n}(\mathbb{N}) \subset \widehat{W}^*_{\tau,n}(\mathcal{Q}) \tag{3.11}$$

as soon as $\tau > 1$. Thus to prove Theorem 3.2.1 in the case $\nu(\mathcal{Q}) = 1$, it suffices to establish that the dimension of the set $\Gamma_n(A,u,v) \cap W_{\tau,n}(\mathbb{N})$ is the expected dimension of the set of $\tau$–well approximable points in dimension $n-1$ ($n \geq 2$).

**Lemma 3.2.4.** *Let $n \geq 2$ be an integer and let $\Gamma_n(A,u,v)$ be a rational hyperplane in $\mathbb{R}^n$ as defined by (3.10). Then, for $\tau > 1 + 1/(n-1)$,*

$$\dim\left(\Gamma_n(A,u,v) \cap W_{\tau,n}(\mathbb{N})\right) = \frac{n}{\tau}\cdot$$

*Proof.* The upper bound for the Hausdorff dimension of the limsup set $\Gamma_n(A,u,v) \cap W_{\tau,n}(\mathbb{N})$ may be computed with a standard covering argument — the details are left to the reader. As for the lower bound, notice that Lemma 3.2.3 implies that

$$\Gamma_n(A,u,v) \cap W_{\tau,n}(\mathbb{N}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \ : \ |\boldsymbol{x} - \boldsymbol{p}/q| < q^{-\tau} \text{ for i.m. } \boldsymbol{p}/q \in \Gamma_n(A,u,v) \right\}.$$

For any vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ ($n \geq 2$), let $\boldsymbol{x_{n-1}}$ be the subvector $\boldsymbol{x_{n-1}} := (x_1, \ldots, x_{n-1})$. Recall that $|\boldsymbol{x}|$ denotes the infinity norm of $\boldsymbol{x}$. Taking $K(A) = \left(\sum_{i=1}^n |a_i|\right) |a_n|^{-1}$, it is then readily verified that

$$|\boldsymbol{x_{n-1}} - \boldsymbol{y_{n-1}}| \ \leq \ |\boldsymbol{x} - \boldsymbol{y}| \ \leq \ K(A) \cdot |\boldsymbol{x_{n-1}} - \boldsymbol{y_{n-1}}| \tag{3.12}$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \Gamma_n(A,u,v)$. Since Hausdorff dimension is invariant under a bi–Lipschitz transformation (see [Prolegomena, Proposition 0.1.3, p.7]), this means that

$$\dim\left(\Gamma_n(A,u,v) \cap W_{\tau,n}(\mathbb{N})\right) = \dim V_\tau\left(\Gamma_n(A,u,v)\right),$$

where

$$V_\tau\left(\Gamma_n(A,u,v)\right) = \left\{ (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \ : \ (x_1, \ldots, x_{n-1}, x_n) \in \Gamma_n(A,u,v) \cap W_{\tau,n}(\mathbb{N}) \right\}.$$

Note that, if $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, then $x_n$ as appearing in the definition of $V_\tau\left(\Gamma_n(A,u,v)\right)$ is uniquely determined.

From (3.12), it should be clear that the set

$$U_{\tau,n-1}\left(K(A)\right) := \Big\{ \boldsymbol{x_{n-1}} = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \ : \ |\boldsymbol{x_{n-1}} - \boldsymbol{p}/q| < \left(K(A)q^{\tau}\right)^{-1} \text{ for i.m. } q \geq 1 \Big\} \quad (3.13)$$

is contained in $V_{\tau}\left(\Gamma_n(A,u,v)\right)$. To conclude the proof of the lemma, it suffices now to invoke the theorem due to Dickinson and Velani stated in [Prolegomena, Theorem 0.1.8, p.10] : in the latter reference, the choices $f(q) = q^s$, $\psi(q) = q^{-\tau+1}K(A)^{-1}$, $m = 1$ and $n = n-1$ yield the equation $\dim U_{\tau,n-1}\left(K(A)\right) = n/\tau$ if $\tau > 1 + 1/(n-1)$, the $n/\tau$–Hausdorff measure of $U_{\tau,n-1}\left(K(A)\right)$ being infinite. Note that this also holds true for the sets $V_{\tau}\left(\Gamma_n(A,u,v)\right)$ and $\Gamma_n(A,u,v) \cap W_{\tau,n}\left(\mathbb{N}\right)$ since a set of infinite Hausdorff measure is transformed under a bi–Lipschitz transformation into a set of infinite Hausdorff measure (see [Prolegomena, Proposition 0.1.3, p.7]). **Q.E.D.**

This completes the proof of Theorem 3.2.1.

### 3.2.2 Focus on dimension 1

Theorem 3.2.1 does not give any information about what happens if $n = 1$ and $\nu\left(\mathcal{Q}\right) = 1$. This situation is not uncommon in Diophantine approximation : some problems are easier to apprehend in higher dimensions than in dimension one (for instance the conjecture of Duffin and Schaeffer). Nevertheless, it is sometimes possible to prove that the set $\widehat{W}^{*}_{\tau,1}\left(\mathcal{Q}\right)$ is not empty, which is not obvious at first sight in the general case.

To illustrate this fact, consider a positive integer $m \neq 1$ and the $\mathbb{N}^*\backslash\mathcal{Q}(m)$–free set

$$\mathcal{Q}(m) := \left\{ n \in \mathbb{N}^* \ : \ \gcd(n,m) = 1 \right\}. \quad (3.14)$$

Assume that the integer $m$ is divisible by exactly $r$ prime numbers $\pi_1, \dots, \pi_r$ and set $\Pi := \{\pi_1, \dots, \pi_r\}$. For simplicity, let $\widehat{W}^{*}_{\tau}(\Pi)$ denote the set $\widehat{W}^{*}_{\tau,1}(\mathcal{Q}(m))$, that is,

$$\widehat{W}^{*}_{\tau}(\Pi) = \left\{ x \in \mathbb{R} \ : \ \left| x - \frac{p}{q} \right| < \frac{1}{q^{\tau}} \text{ for i.m. } q \notin \mathcal{Q}\left( \prod_{\pi \in \Pi} \pi \right) \text{ and f.m. } q \in \mathcal{Q}\left( \prod_{\pi \in \Pi} \pi \right) \right\}. \quad (3.15)$$

Thus an element $x \in \widehat{W}^{*}_{\tau}(\Pi)$ is $\tau$–well approximable by fractions whose denominators, except for a finite number of them, are divisible by a prime in $\Pi$. It will be proved that $\widehat{W}^{*}_{\tau}(\Pi)$ is never empty as soon as $\tau > 2$ if the cardinality of $\Pi$ is greater than or equal to 2.

To this end, the theory of continued fractions is needed : if necessary, the reader is referred to [51, Chapter 1] for an account on the topic. Given an irrational number $x$, denote by $\left(a_s\right)_{s\geq 0}$ the sequence of its partial quotients and by $\left(p_s/q_s\right)_{s\geq 0}$ the sequence of its convergents, which are given by the relations of recurrence

$$p_s = a_s p_{s-1} + p_{s-2}, \quad (3.16)$$

and

$$q_s = a_s q_{s-1} + q_{s-2} \quad (3.17)$$

for $s \geq 1$ along with the initial conditions $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$ and $q_0 = 1$. The convergents of $x \in \mathbb{R}\backslash\mathbb{Q}$ are related to its rational approximations in the following way : if a non–zero rational number $p/q$ satisfies the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}, \tag{3.18}$$

then $p/q$ is a convergent of $x$, i.e. there exists $s \in \mathbb{N}$ such that $p/q = p_s/q_s$.

The following theorem may now be proved :

**Theorem 3.2.5.** *Let* $\Pi$ *be any subset of the primes containing at least two distinct elements. Let* $\tau > 2$ *be a real number. Then the set* $\widehat{W}^*_\tau(\Pi)$ *as defined by (3.15) contains uncountably many Liouville numbers.*

Theorem 3.2.5 will be derived from the following result, proved by Erdős and Mahler [92, Theorem 2]. The notation introduced above is maintained.

**Theorem 3.2.6** (Erdős & Mahler, 1939)**.** *Let $x$ be a real number. Suppose that for an infinity of different indices $s \geq 1$ the denominators $q_{s-1}$, $q_s$, $q_{s+1}$ of three consecutive convergents of $x$ are divisible by only a finite system of primes. Then $x$ is a Liouville number.*

*Proof of Theorem 3.2.5.* Let $\pi_0$ and $\pi_1$ be two distinct primes in $\Pi$. From Theorem 3.2.6 and Property (3.18), it is enough to prove the existence of uncountably many irrationals for which all the denominators $q_s$ of their convergents are only divisible by $\pi_0$ and $\pi_1$ as soon as $s \geq 1$. To this end, first note that the relationships (3.16) and (3.17) may be rewritten more succinctly in the form

$$\begin{pmatrix} p_s & q_s \\ p_{s-1} & q_{s-1} \end{pmatrix} = \prod_{k=0}^{s} \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}$$

when $s \geq 0$. Taking the determinant on both sides of this equation, it appears on the one hand that any sequence of positive integers $(p_s)_{s\geq 0}$ satisfying (3.16) is such that, for any $s \geq 0$, $\gcd(p_s, q_s) = 1$ if the sequence $(q_s)_{s\geq 0}$ satisfies (3.17). It is therefore enough to solve (3.17), where the unknowns are the two sequences $(a_s)_{s\geq 0}$ and $(q_s)_{s\geq 0}$. On the other hand, the same relationship shows that two consecutive denominators $q_s$ and $q_{s+1}$ are coprime : this leads one to choose all the $q_{2s}$ $(s \geq 0)$ as powers of the prime $\pi_0$ and all the $q_{2s+1}$ $(s \geq 0)$ as powers of $\pi_1$ (or conversely).

Thus, let $q_0 = 1$ and $q_1 = a_1 \cdot q_0 = \pi_1$. Setting $q_{2s+i} = \pi_i^{\alpha_{2s+i}}$ for $s \geq 0$ and $i = 0, 1$ with $(\alpha_s)_{s\geq 0}$ an increasing sequence of positive integers such that $\alpha_0 = 0$, equation (3.17) amounts to the following one : for all $s \geq 0$ and all $i \in \{0, 1\}$,

$$a_{2s+i}\, \pi_i^{\alpha_{2s-1+i}} = \pi_{1-i}^{\alpha_{2s-2+i}} \left( \pi_{1-i}^{\alpha_{2s+i} - \alpha_{2s-2+i}} - 1 \right).$$

It should be clear at this stage that both sequences $(a_s)_{s\geq 0}$ and $(q_s)_{s\geq 0}$ will be uniquely determined if the sequence $(\alpha_s)_{s\geq 0}$ (with $\alpha_0 = 0$) is chosen in such a way that, for all $s \in \mathbb{N}^*$,

$$\pi_{1-i}^{\alpha_{2s+i} - \alpha_{2s-2+i}} \equiv 1 \pmod{\pi_i^{\alpha_{2s-1+i}}}$$

for $i = 0, 1$. It is easy to construct such a sequence $(\alpha_s)_{s\geq 0}$ (with $\alpha_0 = 0$) by induction : for two coprime integers $a$ and $b$, denote first by $\omega(a, b)$ the order of $b$ in the multiplicative group $(\mathbb{Z}/a\mathbb{Z})^{\times}$. Then, choose $\alpha_1 \in \mathbb{N}^*$ and set, for any $s \geq 1$ and $i = 0, 1$,

$$\alpha_{2s+i} = \alpha_{2s-2+i} + k_{2s+i}\,\omega\left(\pi_i^{\alpha_{2s-1+i}}, \pi_{1-i}\right),$$

where at each step the integer $k_{2s+i}$ is chosen in such a way that $\alpha_{2s+i} > \alpha_{2s+i-1}$.

This proves the existence of a Liouville number in $\widehat{W}_{\tau}^{*}(\Pi)$ ($\tau > 2$). Since infinitely many choices are possible for the integer $k_{2s+i}$ at each step, $\widehat{W}_{\tau}^{*}(\Pi)$ actually contains uncountably many Liouville numbers as soon as $\tau > 2$. **Q.E.D.**

### 3.2.3 Open problems

To conclude, two notable problems which complement the results of this section are mentioned hereafter.

**Question 3.2.7.** *Let $\Pi$ be any finite system of primes and let $\tau > 2$ be a real number. Define $\widehat{W}_{\tau}^{*}(\Pi)$ as in (3.15).*

*Does $\widehat{W}_{\tau}^{*}(\Pi)$ contain any non–Liouville numbers?*

**Question 3.2.8.** *Let $n \geq 2$ be an integer and let $\tau > 1$ denote a real number. Let $\mathcal{Q}$ be an $\mathbb{N}^*\backslash\mathcal{Q}$–free set of integers such that $\nu(\mathcal{Q}) = 1$. Define $W_{\tau,n}^{*}(\mathcal{Q})$ as in (3.7).*

*Is it true that, if $\boldsymbol{x} = (x_1, \dots, x_n) \in \widehat{W}_{\tau,n}^{*}(\mathcal{Q})$, then $x_1, x_2, \dots, x_n$ are $\mathbb{Q}$–linearly dependent?*

Thus, Question 3.2.8 amounts to establishing the converse inclusion in (3.11) for suitable values of $A$, $u$ and $v$. This is a problem of particular interest when the set $\mathcal{Q}$ is chosen as in (3.14) : indeed, the method used by Babai and Štefankovič in [5] provides in this case a positive answer to Question 3.2.8 if one takes $\epsilon/q$ for a fixed $\epsilon > 0$ as the error function in $\widehat{W}_{\tau,n}^{*}(\mathcal{Q})$ instead of $q^{-\tau}$. However, their method, which consists of investigating certain probability measures on lattices and their Fourier transforms, cannot be extended to a more general class of error functions.

## 3.3 A $p$–adic example

Let $p$ be an arbitrary but fixed prime.

An analogue of Theorem 3.2.1 is now studied in $\mathbb{Q}_p$. Consider first the $p$–adic version of the set of $\tau$–well approximable numbers ($\tau > 0$) in $\mathbb{Q}_p$, namely

$$W_{\tau,n}(p) := \left\{\boldsymbol{x} \in \mathbb{Q}_p^n \;:\; |q\boldsymbol{x} - \boldsymbol{r}|_p < \max\left(|\boldsymbol{r}|, q\right)^{-\tau} \text{ for i.m. } (\boldsymbol{r}, q) \in \mathbb{Z}^n \times \mathbb{N}\right\}. \tag{3.19}$$

Here, $|\boldsymbol{x}|_p$ denotes the supremum of the $p$–adic norms of the components of $\boldsymbol{x} \in \mathbb{Q}_p^n$.

Note that, unlike (3.1), the approximating function depends now both on $|\boldsymbol{r}|$ and $q$ rather than simply $q$. This is due to the fact that, in the $p$–adic setup, given $x \in \mathbb{Z}_p$, a quantity of the form $|qx - r|_p$ can be made arbitrarily small by taking $r$ to be a rational integer with the appropriate number of leading terms taken from the $p$–adic expansion of $qx$. Thus the set of

$\boldsymbol{x} \in \mathbb{Q}_p^n$ such that $|q\boldsymbol{x} - \boldsymbol{r}|_p < q^{-\tau}$ for infinitely many $(\boldsymbol{r}, q) \in \mathbb{Z}^n \times \mathbb{N}$ contains the whole of $\mathbb{Z}_p^n$ and has therefore full Hausdorff dimension regardless of the value of $\tau > 0$.

Another difference with (3.1) is that, in the $p$–adic setup, there is no "normalizing" factor $q$ on the right–hand side of $|q\boldsymbol{x} - \boldsymbol{r}|_p$. This is due to the fact that the $p$–adic norm is an ultra metric. For more details, the limsup set $W_{\tau,n}(p)$ is studied in full generality in [23].

Let $W^*_{\tau,n}(p)$ be the liminf set obtained from (3.19) by imposing the constraint that all the integers $q$ should be divisible by $p$, viz.

$$W^*_{\tau,n}(p) := \Big\{ \boldsymbol{x} \in \mathbb{Q}_p^n \ : \ |q\boldsymbol{x} - \boldsymbol{r}|_p < \max(|\boldsymbol{r}|, q)^{-\tau} \ \text{ for i.m. } (\boldsymbol{r}, q) \in \mathbb{Z}^n \times p\mathbb{N} \text{ and f.m. } (\boldsymbol{r}, q) \notin \mathbb{Z}^n \times p\mathbb{N} \Big\}. \quad (3.20)$$

The set $W^*_{\tau,n}(p)$ may be seen as an analogue of at least two different real liminf sets as introduced in (3.5) : on the one hand, it is defined as the set of elements in $\mathbb{Q}_p^n$ which are $\tau$–well approximable only by integer vectors $(\boldsymbol{r}, q)$ such that $q$ is a multiple of the integer $p$ provided it is large enough. On the other, since the gcd of two $p$–adic integers is the highest power of $p$ dividing both of them (it is defined up to an invertible element), $W^*_{\tau,n}(p)$ is also the set of all elements in $\mathbb{Q}_p^n$ $\tau$–well approximable only by integer vectors $(\boldsymbol{r}, q)$ such that, provided it is large enough, $q$ is *not* coprime to a given non unit $s \in \mathbb{Z}_p$.

The structure of the liminf set $W^*_{\tau,n}(p)$ exhibits very different behaviours depending on whether it is restricted to $\mathbb{Z}_p$ or not.

**Theorem 3.3.1.** *If $\tau > 1 + 1/n$, then*

$$\dim W^*_{\tau,n}(p) = \frac{n+1}{\tau}\cdot$$

*Furthermore, $W^*_{\tau,n}(p) \cap \mathbb{Z}_p^n = \emptyset$ as soon as $\tau \geq 1$.*

Thus the situation is quite original : the liminf set $W^*_{\tau,n}(p)$ has the same Hausdorff dimension as the limsup set $W_{\tau,n}(p)$ when $\tau > 1 + 1/n$ (cf. [23, Theorem 16]) but it contains no $p$–adic integers. This is in sharp contrast with the fact that, when considering the limsup set $W_{\tau,n}(p)$ from a metric point of view, it generally suffices to study its intersection with $\mathbb{Z}_p^n$ as the space $\mathbb{Q}_p^n$ can be written as a countable union of translates of $\mathbb{Z}_p^n$.

The proof of Theorem (3.3.1) rests on the following lemma which uses this definition : a vector $\boldsymbol{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k$ with integer coordinates is said to be $p$–primitive if at least one of the components $n_i$ of $\boldsymbol{n}$ is coprime to $p$.

**Lemma 3.3.2.** *If $\tau \geq 1$, then*

$$W_{\tau,n}(p) = \Big\{ \boldsymbol{x} \in \mathbb{Q}_p^n \ : \ |q\boldsymbol{x} - \boldsymbol{r}|_p < \max(|\boldsymbol{r}|, q)^{-\tau} \ \textit{for i.m. } p\textit{–primitive } (\boldsymbol{r}, q) \in \mathbb{Z}^n \times \mathbb{N} \Big\}.$$

*Proof.* Given $\boldsymbol{x} \in W_{\tau,n}(p)$, let $(u_k := (\boldsymbol{r_k}, q_k))_{k\geq 1}$ be the sequence strictly increasing in $q_k$ of elements of $\mathbb{Z}^n \times \mathbb{N}$ satisfying

$$|q_k\boldsymbol{x} - \boldsymbol{r_k}|_p < \max(|\boldsymbol{r_k}|, q_k)^{-\tau}. \quad (3.21)$$

Note that if $k_0$ and $m$ are positive integers, $mu_{k_0}$ satisfies (3.21) if, and only if,

$$1 \underset{(\tau\geq 1)}{\leq} |m|_p\,|m|^{\tau} < |q_{k_0}\boldsymbol{x} - \boldsymbol{r_{k_0}}|_p^{-1} \max\left(q_{k_0}, |\boldsymbol{r_{k_0}}|\right)^{-\tau}.$$

The first of these inequalities shows that $u_{k_0}$ is a multiple of a $p$–primitive vector $\tilde{u}_{k_0}$ and the second one proves that the number of multiples of $u_{k_0}$ satisfying (3.21) is finite. **Q.E.D.**

**Corollary 3.3.3.** *Assume that* $\tau \geq 1$.

*Then*

$$W^*_{\tau,n}(p) \cap \mathbb{Z}_p^n = \emptyset.$$

*Proof.* Let $\boldsymbol{x} = (x_1, \dots, x_n) \in W^*_{\tau,n}(p) \cap \mathbb{Z}_p^n$ and let $(\boldsymbol{r}, q) \in \mathbb{Z}^n \times \mathbb{N}$ be a vector of approximation of $\boldsymbol{x}$, i.e. a vector satisfying (3.21). From Lemma 3.3.2, $(\boldsymbol{r}, q)$ may be assumed to be $p$–primitive which, from the definition of the liminf set $W^*_{\tau,n}(p)$ and provided that $q$ is large enough, implies on the one hand that $p$ divides $q$ and on the other that $|r_{i_0}|_p = 1$ for some component $r_{i_0} \in \mathbb{Z}$ of the vector $\boldsymbol{r} := (r_1, \dots, r_n) \in \mathbb{Z}^n$. In particular,

$$|qx_{i_0} - r_{i_0}|_p < \max\left(q, |r_{i_0}|\right)^{-\tau}. \tag{3.22}$$

Now, if $1 = |r_{i_0}|_p > |qx_{i_0}|_p$, then (3.22) implies that $|qx_{i_0} - r_{i_0}|_p = |r_{i_0}|_p = 1 < |r_{i_0}|_p^{-\tau}$, which is impossible. If $1 = |r_{i_0}|_p < |qx_{i_0}|_p$, then it follows from (3.22) that $1 < |qx_{i_0}|_p = |qx_{i_0} - r_{i_0}|_p < q^{-\tau}$, which cannot happen. Finally, if $|r_{i_0}|_p = 1 = |qx_{i_0}|_p$, then, since $p$ divides $q$, $1 > |q|_p = |x_{i_0}|_p^{-1} \underset{(x_0\in\mathbb{Z}_p)}{\geq} 1$, which gives again a contradiction. This completes the proof of the corollary. **Q.E.D.**

*Completion of the proof of Theorem 3.3.1.* From the proof of Corollary 3.3.3, it also follows that if $(\boldsymbol{r}, q)$ is a $p$–primitive vector of approximation of $\boldsymbol{x} = (x_1, \cdots, x_n) \in W^*_{\tau,n}(p)$ such that $p$ divides $q$ but does not divide a component $r_{i_0} \in \mathbb{Z}$ of $\boldsymbol{r}$, then, necessarily, $|r_{i_0}|_p = 1 = |qx_{i_0}|_p$. This implies in particular that $x_{i_0} \in \mathbb{Q}_p \backslash \mathbb{Z}_p$. Note also that the condition $|qx_{i_0}|_p = 1$ alone is sufficient to guarantee that $|r_{i_0}|_p = 1$ : indeed, if one had $|r_{i_0}|_p < 1 = |qx_{i_0}|_p$, then one would also have $|qx_{i_0}|_p = |qx_{i_0} - r_{i_0}|_p = 1 < q^{-\tau}$, which cannot be.

Thus, each $p$–primitive vector of approximation $(\boldsymbol{r}, q)$ of $\boldsymbol{x} \in W^*_{\tau,n}(p)$ determines at least one component $x_{i_0}$ of $\boldsymbol{x}$ such that $x_{i_0} \in \mathbb{Q}_p \backslash \mathbb{Z}_p$. Since there are only finitely many components, it follows that

$$W^*_{\tau,n}(p) = \left\{ \boldsymbol{x} \in \mathbb{Q}_p^n \;:\; \exists i_0 \in [\![1, n]\!],\, x_{i_0} \in \mathbb{Q}_p \backslash \mathbb{Z}_p \text{ and } \left\{ \begin{array}{l} |q\boldsymbol{x} - \boldsymbol{r}|_p < \max\left(|\boldsymbol{r}|, q\right)^{-\tau} \\ |qx_{i_0}|_p = |r_{i_0}|_p = 1 \end{array} \right. \text{ i.o.} \right\},$$

where *i.o.* stands for *infinitely often*. Therefore,

$$W^*_{\tau,n}(p) = \bigcup_{i_0=1}^{n} \left\{ \boldsymbol{x} \in \mathbb{Q}_p^n \;:\; x_{i_0} \in \mathbb{Q}_p \backslash \mathbb{Z}_p \text{ and } \left\{ \begin{array}{l} |q\boldsymbol{x} - \boldsymbol{r}|_p < \max\left(|\boldsymbol{r}|, q\right)^{-\tau} \\ |qx_{i_0}|_p = 1 \end{array} \right. \text{ i.o.} \right\}.$$

For any $f \in \mathbb{N}$, denote by $W_{\tau,n}(p, i_0, f)$ the set

$$W_{\tau,n}(p, i_0, f) := \left\{ \boldsymbol{x} \in \mathbb{Q}_p^n \ : \ |x_{i_0}|_p = p^f \text{ and } \left\{ \begin{array}{l} |q\boldsymbol{x} - \boldsymbol{r}|_p < \max\left(|\boldsymbol{r}|, q\right)^{-\tau} \\ |qx_{i_0}|_p = 1 \end{array} \right. \text{ i.o.} \right\}.$$

Then

$$W_{\tau,n}^*(p) = \bigcup_{i_0=1}^{n} \bigcup_{f=1}^{\infty} W_{\tau,n}(p, i_0, f)$$

and it suffices to establish the dimensional result in Theorem 3.3.1 for any of the sets $W_{\tau,n}(p, i_0, f)$.

Fix $f \geq 1$ and $i_0 \in [\![1, n]\!]$. Given $(\boldsymbol{r}, q) \in \mathbb{Z}^n \times \mathbb{N}$, let $\nu(\boldsymbol{r}, q) := \max(q, |\boldsymbol{r}|)$ and let

$$B(\boldsymbol{x}, \rho) := \left\{ \boldsymbol{a} \in \mathbb{Q}_p^n \ : \ |\boldsymbol{x} - \boldsymbol{a}|_p < \rho \right\}$$

denote the open ball of radius $\rho > 0$ centered at $\boldsymbol{x} \in \mathbb{Q}_p^n$. It should then be clear that a cover for $W_{\tau,n}(p, i_0, f)$ is given by

$$\bigcap_{N=1}^{\infty} \bigcup_{\nu > N} \bigcup_{\nu(\boldsymbol{r},q) \in \mathcal{A}_\nu(i_0,f)} B\left(\frac{\boldsymbol{r}}{q}, \nu^{-\tau} p^f\right),$$

where

$$\mathcal{A}_\nu(i_0, f) := \left\{ (\boldsymbol{r}, q) \in \mathbb{Z}^n \times \mathbb{N} \ : \ |q|_p = p^{-f}, |r_{i_0}|_p = 1 \text{ and } \nu(\boldsymbol{r}, q) = \nu \right\}.$$

Furthermore, it is readily checked that $\#\mathcal{A}_\nu(i_0, f) \asymp \nu^n$, where the implicit constants depend on $n$ and $p$ (here, $\#\mathcal{A}_\nu(i_0, f)$ denotes the cardinality of the set $\mathcal{A}_\nu(i_0, f)$). Hence, for any $N > 0$,

$$\mathcal{H}^s(W_{\tau,n}(p, i_0, f))) \ll \sum_{\nu > N} \sum_{\nu(\boldsymbol{r},q) \in \mathcal{A}_\nu(i_0,f)} \nu^{-\tau s} \ll \sum_{\nu > N} \nu^{-\tau s + n},$$

which is finite as soon as $s > (n+1)/\tau$, so that $\dim(W_{\tau,n}(p, i_0, f)) \leq (n+1)/\tau$.

The proof that $\dim(W_{\tau,n}(p, i_0, f)) \geq (n+1)/\tau$ is very similar to the corresponding result for the limsup set $W_{\tau,n}(p)$ as defined in (3.19) and will therefore not be given : this is due to the fact that $W_{\tau,n}(p, i_0, f)$ is itself a limsup set. For further details, the reader is referred to [1, 23].

This completes the proof of Theorem 3.3.1. **Q.E.D.**

## 3.4 Notes for the chapter

- It will be proved in [Chapter 4, Theorem 4.1.1] that for any infinite subset $\mathcal{Q} \subset \mathbb{N}^*$ such that $\nu(\mathcal{Q}) = 1$ (recall that $\nu(\mathcal{Q})$ denotes the exponent of convergence of $\mathcal{Q}$ as defined in (3.4)), $\dim W_{\tau,n}^*(\mathcal{Q}) = \dim W_{\tau,n}(\mathbb{N}) = (n+1)/\tau$ for all $\tau > 2 + 1/n$. The proof is non–constructive but provides a positive answer to Question 3.2.7 and a negative one to Question 3.2.8.

- Y. Sun has recently provided a positive answer to Question 3.2.7. Indeed, the following stronger result is established in [185].

  **Theorem 3.4.1** (Y. Sun, [185]). *Let $m \geq 1$ be an integer divisible by at least two distinct primes $\pi_1$ and $\pi_2$. Let $\Pi := \{\pi_1, \pi_2\}$. Define $\widehat{W}_\tau^*(\Pi)$ in the same way as in subsection 3.2.2. Then for $\tau \geq 2$, the set $\widehat{W}_\tau^*(\Pi)$ contains uncountably many reals whose measure of irrationality is exactly $\tau$.*

  The proof is constructive in the sense that a real number satisfying the conclusion of this theorem may explicitly be given in terms of its continued fraction expansion.

- B. Wang, Z. Wen and J. Wu [197] have recently proved in the case when $n = 1$ that, for any $\mathbb{N}^*\backslash\mathcal{Q}$–free set $\mathcal{Q} \subset \mathbb{N}^*$, $\dim \widehat{W}_{\tau,1}^*(\mathcal{Q}) = 2/\tau$ for any $\tau > 2$ (note that this does not solve the case $\tau = 2$ as the sets $\widehat{W}_{\tau,1}^*(\mathcal{Q})$ are not decreasing for inclusion when $\tau$ increases). Their result is more precise in this particular setup than the one of the next chapter (cf. [Chapter 4, Theorem 4.1.1]) which is only valid, when $n = 1$, for $\tau > 3$. Nevertheless, the proof given by B. Wang, Z. Wen and J. Wu strongly rests on the theory of continued fractions and cannot be naturally generalized to higher dimensions.

# Chapter 4

# Liminf Sets in simultaneous Diophantine Approximation : a Jarník–Besicovitch Type Theorem


**Abstract**

Let $\mathcal{Q}$ be an infinite set of positive integers. Denote by $W^*_{\tau,n}(\mathcal{Q})$ the set of $n$–tuples of real numbers simultaneously $\tau$–well approximable by infinitely many rationals with denominators in $\mathcal{Q}$ but by only finitely many rationals with denominators in the complement of $\mathcal{Q}$. The Hausdorff dimension of the liminf set $W^*_{\tau,n}(\mathcal{Q})$ is determined when $n \geq 1$ and $\tau > 2 + 1/n$.


## 4.1 Introduction

Let $n \geq 1$ be an integer and $\tau > 1$ be a real number. Given an infinite set of positive integers $\mathcal{Q}$, denote by $W_{\tau,n}(\mathcal{Q})$ the set of points in dimension $n \geq 1$ approximable at order $\tau$ by infinitely many rationals with denominators in $\mathcal{Q}$, i.e. the limsup set

$$W_{\tau,n}(\mathcal{Q}) := \left\{ \boldsymbol{x} \in \mathbb{R}^n \ : \ |\boldsymbol{x} - \boldsymbol{p}/q| < q^{-\tau} \text{ for i.m. } (\boldsymbol{p}, q) \in \mathbb{Z}^n \times \mathcal{Q} \right\}. \tag{4.1}$$

Here and throughout, *i.m.* stands for *infinitely many*, $|\boldsymbol{x}|$ is the usual supremum norm of a vector $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{p}/q$ is shorthand notation for the rational vector $(p_1/q, \dots, p_n/q)$, where $\boldsymbol{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n$.

As mentioned in Chapter 3, Borosh and Fraenkel [48] have generalized the classical theorem of Jarník and Besicovitch [Prolegomena, Theorem 0.1.5, p.9] in metric Diophantine approximation by proving that

$$\dim W_{\tau,n}(\mathcal{Q}) = \frac{n+\nu(\mathcal{Q})}{\tau} \tag{4.2}$$

when $\tau > 1 + \nu(\mathcal{Q})/n$, where $\nu(\mathcal{Q})$ is the exponent of convergence of $\mathcal{Q}$ defined as

$$\nu(\mathcal{Q}) := \inf\left\{\nu > 0 \ : \ \sum_{q\in\mathcal{Q}} q^{-\nu} < \infty\right\} \in [0,1]. \tag{4.3}$$

In order to pursue the study undertaken in Chapter 3, consider the liminf set

$$W^*_{\tau,n}(\mathcal{Q}) := W_{\tau,n}(\mathbb{N}) \setminus W_{\tau,n}\left(\mathbb{N}\backslash\mathcal{Q}\right) = W_{\tau,n}(\mathcal{Q}) \setminus W_{\tau,n}\left(\mathbb{N}\backslash\mathcal{Q}\right). \tag{4.4}$$

Recall that this is the set of all those vectors $\boldsymbol{x}$ in $\mathbb{R}^n$ which admit infinitely many approximations at order $\tau$ as in (4.1) by rational vectors $(\boldsymbol{p}, q)$ whose denominators $q$ lie in $\mathcal{Q}$, but only finitely many approximations by rational vectors whose denominators do not lie in $\mathcal{Q}$.

The previous chapter was partly concerned with the case when the set $\mathcal{Q}$ was a so–called $\mathbb{N}^*\backslash\mathcal{Q}$–free set (that is, a set whose elements are divisible by no integer in its complement). A non–trivial lower bound for $\dim W^*_{\tau,n}(\mathcal{Q})$ was then exhibited when $n \geq 2$ and $\tau > 1+1/(n-1)$. A construction, explicit in terms of the continued fraction expansion, of uncountably many Liouville numbers lying in the set $W^*_{\tau,1}\left(\pi_1\mathbb{N}\cup\pi_2\mathbb{N}\right)$, where $\pi_1$ and $\pi_2$ denote primes, was also provided when $\tau > 2$.

It is not clear that the set $W^*_{\tau,n}(\mathcal{Q})$ should be non–empty for a general infinite subset $\mathcal{Q} \subset \mathbb{N}^*$. This is in particular implied by the following much stronger statement which is the main result of this chapter.

**Theorem 4.1.1.** *Let $\mathcal{Q} \subset \mathbb{N}^*$ be infinite. Assume that $n \geq 1$ is an integer and that $\tau > 2+1/n$ is a real number. Then*

$$\dim W^*_{\tau,n}(\mathcal{Q}) = \frac{n+\nu(\mathcal{Q})}{\tau}\cdot$$

Thus when $\tau > 2 + 1/n$, the limsup set $W_{\tau,n}(\mathcal{Q})$ and the corresponding liminf set $W^*_{\tau,n}(\mathcal{Q})$ actually share the same Hausdorff dimension. This leaves a gap corresponding to the case where $\tau$ lies in the interval $(1 + \nu(\mathcal{Q})/n\, ,\, 2 + 1/n]$. The nature of this restriction will clearly appear in the course of the proof and will be discussed in the concluding notes for the chapter. It is however worth mentioning at this stage that the underlying difficulty does not seem easy to overcome and may be linked to some deep problems in the metric theory of numbers.

### Notation

In addition to that already introduced, the following notation will be used throughout this chapter :

- $x \ll y$ (resp. $x \gg y$, where $x, y \in \mathbb{R}$) : there exists a constant $c > 0$ such that $x \leq cy$ (resp. $x \geq cy$).
- $x \asymp y$ $(x, y \in \mathbb{R})$ means both $x \ll y$ and $x \gg y$.

- $[\![x, y]\!]$ $(x, y \in \mathbb{R},\ x \leq y)$ : interval of integers, i.e. $[\![x, y]\!] = \{n \in \mathbb{Z} \ : \ x \leq n \leq y\}$.

- $\lambda_n$ : the $n$–dimensional Lebesgue measure (for simplicity, $\lambda := \lambda_1$).

- $\#X$ : the cardinality of a finite set $X$.

- $|U|$ : the diameter of a bounded set $U \subset \mathbb{R}^n$.

- $\delta_x(\mathcal{S}) := \#\left(\mathcal{S} \cap [\![1, x]\!]\right)$ for any subset $\mathcal{S} \subset \mathbb{N}$ and any real $x \geq 1$.

- $I_\tau\!\left(\frac{p}{q}\right) := \left(\frac{p}{q} - \frac{1}{q^\tau}\, ,\, \frac{p}{q} + \frac{1}{q^\tau}\right)$, where $\frac{p}{q} \in \mathbb{Q}$.

- $C_\tau\!\left(\frac{\boldsymbol{p}}{q}\right) := \prod_{i=1}^{n} I_\tau\!\left(\frac{p_i}{q}\right)$, where $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ (note that with this convention, $C_\tau\!\left(\frac{2\boldsymbol{p}}{2q}\right)$ is strictly contained in $C_\tau\!\left(\frac{\boldsymbol{p}}{q}\right)$).

## 4.2 Auxiliary lemmata

In this section, $\mathcal{S}$ denotes an arbitrary infinite set of non–zero natural integers.

### 4.2.1 On the logarithmic density of a subset of integers

As is well–known, the exponent of convergence of the set $S$ as defined by (4.3) is related to its logarithmic density in the following way (see, e.g., [107] for a proof) :

$$\nu(\mathcal{S}) \;=\; \limsup_{n\to\infty} \left(\frac{\log \delta_n(\mathcal{S})}{\log n}\right). \tag{4.5}$$

The next lemma provides a similar formula for $\nu(\mathcal{S})$.

**Lemma 4.2.1.** *The following equation holds :*

$$\nu(\mathcal{S}) \;=\; \limsup_{n\to\infty} \left(\frac{\log\left(\delta_{2n}(\mathcal{S}) - \delta_n(\mathcal{S})\right)}{\log n}\right).$$

*Proof.* First note that, for $n \geq 2$,

$$\frac{\log\left(\delta_{2n}(\mathcal{S}) - \delta_n(\mathcal{S})\right)}{\log n} \;\leq\; \frac{\log \delta_{2n}(\mathcal{S})}{\log n} \;\underset{n\to\infty}{\sim}\; \frac{\log \delta_{2n}(\mathcal{S})}{\log 2n}.$$

Upon taking the limsup on both sides of this inequality, it is easily seen that (4.5) implies that

$$\limsup_{n\to\infty} \left(\frac{\log\left(\delta_{2n}(\mathcal{S}) - \delta_n(\mathcal{S})\right)}{\log n}\right) \;\leq\; \nu(\mathcal{S}).$$

This suffices to prove the result in the case $\nu(\mathcal{S}) = 0$ since, the set $\mathcal{S}$ being infinite, $\delta_{2n}(\mathcal{S}) - \delta_n(\mathcal{S}) \geq 1$ for infinitely many $n \in \mathbb{N}$. Therefore, assume from now on that $\nu(\mathcal{S}) > 0$. Then (4.5) shows the existence of a sequence $(n_k)_{k\geq 0}$ of positive integers such that

$$\log \delta_{n_k}(\mathcal{S}) \;\underset{n\to\infty}{\sim}\; \nu(\mathcal{S}) \log n_k. \tag{4.6}$$

For a fixed $k \in \mathbb{N}$, consider the following partition of the interval $[\![2, n_k]\!]$ into $u_k := \lfloor \log n_k / \log 2 \rfloor$ subintervals :

$$[\![2, n_k]\!] = \bigcup_{r=0}^{u_k} \left[\!\!\left[ \frac{n_k}{2^{r+1}} + 1, \frac{n_k}{2^r} \right]\!\!\right].$$

From the definition of the integer $\delta_{n_k}(\mathcal{S})$, at least one of these intervals contains more than $(\delta_{n_k}(\mathcal{S})-1)/(u_k+1)$ elements of $\mathcal{S}$, which determines a rational number $l_k$ of the form $n_k/2^{a+1}$ $(0 \leq a \leq u_k - 1)$ such that

$$\frac{\delta_{n_k}(\mathcal{S}) - 1}{u_k + 1} \leq 2l_k - (l_k+1) + 1 = l_k \leq \frac{n_k}{2} \quad \text{and} \quad \frac{\delta_{n_k}(\mathcal{S}) - 1}{u_k + 1} \leq \delta_{2l_k}(\mathcal{S}) - \delta_{l_k}(\mathcal{S}). \tag{4.7}$$

From (4.6) and from the definition of $u_k$, one deduces on the one hand that the first inequality in (4.7) implies that the sequence $(l_k)_{k \geq 1}$ tends to infinity and that, on the other,

$$\frac{\log(\delta_{n_k}(\mathcal{S}) - 1) - \log(u_k+1)}{\log n_k} \underset{n \to \infty}{\sim} \frac{\log \delta_{n_k}(\mathcal{S})}{\log n_k} \underset{n \to \infty}{\sim} \nu(\mathcal{S}).$$

Furthermore, it follows from (4.7) that

$$\frac{\log(\delta_{n_k}(\mathcal{S}) - 1) - \log(u_k+1)}{\log n_k} \leq \frac{\log(\delta_{2l_k}(\mathcal{S}) - \delta_{l_k}(\mathcal{S}))}{\log l_k}.$$

Combining these last two inequalities leads to the relationship

$$\nu(\mathcal{S}) \leq \limsup_{n \to \infty} \left( \frac{\log(\delta_{2l_k}(\mathcal{S}) - \delta_{l_k}(\mathcal{S}))}{\log l_k} \right),$$

which completes the proof. **Q.E.D.**

One key–step in the proof of Theorem 4.1.1 is to approximate an infinite set of positive integers by arbitrarily large subsets, the size of a subset being measured by its exponent of convergence. In this respect, the following proposition will turn out to be very useful.

**Proposition 4.2.2.** *Assume that $\nu(\mathcal{S}) > 0$ and let $\nu \in (0, \nu(\mathcal{S}))$. Furthermore, let $(\alpha_n)_{n \geq 0}$ be a sequence of positive reals such that the sequence $(n^\nu \alpha_n)_{n \geq 0}$ is increasing and such that $(\log \alpha_n / \log n)_{n \geq 2}$ tends to 0 as n goes to infinity.*

*Then, there exists a subset $\mathcal{S}_\nu \subset \mathcal{S}$ such that :*

- *for all $n \geq 1$, $\delta_{2n}(\mathcal{S}_\nu) - \delta_n(\mathcal{S}_\nu) \leq n^\nu \alpha_n$.*
- *there exists a strictly increasing sequence of positive integers $(n_k)_{k \geq 0}$ satisfying*

$$\delta_{2n_k}(\mathcal{S}_\nu) - \delta_{n_k}(\mathcal{S}_\nu) \underset{k \to \infty}{\sim} n_k^\nu \alpha_{n_k}.$$

*In particular, $\nu(\mathcal{S}_\nu) = \nu$.*

*Proof.* The fact that $\nu(\mathcal{S}_\nu) = \nu$ follows immediately from Lemma 4.2.1. Note that this lemma applied to the set $\mathcal{S}$ amounts to claiming the existence of a sequence of real numbers $(\beta_n)_{n\geq 0}$ tending to zero and of a strictly increasing sequence of positive integers $(p_k)_{k\geq 0}$ satisfying

$$\delta_{2n}(\mathcal{S}) - \delta_n(\mathcal{S}) \leq n^{\nu(\mathcal{S})+\beta_n} \text{ for all } n \in \mathbb{N} \text{ and } \delta_{2p_k}(\mathcal{S}) - \delta_{p_k}(\mathcal{S}) \underset{k\to\infty}{\sim} p_k^{\nu(\mathcal{S})+\beta_{p_k}}. \quad (4.8)$$

Also, the assumption that $\log \alpha_n / \log n$ tends to zero amounts to the fact that $\alpha_n = o(n^\epsilon)$ for any $\epsilon > 0$. Thus, the second relationship in (4.8) and the fact that $\nu < \nu(\mathcal{S})$ guarantee the existence of a smallest positive integer $n_1$ such that

$$\lfloor n_1^\nu \alpha_{n_1} \rfloor \; < \; \delta_{2n_1}(\mathcal{S}) - \delta_{n_1}(\mathcal{S}) := r_1.$$

Now remove $r_1 - \lfloor n_1^\nu \alpha_{n_1} \rfloor$ elements of $\mathcal{S}$ from the interval $[\![n_1 + 1, 2n_1]\!]$ to define a subset $\mathcal{S}_\nu^{(1)} \subset \mathcal{S}$ satisfying the following properties :

- $\mathcal{S}_\nu^{(1)}$ and $\mathcal{S}$ coincide on the intervals $[\![1, n_1]\!]$ and $\mathbb{N}\backslash[\![1, 2n_1]\!]$,
- for all $n \in [\![1, n_1]\!]$, $\delta_{2n}\left(\mathcal{S}_\nu^{(1)}\right) - \delta_n\left(\mathcal{S}_\nu^{(1)}\right) \; \leq \; n^\nu \alpha_n$,
- $\delta_{2n_1}\left(\mathcal{S}_\nu^{(1)}\right) - \delta_{n_1}\left(\mathcal{S}_\nu^{(1)}\right) \; = \; \lfloor n_1^\nu \alpha_{n_1} \rfloor$.

Consider then the smallest integer $n_2 > n_1$ such that

$$\lfloor n_2^\nu \alpha_{n_2} \rfloor \; < \; \delta_{2n_2}\left(\mathcal{S}_\nu^{(1)}\right) - \delta_{n_2}\left(\mathcal{S}_\nu^{(1)}\right) := r_2.$$

Since for $n \geq 2n_1+1$, $\delta_{2n}\left(\mathcal{S}_\nu^{(1)}\right) - \delta_n\left(\mathcal{S}_\nu^{(1)}\right) = \delta_{2n}(\mathcal{S}) - \delta_n(\mathcal{S})$, the existence of $n_2$ is guaranteed in the same way as for $n_1$.

Defining $u_1 := \max\{n_2, 2n_1\}$, remove $r_2 - \lfloor n_2^\nu \alpha_{n_2} \rfloor$ elements of $\mathcal{S}$ from the interval $[\![u_1 + 1, 2n_2]\!]$. This is clearly possible if $n_2 \geq 2n_1$ as there is no overlap in this case between the intervals $[\![n_1, 2n_1]\!]$ and $[\![u_1+1, 2n_2]\!]$. But this is also possible if $n_1 < n_2 < 2n_1$ : indeed, if the interval $[\![u_1+1, 2n_2]\!] = [\![2n_1+1, 2n_2]\!]$ was to contain strictly less than $r_2 - \lfloor n_2^\nu \alpha_{n_2} \rfloor$ elements, one would have :

$$\begin{aligned}
r_2 := \delta_{2n_2}\left(\mathcal{S}_\nu^{(1)}\right) - \delta_{n_2}\left(\mathcal{S}_\nu^{(1)}\right) &= \delta_{2n_2}\left(\mathcal{S}_\nu^{(1)}\right) - \delta_{2n_1}\left(\mathcal{S}_\nu^{(1)}\right) + \delta_{2n_1}\left(\mathcal{S}_\nu^{(1)}\right) - \delta_{n_2}\left(\mathcal{S}_\nu^{(1)}\right) \\
&= \delta_{2n_2}(\mathcal{S}) - \delta_{2n_1}(\mathcal{S}) + \delta_{2n_1}\left(\mathcal{S}_\nu^{(1)}\right) - \delta_{n_2}\left(\mathcal{S}_\nu^{(1)}\right) \\
&\qquad \left(\text{as } \mathcal{S}_\nu^{(1)} \cap \{n \geq 2n_1+1\} = \mathcal{S} \cap \{n \geq 2n_1+1\}\right) \\
&\leq \delta_{2n_2}(\mathcal{S}) - \delta_{2n_1}(\mathcal{S}) + \delta_{2n_1}\left(\mathcal{S}_\nu^{(1)}\right) - \delta_{n_1}\left(\mathcal{S}_\nu^{(1)}\right) \\
&< r_2 - \lfloor n_2^\nu \alpha_{n_2} \rfloor + \lfloor n_1^\nu \alpha_{n_1} \rfloor \\
&\leq r_2
\end{aligned}$$

since the sequence $(n^\nu \alpha_n)_{n\geq 0}$ is increasing. This contradiction shows that one can find a subset $\mathcal{S}_\nu^{(2)} \subset \mathcal{S}_\nu^{(1)}$ such that :

- $\mathcal{S}_\nu^{(1)}$ and $\mathcal{S}_\nu^{(2)}$ coincide on the intervals $[\![1, n_2]\!]$ and $\mathbb{N}\backslash[\![1, 2n_2]\!]$,

- for all $n \in [\![1, n_2]\!]$, $\delta_{2n}\left(\mathcal{S}_\nu^{(2)}\right) - \delta_n\left(\mathcal{S}_\nu^{(2)}\right) \leq n^\nu \alpha_n$,
- $\delta_{2n_2}\left(\mathcal{S}_\nu^{(2)}\right) - \delta_{n_2}\left(\mathcal{S}_\nu^{(2)}\right) = \lfloor n_2^\nu \alpha_{n_2} \rfloor$.

By induction, one can thus construct a decreasing sequence $\left(\mathcal{S}_\nu^{(k)}\right)_{k\geq 1}$ of subsets of $\mathcal{S}$ and a strictly increasing sequence of natural integers $(n_k)_{k\geq 1}$ such that, for all $k \geq 2$,

- $\mathcal{S}_\nu^{(k-1)}$ and $\mathcal{S}_\nu^{(k)}$ coincide on $[\![1, n_k]\!]$ and $\mathbb{N}\backslash[\![1, 2n_k]\!]$,
- for all $n \in [\![1, n_k]\!]$, $\delta_{2n}\left(\mathcal{S}_\nu^{(k)}\right) - \delta_n\left(\mathcal{S}_\nu^{(k)}\right) \leq n^\nu \alpha_n$,
- $\delta_{2n_k}\left(\mathcal{S}_\nu^{(k)}\right) - \delta_{n_k}\left(\mathcal{S}_\nu^{(k)}\right) = \lfloor n_k^\nu \alpha_{n_k} \rfloor$.

By construction, the set $\mathcal{S}_\nu := \cap_{k=1}^\infty \mathcal{S}_\nu^{(k)}$ satisfies the conclusions of the proposition. **Q.E.D.**

### 4.2.2 Steps to the construction of a Cantor set

Theorem 4.1.1 will be proved by exhibiting nice Cantor sets contained in the liminf set under consideration. To this end, a few auxiliary results are gathered in this subsection. They are preceded by two definitions which will be used throughout the rest of this chapter.

**Definition 4.2.3.** *A vector $\boldsymbol{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n$ is* $q$–primitive *(where $q \in \mathbb{N}^*$) if at least one of the components $p_i$ of $\boldsymbol{p}$ is coprime to $q$. The vector $\boldsymbol{p}$ is* absolutely $q$–primitive *if all its components are coprime to $q$.*

**Definition 4.2.4.** *Given $\tau > 1$, $\boldsymbol{p_0} \in \mathbb{Z}^n$ and $q_0 \in \mathbb{N}^*$, a* hypercube of new generation in $C_\tau\left(\frac{\boldsymbol{p_0}}{q_0}\right)$ *is a hypercube of the form $C_\tau\left(\frac{\boldsymbol{p}}{q}\right)$ contained in $C_\tau\left(\frac{\boldsymbol{p_0}}{q_0}\right)$ such that $\boldsymbol{p} \in \mathbb{Z}^n$ is absolutely $q$–primitive ($q \in \mathbb{N}^*$) and such that for any $q_1 \in [\![q_0 + 1, q - 1]\!]$ and any $\boldsymbol{p_1} \in \mathbb{Z}^n$,*

$$C_\tau\left(\frac{\boldsymbol{p_1}}{q_1}\right) \cap C_\tau\left(\frac{\boldsymbol{p}}{q}\right) = \emptyset.$$

Thus, the concept of a hypercube of new generation renders the idea that such a polytope covers a volume inside a given hypercube which has been covered by no other. The next proposition counts the number of such hypercubes and constitutes a problem specific to the liminf setup in Diophantine approximation. It is preceded by a well–known lemma on the repartition of integers coprime to a given natural number.

**Lemma 4.2.5.** *Let $q$ be a positive integer and $\eta$ be any positive real number. Denote by $\varphi_\eta(q)$ the number of integers less than $\eta q$ and coprime to $q$. Then, for any $\epsilon > 0$,*

$$\varphi_\eta(q) = \varphi(q)\left(\eta + o\left(q^{-1+\epsilon}\right)\right),$$

*where $\varphi$ denotes Euler's totient function.*

*In particular, if $\epsilon \in (0, 1)$, $\eta > q^{-1+\epsilon}$ and $q$ is large enough, then for any $\gamma \geq 0$,*

$$\#\left\{ p \in [\![\gamma q, (\gamma + \eta)q]\!] \ : \ \gcd(p, q) = 1 \right\} \asymp \eta\varphi(q),$$

*where the implicit constants depend only on $\epsilon$.*

*Proof.* This follows easily from the inclusion–exclusion principle and some standard estimates of arithmetical functions. See, e.g., [83, Lemma III] for details. **Q.E.D.**

**Proposition 4.2.6.** *Let $\tau > 2+1/n$, $\boldsymbol{p_0} \in \mathbb{Z}^n$ and $q_0 \in \mathbb{N}^*$. Assume that $C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right) \subset (0,1)^n$ and that $q > q_0^{\tau^3}$ has been chosen large enough. Denote furthermore by $\mathcal{N}\left(q, \frac{\boldsymbol{p_0}}{q_0}, \tau\right)$ the cardinality of the set of hypercubes of new generation in $C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)$ of the form $C_\tau\!\left(\frac{\boldsymbol{p}}{q}\right)$ for some $\boldsymbol{p} \in \mathbb{Z}^n$.*

*Then, provided that $q_0$ is larger than some constant (independent of $q$),*

$$\mathcal{N}\left(q, \frac{\boldsymbol{p_0}}{q_0}, \tau\right) \;\geq\; \frac{\varphi(q)^n\,\lambda_n\!\left(C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)\right)}{2^{n+1}}.$$

*Proof.* Set $\widetilde{C}_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right) := C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right) \Big\backslash C_\tau\!\left(\frac{2\boldsymbol{p_0}}{2q_0}\right)$. If $q > q_0$ is large enough, the number of absolutely $q$–primitive vectors $\boldsymbol{p} \in \mathbb{Z}^n$ such that $C_\tau\!\left(\frac{\boldsymbol{p}}{q}\right) \subset \widetilde{C}_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)$ is certainly bigger than

$$2^n\left(1-\frac{1}{2^\tau}\right)^n \frac{\varphi(q)^n}{2^n}\,\lambda_n\!\left(C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)\right) \underset{(\tau>1)}{\geq} \frac{\varphi(q)^n}{2^n}\lambda_n\!\left(C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)\right) \tag{4.9}$$

(this follows for instance from Lemma 4.2.5).

Assume now that there exist an integer $q_1 > q_0$ and $\boldsymbol{p_1} \in \mathbb{Z}^n$ such that $\widetilde{C}_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right) \cap C_\tau\!\left(\frac{\boldsymbol{p_1}}{q_1}\right) \neq \emptyset$. In particular, $\boldsymbol{p_1}/q_1 \neq \boldsymbol{p_0}/q_0$, whence

$$\frac{1}{q_0 q_1} \leq \left|\frac{\boldsymbol{p_0}}{q_0} - \frac{\boldsymbol{p_1}}{q_1}\right| < \frac{2}{q_0^\tau}.$$

This means that, when computing the number of hypercubes $C_\tau\!\left(\frac{\boldsymbol{p}}{q}\right)$ of new generation in $\widetilde{C}_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)$ ($\boldsymbol{p} \in \mathbb{Z}^n$), it suffices to consider those hypercubes of this form which have no overlap with any hypercube of the form $C_\tau\!\left(\frac{\boldsymbol{p_1}}{q_1}\right)$, where $\boldsymbol{p_1} \in \mathbb{Z}^n$ and $q_1 > q_0^{\tau-1}/2$.

Given this, let us now count the number of integer vectors $\boldsymbol{p} \in \mathbb{Z}^n$ such that $C_\tau\!\left(\frac{\boldsymbol{p}}{q}\right)$ has a non–empty intersection with a hypercube $C_\tau\!\left(\frac{\boldsymbol{p_1}}{q_1}\right)$ contained in $C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)$, where $\boldsymbol{p_1} \in \mathbb{Z}^n$ and $\frac{q_0^{\tau-1}}{2} < q_1 < q$.

*First case :* $\frac{q_0^{\tau-1}}{2} < q_1 \leq \frac{q_0^\tau}{4}$. Fix an integer $q_1$ in this range. Then there exists at most one integer vector $\boldsymbol{p_1} \in \mathbb{Z}^n$ such that $C_\tau\!\left(\frac{\boldsymbol{p_1}}{q_1}\right) \cap C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right) \neq \emptyset$. Indeed, should there exist another one $\boldsymbol{p_1'} \in \mathbb{Z}^n$, one would have

$$\frac{1}{q_1} \leq \left|\frac{\boldsymbol{p_1}}{q_1} - \frac{\boldsymbol{p_1'}}{q_1}\right| \leq \left|\frac{\boldsymbol{p_1}}{q_1} - \frac{\boldsymbol{p_0}}{q_0}\right| + \left|\frac{\boldsymbol{p_0}}{q_0} - \frac{\boldsymbol{p_1'}}{q_1}\right| < \frac{4}{q_0^\tau},$$

contradicting the assumption on $q_1$.

Suppose now that there does exist $\boldsymbol{p_1} = (p_{1,i})_{1\leq i\leq n} \in \mathbb{Z}^n$ satisfying $C_\tau\!\left(\frac{\boldsymbol{p_1}}{q_1}\right) \cap C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right) \neq \emptyset$. If, furthermore, $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{Z}^n$ is an absolutely $q$–primitive vector such that $C_\tau\!\left(\frac{\boldsymbol{p}}{q}\right) \cap$

$C_\tau\left(\frac{\boldsymbol{p_1}}{q_1}\right) \neq \emptyset$, then, for any $i \in [\![1, n]\!]$,

$$\left| p_i - \frac{p_{1,i}}{q_1} q \right| < \frac{2q}{q_1^\tau}. \tag{4.10}$$

Under the assumption that $q \geq q_0^{\tau^3}$ and $q_1 \leq q_0^\tau/4$, it follows from Lemma 4.2.5 that, if $q_0$ is chosen large enough, the number of such absolutely $q$–primitive vectors $\boldsymbol{p} \in \mathbb{Z}^n$ is less than

$$K \frac{\varphi(q)^n}{q_1^{n\tau}}$$

for some constant $K > 0$ depending on $n$.

Summing over all the possible values of $q_1$, the number of hypercubes $C_\tau\left(\frac{\boldsymbol{p}}{q}\right)$ with $\boldsymbol{p} \in \mathbb{Z}^n$ absolutely $q$–primitive having a non–empty intersection with a hypercube of the form $C_\tau\left(\frac{\boldsymbol{p_1}}{q_1}\right)$ is seen to be less that

$$K\varphi(q)^n \sum_{q_0^{\tau-1}/2 < q_1 \leq q_0^\tau/4} q_1^{-n\tau} \;\leq\; K\varphi(q)^n \sum_{q_1 > q_0^{\tau-1}/2} q_1^{-n\tau} \;\leq\; c_1 \frac{\varphi(q)^n}{q_0^{(\tau-1)(n\tau-1)}} \tag{4.11}$$

for some $c_1 > 0$ depending on $\tau$ and $n$.

*Second case :* $\frac{q_0^\tau}{4} < q_1 < q$. Fix an integer $q_1$ in this range and assume that $C_\tau\left(\frac{\boldsymbol{p_1}}{q_1}\right) \cap C_\tau\left(\frac{\boldsymbol{p}}{q}\right) \neq \emptyset$ for some $\boldsymbol{p} \in \mathbb{Z}^n$ absolutely $q$–primitive and some $\boldsymbol{p_1} \in \mathbb{Z}^n$. Then

$$\frac{1}{qq_1} \leq \left| \frac{\boldsymbol{p}}{q} - \frac{\boldsymbol{p_1}}{q_1} \right| < \frac{2}{q_1^\tau}, \qquad \text{whence} \qquad q_1^{\tau-1} < 2q. \tag{4.12}$$

Furthermore, inequality (4.10) still holds.

Given $\epsilon > 0$ and $i \in [\![1, n]\!]$, it follows from Lemma 4.2.5 that the number of solutions in $p_i$ to (4.10) is

$$\varphi(q) \left( \frac{4}{q_1^\tau} + o\left( \frac{1}{q^{1-\epsilon}} \right) \right) \underset{(4.12)}{\leq} \frac{6\varphi(q)}{q_1^{(\tau-1)(1-\epsilon)}} \tag{4.13}$$

for $q_0$ (and so $q_1$ and $q$) large enough depending on the choice of $\epsilon > 0$ (note that the error term in Lemma 4.2.5 is independent of $\eta > 0$). Now, if there is an overlap between $C_\tau\left(\frac{\boldsymbol{p_1}}{q_1}\right)$ and $C_\tau\left(\frac{\boldsymbol{p_0}}{q_0}\right)$, it is easily seen that $p_{1,i}$ can assume at most $8q_1\,\lambda\left(I_\tau\left(\frac{p_{0,i}}{q_0}\right)\right)$ values (where $\boldsymbol{p_0} = (p_{0,i})_{1 \leq i \leq n}$). Therefore, the number of solutions to (4.10) in $\boldsymbol{p} \in \mathbb{Z}^n$ absolutely $q$–primitive is at most

$$8^n \left( \frac{6\varphi(q)}{q_1^{(\tau-1)(1-\epsilon)-1}} \right)^n \lambda_n\left( C_\tau\left( \frac{\boldsymbol{p_0}}{q_0} \right) \right)$$

for $q_0$ large enough.

Summing over all the possible values for $q_1$, the number of hypercubes $C_\tau\left(\frac{\boldsymbol{p}}{q}\right)$ with $\boldsymbol{p} \in \mathbb{Z}^n$ absolutely $q$–primitive having a non–empty intersection with a hypercube of the form $C_\tau\left(\frac{\boldsymbol{p_1}}{q_1}\right)$

is seen to be less that

$$48^n\varphi(q)^n\,\lambda_n\left(C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)\right)\sum_{q_0^\tau/4<q_1\leq(2q)^{1/(\tau-1)}} q_1^{-n((\tau-1)(1-\epsilon)-1)}$$

$$\leq\; 48^n\varphi(q)^n\,\lambda_n\left(C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)\right)\sum_{q_1>q_0^\tau/4} q_1^{-n((\tau-1)(1-\epsilon)-1)}\;\leq\;\frac{c_2\varphi(q)^n\,\lambda_n\left(C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)\right)}{q_0^{\tau(n((\tau-1)(1-\epsilon)-1)-1)}} \tag{4.14}$$

for $q_0$ large enough depending on the choice of an arbitrarily small $\epsilon>0$ and for some $c_2>0$ depending on $\tau$ and $n$.

*Conclusion.* Taking into account (4.9), (4.11) and (4.14), for $q>q_0^{\tau^3}$ large enough,

$$\mathcal{N}\left(q,\frac{\boldsymbol{p_0}}{q_0},\tau\right)\;\geq\;\frac{\varphi(q)^n}{2^n}\,\lambda_n\left(C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)\right)\left[1-2^n\frac{q_0^{n\tau}}{2^n}\frac{c_1}{q_0^{(\tau-1)(n\tau-1)}}-\frac{2^nc_2}{q_0^{\tau(n((\tau-1)(1-\epsilon)-1)-1)}}\right]$$

(we used the fact that $\lambda_n\left(C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)\right)=2^n/q_0^{n\tau}$). This holds provided that $q_0$ satisfies the assumptions of (4.9), (4.11) and (4.14).

Now, if $\epsilon>0$ has been chosen small enough, this last quantity is bigger than $\varphi(q)^n\,\lambda_n\left(C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)\right)/2^{n+1}$ for $q_0$ large enough if $\tau>1+(1+1/n)/(1-\epsilon)$. The result follows on letting $\epsilon$ tend to zero. **Q.E.D.**

The last result of this subsection contains the main feature of the proof of Theorem 4.1.1 and should be compared with [48, Lemma 4].

**Lemma 4.2.7.** *Let $\tau>2+1/n$, $\boldsymbol{p_0}\in\mathbb{Z}^n$ and $q_0\in\mathbb{N}$ such that $C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)\subset(0,1)^n$ and such that Proposition 4.2.6 applies. Assume furthermore that $\nu(\mathcal{S})>0$ and that $\delta_{2k}(\mathcal{S})-\delta_k(\mathcal{S})=o\left(\frac{k^{\nu(\mathcal{S})}}{(\log\log k)^n}\right)$.*

*Then for any $k>q_0$ sufficiently large, there exists a set $\mathcal{E}_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)$ of rational vectors contained in $C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)$ such that :*

*i) for any $\frac{\boldsymbol{p}}{q}\in\mathcal{E}_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)$, $\boldsymbol{p}\in\mathbb{Z}^n$ is absolutely $q$–primitive, $q\in\mathcal{S}$ and $k<q\leq 2k$ ;*

*ii) for any two distinct elements $\frac{\boldsymbol{p_1}}{q_1}$ and $\frac{\boldsymbol{p_2}}{q_2}$ in $\mathcal{E}_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)$ such that $q_1\leq q_2$,*

$$\left|\frac{\boldsymbol{p_1}}{q_1}-\frac{\boldsymbol{p_2}}{q_2}\right|\;\geq\;\frac{1}{q_1^{1+\nu(\mathcal{S})/n}}\ ;$$

*iii) for any $\frac{\boldsymbol{p}}{q}\in\mathcal{E}_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)$, $C_\tau\!\left(\frac{\boldsymbol{p}}{q}\right)$ is a hypercube of new generation in $C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)$ ;*

*iv) the following inequalities hold :*

$$\#\mathcal{E}_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)\;\geq\;\frac{\lambda_n\left(C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)\right)}{2^{n+2}}\sum_{\substack{k<q\leq 2k\\ q\in\mathcal{S}}}\varphi(q)^n\;\gg\;\lambda_n\left(C_\tau\!\left(\frac{\boldsymbol{p_0}}{q_0}\right)\right)\frac{k^n\left(\delta_{2k}(\mathcal{S})-\delta_k(\mathcal{S})\right)}{(\log\log k)^n},$$

*where the implicit constant depends only on n.*

*Proof.* For the sake of simplicity of notation, let $C$ denote the hypercube $C_\tau\left(\frac{\boldsymbol{p_0}}{q_0}\right)$ in this proof only. Let $\mathcal{F}(C)$ be the set of rational vectors $\frac{\boldsymbol{p}}{q}$ such that :

1) $q \in \mathcal{S}$ and $k < q \leq 2k$;

2) $\boldsymbol{p} \in \mathbb{Z}^n$ is absolutely $q$–primitive;

3) $C_\tau\left(\frac{\boldsymbol{p}}{q}\right)$ is a hypercube of new generation in $C$.

If $\frac{\boldsymbol{p}}{q}$ and $\frac{\boldsymbol{p'}}{q'}$ are two rational vectors satisfying *1)* and *2)*, and if furthermore $C_\tau\left(\frac{\boldsymbol{p}}{q}\right) \cap C_\tau\left(\frac{\boldsymbol{p'}}{q'}\right) \neq \emptyset$, then

$$\frac{1}{4k^2} \leq \left|\frac{\boldsymbol{p}}{q} - \frac{\boldsymbol{p'}}{q'}\right| \leq \frac{1}{k^\tau},$$

which cannot happen if $\tau > 2$ and $k > q_0$ is chosen large enough. This shows together with Proposition 4.2.6 that for such an integer $k$,

$$\#\mathcal{F}(C) \geq \frac{\lambda_n(C)}{2^{n+1}} \sum_{\substack{k<q\leq 2k \\ q\in\mathcal{S}}} \varphi(q)^n. \tag{4.15}$$

Let $\mathcal{E}(C)$ denote the subset of $\mathcal{F}(C)$ from which one excludes all the rational vectors $\frac{\boldsymbol{p}}{q}$ for which there exists an integer $q_1 \in [\![k+1, q-1]\!]$ and an element $\frac{\boldsymbol{p_1}}{q_1} \in \mathcal{F}(C)$ satisfying

$$\left|\frac{\boldsymbol{p_1}}{q_1} - \frac{\boldsymbol{p}}{q}\right| < \frac{1}{(q_1)^{1+\nu(\mathcal{S})/n}}.$$

It should be clear that $\mathcal{E}(C)$ defined this way satisfies the conclusions *i)* to *iii)* of the lemma. It remains to evaluate its cardinality.

Let $q_1, q \in \mathcal{S}$, $k < q_1 < q \leq 2k$. When $q$ is fixed, denote by $N_i(q, q_1)$ $(1 \leq i \leq n)$ the number of integers $p_i$ such that there exists an integer $p_{1,i}$ satisfying

$$|p_{1,i}q - p_i q_1| < \frac{q}{q_1^{\nu(\mathcal{S})/n}}. \tag{4.16}$$

Let furthermore $N(q)$ be the number of rational vectors in $\mathcal{F}(C) \backslash \mathcal{E}(C)$ with denominator $q$ : it should be clear that

$$N(q) \leq \sum_{\substack{k<q_1<q \\ q\in\mathcal{S}}} \prod_{i=1}^{n} N_i(q, q_1).$$

From a familiar argument in elementary number theory (see, e.g., [83, Lemma I]), the number of solutions $N_i(q, q_1)$ in $p_i$ to (4.16) is bounded above by $2q \cdot q_1^{-\nu(\mathcal{S})/n}$, whence

$$N(q) \leq 2^n q^n \sum_{\substack{k<q_1<q \\ q\in\mathcal{S}}} \frac{1}{(q_1)^{\nu(\mathcal{S})}} \leq 2^n q^n \frac{\delta_{2k}(\mathcal{S}) - \delta_k(\mathcal{S})}{k^{\nu(\mathcal{S})}}.$$

Using the well–known result

$$\liminf_{m\to\infty}\left(\frac{\varphi(m)\log\log m}{m}\right)=e^{-\gamma}, \tag{4.17}$$

where $\gamma$ is the Euler–Mascheroni constant, this also leads to the following estimate valid for $k$ large enough :

$$N(q)\ \le\ 2^n e^{\gamma n}2^n\varphi\left(q\right)^n\left(\log\log k\right)^n\frac{\delta_{2k}\left(\mathcal{S}\right)-\delta_k\left(\mathcal{S}\right)}{k^{\nu(\mathcal{S})}}\cdot$$

Now, by the assumption made on the sequence $\left(\delta_{2k}\left(\mathcal{S}\right)-\delta_k\left(\mathcal{S}\right)\right)_{k\ge1}$, for $k$ large enough, $2^n e^{\gamma n}\left(\log\log k\right)^n\left(\delta_{2k}\left(\mathcal{S}\right)-\delta_k\left(\mathcal{S}\right)\right)k^{-\nu(\mathcal{S})}\ \le\ \lambda_n(C)/2^{2n+2}$, so that

$$N(q)\ \le\ \frac{\varphi(q)^n\lambda_n(C)}{2^{n+2}}\cdot \tag{4.18}$$

For such an integer $k$, from (4.15) and (4.18),

$$\begin{aligned}\#\mathcal{E}\left(C\right)=\#\mathcal{F}\left(C\right)-\#\left(\mathcal{F}\left(C\right)\backslash\mathcal{E}\left(C\right)\right)\ &\ge\ \sum_{\substack{k<q\le 2k\\ q\in\mathcal{S}}}\left(\frac{\varphi(q)^n\lambda_n(C)}{2^{n+1}}-N(q)\right)\\ &\ge\ \frac{\lambda_n(C)}{2^{n+2}}\sum_{\substack{k<q\le 2k\\ q\in\mathcal{S}}}\varphi(q)^n\\ &\underset{(4.17)}{\gg}\ \lambda_n(C)\,\frac{k^n}{(\log\log k)^n}\left(\delta_{2k}\left(\mathcal{S}\right)-\delta_k\left(\mathcal{S}\right)\right).\end{aligned}$$

**Q.E.D.**

## 4.3 The Jarník–Besicovitch type Theorem

Theorem 4.1.1 will now be proved for a given infinite set of positive integers $\mathcal{Q}$. As should be clear, it is enough to establish the result for the set $W^*_{\tau,n}(\mathcal{Q})\cap[0,1]^n$ which, for the sake of simplicity, will still be denoted by $W^*_{\tau,n}(\mathcal{Q})$ in what follows. The same convention of notation will be applied to the limsup set $W_{\tau,n}(\mathcal{Q})$ when needed.

### 4.3.1 The upper bound

Proving that $\dim W^*_{\tau,n}(\mathcal{Q})\le\left(1+\nu(\mathcal{Q})\right)/\tau$ is almost trivial : for any $N\ge1$,

$$\bigcup_{\substack{q\ge N\\ q\in\mathcal{Q}}}\ \bigcup_{\boldsymbol{p}\in[\![0,q]\!]^n} C_\tau\!\left(\frac{\boldsymbol{p}}{q}\right)$$

is a cover for the limsup set $W_{\tau,n}(\mathcal{Q})$, and consequently in particular for the liminf set $W^*_{\tau,n}(\mathcal{Q})$. Consequently, for any $N\ge1$, the $s$–dimensional Hausdorff measure $\mathcal{H}^s\big(W^*_{\tau,n}(\mathcal{Q})\big)$ of the set $W^*_{\tau,n}(\mathcal{Q})$ satisfies

$$\mathcal{H}^s\big(W^*_{\tau,n}(\mathcal{Q})\big)\ \le\ \sum_{\substack{q\ge N\\ q\in\mathcal{Q}}}\frac{(q+1)^n}{q^{s\tau}}\cdot$$

The right–hand side of this inequality is finite as soon as $s > (n + \nu(\mathcal{Q}))/\tau$, which implies that $\dim W^*_{\tau,n}(\mathcal{Q}) \leq (n + \nu(\mathcal{Q}))/\tau$ for any $\tau > 1 + \nu(\mathcal{Q})/n$.

### 4.3.2 The lower bound

The core of the proof of Theorem 4.1.1 consists of establishing the correct lower bound for $\dim W^*_{\tau,n}(\mathcal{Q})$. The ideas developed here are inspired by [95, Chap. 1 & 4] and by [48] (which is based itself on the pioneering work of Jarník [126]).

Recall first the construction of a *level set* $E$ in $[0,1]^n$ : let

$$[0,1]^n = E_0 \supset E_1 \supset E_2 \supset \dots$$

be a decreasing sequence of sets such that each $E_k$ is a finite union of disjoint and closed hypercubes. Assume furthermore that each hypercube of $E_k$ contains $m_k \geq 2$ hypercubes from $E_{k+1}$ and that the maximal diameter of the hypercubes of level $k$ (i.e. in $E_k$) tends to 0 as $k$ goes to infinity. Then

$$E := \bigcap_{k=0}^{\infty} E_k \tag{4.19}$$

is a totally disconnected subset of $[0,1]^n$ — a Cantor set — referred to as a level set.

It is possible to equip such a level set $E$ with a measure $\mu$ supported on it in the following way : let $\mu_0$ be the uniform distribution on $E_0 = [0,1]^n$. If $\mu_{k-1}$ is a measure supported on $E_{k-1}$, let $\mu_k$ be the measure supported on $E_k$ assigning a mass of $(m_1 \dots m_k)^{-1}$ to each of the $m_1 \dots m_k$ hypercubes of $E_k$, the distribution of $\mu_k$ on each of these hypercubes being uniform. Denote by $\mathcal{E}$ the set of hypercubes of all levels used to construct $E$. For any $U \in \mathcal{E}$ of level $k$, let $\mu(U) := \mu_k(U) = (m_1 \dots m_k)^{-1}$. If one sets, for a given subset $A \subset \mathbb{R}^n$,

$$\mu(A) := \inf\left\{ \sum_{l=0}^{\infty} \mu(U_l) \ : \ A \cap E \subset \bigcup_{l=0}^{\infty} U_l \text{ and } U_l \in \mathcal{E} \right\}, \tag{4.20}$$

then $\mu$ defines a probability measure supported on $E$ (see [95, Chap. 1] for details).

Such a measure often turns out to be useful when establishing a lower bound for $\dim E$ by virtue of the well–known Mass Distribution Principle which is now recalled (cf., e.g., [95] for a proof).

**Theorem 4.3.1** (Mass Distribution Principle)**.** *Let $E$ be a level set as described above supporting a probability measure $\mu$. Assume furthermore that for some $s \geq 0$, there exist numbers $c, \kappa > 0$ such that*

$$\mu(U) \ \leq \ c\,|U|^s \tag{4.21}$$

*for all hypercubes $U \in \mathbb{R}^n$ satisfying $|U| \leq \kappa$ (recall that $|U|$ denotes the diameter of $U$).*

*Then*

$$\dim E \geq s.$$

This principle will now be used to determine the Hausdorff dimension of sufficiently large level sets contained in $W^*_{\tau,n}(\mathcal{Q})$.

#### 4.3.2.1 The case $\nu(\mathcal{Q}) > 0$

Assume first that $\nu(\mathcal{Q}) > 0$ and let $\delta \in (0, \nu(\mathcal{Q})/2)$.

Since the sequence $\left(n^{\nu(\mathcal{Q})-\delta}/\log n\right)_{n\geq 2}$ is increasing for $n$ large enough, Proposition 4.2.2 guarantees the existence of a subset $\mathcal{Q}_\delta \subset \mathcal{Q}$ for which one can find a strictly increasing sequence of natural integers $(n_k)_{k\geq 1}$ satisfying

$$\delta_{2n_k}(\mathcal{Q}_\delta) - \delta_{n_k}(\mathcal{Q}_\delta) \underset{k\to\infty}{\sim} \frac{n_k^{\nu(\mathcal{Q}_\delta)}}{\log n_k},$$

where $\nu(\mathcal{Q}_\delta) = \nu(\mathcal{Q}) - \delta$.

In the general construction of a level set, let $E_0 := [0,1]^n$ and, for $q_1 \in \mathcal{Q}_\delta$, $q_1 \geq 2$,

$$E_1 = \bigcup_{\boldsymbol{p_1} \in [\![1, q_1 - 1]\!]^n} C_\tau\!\left(\frac{\boldsymbol{p_1}}{q_1}\right).$$

If $E_{k-1}$ $(k \geq 2)$ has been defined, let $C_\tau\!\left(\frac{\boldsymbol{p_{k-1}}}{q_{k-1}}\right)$ be one of its connected components contained in $(0,1)^n$. From Lemma 4.2.7, there exists an element $q_k > q_{k-1}$ in the sequence $(n_k)_{k\geq 1}$ and

$$m_k \gg \lambda_n\!\left(C_\tau\!\left(\frac{\boldsymbol{p_{k-1}}}{q_{k-1}}\right)\right) \sum_{\substack{q_k < q \leq 2q_k \\ q \in \mathcal{Q}_\delta}} \varphi(q)^n \gg \frac{q_k^{n+\nu(\mathcal{Q}_\delta)}}{q_{k-1}^{n\tau}(\log q_k)(\log\log q_k)^n} \tag{4.22}$$

hypercubes of new generation in $C_\tau\!\left(\frac{\boldsymbol{p_{k-1}}}{q_{k-1}}\right)$ of the form $C_\tau\!\left(\frac{\boldsymbol{p}}{q}\right)$ with $q_k < q \leq 2q_k$ and $q \in \mathcal{Q}_\delta$. Furthermore, the distance between these hypercubes is at least

$$\epsilon_k := \frac{1}{2(q_k)^{1+\nu(\mathcal{Q}_\delta)/n}} \tag{4.23}$$

(by convention, $\epsilon_0 := 1$).

Let then $E_k$ be defined as the union of all these hypercubes over all the connected components of $E_{k-1}$ and let $E$ be as in (4.19). By construction, $E \subset W^*_{\tau,n}(\mathcal{Q}_\delta) \subset W^*_{\tau,n}(\mathcal{Q})$ and $E$ supports a probability measure $\mu$ as defined in (4.20).

**Remark 4.3.2.** The connected components of $E_k$ $(k \geq 1)$ are of the form $C_\tau\!\left(\frac{\boldsymbol{p}}{q}\right)$ for some $\boldsymbol{p} \in \mathbb{Z}^n$ and $q \in \mathcal{Q}$ and therefore are not closed as in the definition of a level set. This difficulty can easily be overcome by redefining them as the closure of the same hypercubes whose side lengths are shrunk by a factor $1-\eta$ for some $\eta < 1/2$. It is then readily checked that Proposition 4.2.6 and Lemma 4.2.7 remain true up to an additional multiplicative constant which will not cause any trouble at all in the rest of the proof. For the sake of simplicity of notation, such detail will be omitted in what follows.

Letting

$$\rho := \frac{n + \nu(\mathcal{Q}_\delta) - \delta}{\tau} = \frac{n + \nu(\mathcal{Q}) - 2\delta}{\tau},$$

it will now be shown by induction on $k \geq 0$ that the sequence $(q_k)_{k\geq 0}$ may be chosen in such a way that, for any hypercube $U \subset \mathbb{R}^n$, (4.21) holds with $s = \rho$ for some real $c > 0$ to be defined later. The following simplifies a great deal the method of [48].

Let $U$ be a hypercube in $\mathbb{R}^n$ and let $k \geq 0$ be such that $\epsilon_{k+1} \leq |U| < \epsilon_k$ (this comes down to taking $\kappa = \epsilon_0 = 1$ in Theorem 4.3.1). Then $U$ intersects at most one connected component of $E_k$ and, since the measure $\mu$ is supported on $E$, there is no loss of generality in assuming that it is actually contained in this connected component. Furthermore, it may also be assumed that $U$ intersects $E_{k+1}$ (otherwise $\mu(U) = 0$ again from (4.20) and the result to prove is trivial). Thus, under these conditions, it follows from (4.20) that

$$\mu(U) \leq \mu_{k+1}(U),$$

where $\mu_{k+1}$ is the uniform distribution supported by $E_{k+1}$.

All this shows that it is enough to prove by induction on $k \geq 0$ the following statement : $(H_k)$ : *For any hypercube $U$ contained in a connected component of $E_k$, having a non–empty intersection with $E_{k+1}$ and satisfying furthermore* the inequalities $\epsilon_{k+1} \leq |U| < \epsilon_k$,

$$\frac{\mu_{k+1}(U)}{|U|^\rho} \leq c.$$

Note that for any hypercube $U \subset [0,1]^n$,

$$\frac{\mu_0(U)}{|U|^\rho} = \frac{\lambda_n(U)}{|U|^\rho} < |U|^{n-\rho} \leq 1.$$

Therefore, it will be assumed that $c \geq 1$.

Consider now an integer $k \geq 0$ and a hypercube $U$ satisfying the assumptions of $(H_k)$. Let $C_k$ be the connected component of $E_k$ containing $U$ and let $N_U$ denote the number of connected components of $E_{k+1}$ having a non–empty intersection with $U$. By assumption, $N_U \geq 1$. The conclusion of $(H_k)$ is proved by distinguishing two subcases.

*First subcase :* $|U| \geq (q_{k+1})^{-1/2}$. Under this assumption, if $q_1$ is chosen large enough so that Lemma 4.2.5 applies with $\epsilon = 1/2$, then, for all $k \geq 0$,

$$N_U \ll |U|^n \sum_{\substack{q_{k+1} < q \leq 2q_{k+1} \\ q \in \mathcal{Q}_\delta}} \varphi(q)^n, \tag{4.24}$$

hence

$$\begin{aligned}
\frac{\mu_{k+1}(U)}{|U|^\rho} &\leq \frac{\mu_{k+1}(C_k)}{|U|^\rho} = \frac{\mu_k(C_k)}{m_{k+1}} \frac{1}{|U|^\rho} \leq \frac{\mu_k(C_k)}{m_{k+1}} \frac{N_U}{|U|^\rho} \\
&\underset{(4.22)\,\&\,(4.24)}{\ll} \frac{\mu_k(C_k)}{|U|^\rho} \frac{|U|^n}{|C_k|^n} \left( \sum_{\substack{q_{k+1} < q \leq 2q_{k+1} \\ q \in \mathcal{Q}_\delta}} \varphi(q)^n \right) \cdot \left( \sum_{\substack{q_{k+1} < q \leq 2q_{k+1} \\ q \in \mathcal{Q}_\delta}} \varphi(q)^n \right)^{-1} \\
&= \frac{\mu_k(C_k)}{|C_k|^\rho} \left( \frac{|U|}{|C_k|} \right)^{n-\rho} \\
&\leq \frac{\mu_k(C_k)}{|C_k|^\rho}.
\end{aligned}$$

If $k = 0$, this means that there exists a constant $K \geq 1$ such that

$$\frac{\mu_1(U)}{|U|^\rho} \leq K \frac{\mu_0(C_0)}{|C_0|^\rho},$$

where $C_0 = [0,1]^n$. Choosing $c$ bigger than this last quantity proves the result in this case.

If $k \geq 1$, then, denoting by $C_{k-1}$ the connected component of $E_{k-1}$ containing $C_k$,

$$\frac{\mu_k(C_k)}{|C_k|^\rho} = \frac{\mu_{k-1}(C_{k-1})}{m_k |C_k|^\rho} \ll \frac{\mu_{k-1}(C_{k-1})\, q_{k-1}^{n\tau} (\log q_k)(\log\log q_k)^n}{q_k^\delta},$$

the last inequality following from (4.22) and the fact that $|C_k| \asymp q_k^{-\tau}$. Choosing $q_k$ large enough in the previous step, this quantity can be made arbitrarily small.

*Second subcase :* $|U| \leq (q_{k+1})^{-1/2}$. By assumption, $\epsilon_{k+1} \leq |U|$. Since two connected components of $E_{k+1}$ are distant by at least $\epsilon_{k+1}$, inequality (4.23) implies that

$$N_U \ll |U|^n (q_{k+1})^{n+\nu(\mathcal{Q}_\delta)}.$$

Therefore, denoting by $C_{k+1}$ any connected component of $E_{k+1}$,

$$\begin{aligned}\frac{\mu_{k+1}(U)}{|U|^\rho} &\leq \frac{\mu_{k+1}(C_{k+1})\, N_U}{|U|^\rho} \ll \frac{\mu_k(C_k)}{m_{k+1}} q_{k+1}^{n+\nu(\mathcal{Q}_\delta)} q_{k+1}^{-(n-\rho)/2}\\ &\underset{(4.22)}{\ll} \frac{\mu_k(C_k)\, q_k^{n\tau} (\log q_{k+1})(\log\log q_{k+1})^n}{q_{k+1}^{(n-\rho)/2}}\end{aligned}$$

(for the second inequality, we used the fact that $C_{k+1} \subset C_k$). Choosing $q_{k+1}$ large enough, this quantity can be made arbitrarily small.

*Conclusion :* From the Mass Distribution Principle (Theorem 4.3.1), for any $\delta \in (0, \nu(\mathcal{Q})/2)$,

$$\dim W^*_{\tau,n}(\mathcal{Q}) \geq \dim E \geq \frac{n + \nu(\mathcal{Q}) - 2\delta}{\tau}.$$

Letting $\delta$ tend to zero completes the proof of Theorem 4.1.1 in the case $\nu(\mathcal{Q}) > 0$.

#### 4.3.2.2 The case $\nu(\mathcal{Q}) = 0$

The proof in the case $\nu(\mathcal{Q}) = 0$ is a simplified version of the previous one. We just mention hereafter the changes to make in the latter : in the construction of the level set $E$, assume that $E_{k-1}$ ($k \geq 2$) has been defined and let $C_\tau\left(\frac{\boldsymbol{p_{k-1}}}{q_{k-1}}\right)$ be one of its connected components. For $q_k > q_{k-1}$ large enough, $q_k \in \mathcal{Q}$, Proposition 4.2.6 guarantees the existence of at least

$$m_k \gg \varphi(q_k)^n \lambda_n\left(C_\tau\left(\frac{\boldsymbol{p_{k-1}}}{q_{k-1}}\right)\right) \underset{(4.17)}{\gg} \frac{q_k^n}{q_{k-1}^{n\tau} (\log\log q_k)^n}$$

hypercubes of new generation in $C_\tau\left(\frac{\boldsymbol{p_{k-1}}}{q_{k-1}}\right)$ of the form $C_\tau\left(\frac{\boldsymbol{p}}{q_k}\right)$ ($\boldsymbol{p} \in \mathbb{Z}^n$) which are furthermore evidently at least

$$\epsilon_k := \frac{1}{2q_k^n}$$

apart. The set $E_k$ is then defined as the union of all these hypercubes over all the connected components of $E_{k-1}$.

The level set $E$ obtained this way may again be equipped with a probability measure supported on it. Given $\delta > 0$, the same reasoning as in the case $\nu(\mathcal{Q}) > 0$ shows that the sequence $(q_k)_{k\geq 0}$ may be chosen in such a way that the Mass Distribution Principle (Theorem 4.3.1) leads to the estimate

$$\dim E \;\geq\; \rho := \frac{n-2\delta}{\tau}\cdot \tag{4.25}$$

It should however be mentioned that inequality (4.24) must now be replaced with the following one :

$$N_U \;\ll\; |U|^n\, \varphi\left(q_{k+1}\right)^n .$$

Letting $\delta$ tend to zero in (4.25) completes the proof of Theorem 4.1.1 in this case also.

## 4.4 Notes for the chapter

- Proposition 4.2.6 imposes the constraint $\tau > 2 + 1/n$ in the statement of Theorem 4.1.1. The nature of this constraint appears to be twofold : on the one hand, one could expect to improve inequalities (4.11) by restricting the summation over only those integers $q_1$ for which there exists, in the first case of the proof, an overlap between $C_\tau\left(\frac{\boldsymbol{p_0}}{q_0}\right)$ and $C_\tau\left(\frac{\boldsymbol{p_1}}{q_1}\right)$ for some $\boldsymbol{p_1} \in \mathbb{Z}^n$. On the other hand, in the second case of the proof, Lemma 4.2.5 does not give enough information about the distribution of integers coprime to $q$ in very short intervals, so that estimate (4.13) leads to some loss of accuracy.

  It is not clear however whether improvements on these inequalities will extend the result of Theorem 4.1.1 to the case of any $\tau$ lying in the interval $(1+\nu\left(\mathcal{Q}\right)/n\,,\,2+1/n)$. Indeed, depending on the choice of $\mathcal{Q}$, one could also expect the Hausdorff dimension of liminf sets such as those under consideration to admit a "phase of transition" at a critical value $\tau_0 \in (1+\nu\left(\mathcal{Q}\right)/n\,,\,2+1/n]$; that is, the value of this dimension will be given by different expressions depending on whether $\tau$ is bigger or smaller than $\tau_0$. Such a phenomenon has been conjectured in other situations — see, e.g., [53, Conjecture 1].

- Restricting to the case $n = 1$ for the sake of simplicity, denote for all integers $q \geq 1$ by $\mathcal{E}_\tau(q)$ the intersection with the unit interval of the union over all $p$ of the intervals $I_\tau\left(\frac{p}{q}\right)$. The main difficulty underlying the proof of Theorem 4.1.1 turns out to be the control of the intersection between $\mathcal{E}_\tau(q)$ and $\mathcal{E}_\tau(q_1)$ (see the proof of Proposition 4.2.6). This is also the notorious issue in proving the Duffin–Schaeffer conjecture : as pointed out (and explained in more detail) in [25], this happens to be not just a deficiency in our knowledge but a *real* problem in the sense that the intersection $\mathcal{E}_\tau(q)\bigcap\mathcal{E}_\tau(q_1)$ may well have a measure much bigger than the expected value $\lambda(\mathcal{E}_\tau(q))\times\lambda(\mathcal{E}_\tau(q_1))$ depending on the values taken by $q$ and $q_1$. The problem is actually even more specific in the proof of

Proposition 4.2.6 as one wishes to control the intersection between $I_\tau\left(\frac{p}{q}\right)$ and a suitable subset of intervals in $\mathcal{E}_\tau(q_1)$ for fixed integers $p$ and $q$ and some $q_1 > q$. It is likely that any substantial improvement on the bound for $\tau > 1+\nu\left(\mathcal{Q}\right)/n$ in Proposition 4.2.6 would require ideas very closely related to the problem of Duffin and Schaeffer.

- The Theorem of Borosh and Fraenkel as mentioned in (4.2) together with equation (3.8) in [Chap. 3, p.85] show that $\dim W^*_{\tau,n}(\mathcal{Q}) = (n+1)/\tau$ for *any* $\tau > 1 + 1/n$ whenever $\nu\left(\mathbb{N}^*\backslash\mathcal{Q}\right) < 1$. On the other hand, when $\mathcal{Q}$ is an $\mathbb{N}^*\backslash\mathcal{Q}$–free set as defined in Chapter 3, the result of B. Wang, Z. Wen and J. Wu [197] mentioned in [Chap. 3, p.94] shows that $\dim W^*_{\tau,1}(\mathcal{Q}) = 2/\tau$ in dimension one ($\tau > 2$), even in the case when $\nu\left(\mathbb{N}^*\backslash\mathcal{Q}\right) = 1$. Their proof however strongly rests on the property defining an $\mathbb{N}^*\backslash\mathcal{Q}$–free set $\mathcal{Q}$.

  These considerations could lead one to conjecture that the Hausdorff dimension of the liminf set $W^*_{\tau,n}(\mathcal{Q})$ ($n \geq 1$) might depend, for $\tau$ smaller than a critical value $\tau_0 \leq 2+1/n$, not only on the "size" of the set $\mathcal{Q}$ measured by its exponent of convergence, but also on some of its arithmetical properties.

- An intriguing problem would be to establish under which conditions on the infinite subset $\mathcal{Q} \subset \mathbb{N}^*$ the one–dimensional liminf set $W^*_{\tau,1}(\mathcal{Q})$ is non–empty when $\tau$ is less than the Dirichlet bound (i.e. when $\tau \leq 2$). The case $\tau = 2$ is already interesting in its own right as it is covered by none of the results seen in Chapter 3 or in this one.

# Chapter 5

# Rational Approximation and arithmetic Progressions

**Abstract**

A reasonably complete theory of the approximation of an irrational by rational fractions whose numerators and denominators lie in prescribed arithmetic progressions is developed in this chapter. Results are both, on the one hand, from a metrical and a non–metrical point of view and, on the other, from an asymptotic and also a uniform point of view. The principal novelty is a Khintchine type theorem for uniform approximation in this context. Some applications of this theory are also discussed.

## 5.1 Introduction

Let $\xi$ denote an irrational number.

The celebrated theorem of Dirichlet in Diophantine approximation asserts that, for any real number $Q \geq 1$, there exist integers $p, q \in \mathbb{Z}$ such that

$$\left|\xi - \frac{p}{q}\right| \leq \frac{1}{qQ} \quad \text{and} \quad 1 \leq q \leq Q. \tag{5.1}$$

This *uniform* version implies in particular an *asymptotic* one, namely the fact that there exist arbitrarily large integer values of $q$ such that the inequality $|\xi - p/q| < q^{-2}$ holds for some integer $p$ depending on $q$. Hurwitz [120] has shown that the stronger inequality

$$\left|\xi - \frac{p}{q}\right| \leq \frac{1}{\sqrt{5}q^2}$$

happens infinitely often and that the constant $1/\sqrt{5}$ in the right–hand side could not be chosen any smaller for the result to hold true for all irrationals.

In general, establishing a result concerning asymptotic approximation (and, *a fortiori*, uniform approximation) when the numerators and/or the denominators of the rational approximants lie in given infinite sets turns out to be difficult (see [113, Chap. 4] or [103] and the references therein for some examples). For more details about the concepts of uniform and asymptotic approximations, the reader is referred to [Prolegomena, subsection 0.2.1]. This chapter is concerned with the case when both the numerators and the denominators belong to prescribed arithmetic progressions. The results known so far in this context (which will be recalled), whether they are metrical, non–metrical, uniform or asymptotic, are very incomplete.

First some notation is fixed : throughout this chapter, $a, b, r$ and $s$ will refer to integers satisfying the constraints

$$a \geq 1, \quad b \geq 1, \quad 0 \leq r \leq a-1 \quad \text{and} \quad 0 \leq s \leq b-1. \tag{5.2}$$

The problem under consideration amounts to finding rational approximations to an irrational $\xi$ with numerators (resp. denominators) of the form $am+r$ (resp. $bn+s$) for integers $n$ and $m$. Note that the case $r = s = 0$ is settled in a straightforward manner : applying Dirichlet's and Hurwitz's theorems to the irrational $b\xi/a$, it is easy to see that, on the one hand, for any integer $Q \geq b$, there exist integers $m$ and $n$ such that

$$\left|\xi - \frac{am}{bn}\right| \leq \frac{ab}{(bn)Q} \quad \text{and} \quad 1 \leq bn \leq Q \tag{5.3}$$

and that, on the other, there exist infinitely many integers $m$ and $n$ such that the inequality

$$\left|\xi - \frac{am}{bn}\right| \leq \frac{ab}{\sqrt{5}(bn)^2} \tag{5.4}$$

holds true infinitely often, the constant $(ab)/\sqrt{5}$ being optimal uniformly in $\xi \in \mathbb{R}\backslash\mathbb{Q}$. As will be apparent from the various results stated below, the fact that the constant $ab$ in the right–hand side of (5.3) may be chosen uniformly in $\xi \in \mathbb{R}\backslash\mathbb{Q}$ is typical of the "homogeneous" case $r = s = 0$.

It is stressed that not all the theorems in this introduction are stated in full generality in order to keep the discourse coherent with respect to the problem under consideration.

### 5.1.1 The theory of asymptotic approximation

The first result deals with non–metrical asymptotic approximation.

**Theorem 5.1.1.** *Given an irrational $\xi$, there exist infinitely many integers $m$ and $n$ such that*

$$\left|\xi - \frac{am+r}{bn+s}\right| \leq \frac{ab}{4(bn+s)^2} \tag{5.5}$$

*provided that $(r, s) \neq (0, 0)$.*

This theorem has already been proved in some particular cases, for example with the additional constraint $a = b$ (cf. [187]) or with a constant weaker than $(ab)/4$ on the right–hand side of (5.5) (cf. [114]). See also [89] and the references therein for further details and partial results in this direction.

**Remark 5.1.2.** Given the trivial relation $|u/v - p/q| \geq 1/(vq)$ satisfied by any two distinct rationals $u/v$ and $p/q$, an inequality as in (5.5) can be satisfied by a rational $\xi = u/v$ infinitely often if, and only if, there exists $\alpha \in \mathbb{Z}$ (and hence infinitely many of those) such that $\alpha u \equiv r \pmod{a}$ and $\alpha v \equiv s \pmod{b}$; that is, from Lemma 5.3.2 in subsection 5.3.1 below, if, and only if, the three conditions $\gcd(bu, av) \,|\, (us - vr)$, $\gcd(u, a) \,|\, r$ and $\gcd(v, b) \,|\, s$ are simultaneously met.

The next theorem deals with asymptotic approximation from a metrical point of view : it provides a Khintchine type result in the setup under consideration. In what follows, $\lambda$ denotes the one–dimensional Lebesgue measure. As usual, a set is said to be of *full measure* if the measure of its complement is null.

**Theorem 5.1.3.** *Let* $\Psi\ : [1, \infty) \to (0, \infty)$ *be a non–increasing continuous function. Set*

$$\mathcal{K}(\Psi) := \left\{ \xi \in \mathbb{R} \ : \ \left| \xi - \frac{am + r}{bn + s} \right| \ < \ \Psi(bn + s) \ i.o. \right\},$$

*where "i.o." stands for "infinitely often".*

*Then,*

$$\lambda\left(\mathcal{K}(\Psi)\right) = \begin{cases} \texttt{ZERO} \quad \text{if } \sum_{n=1}^{\infty} n\Psi(bn + s) < \infty, \\ \\ \texttt{FULL} \quad \text{if } \sum_{n=1}^{\infty} n\Psi(bn + s) = \infty. \end{cases}$$

*Furthermore, the result still holds if the additional condition* $\gcd(am + r, bn + s) = \gcd(a, b, r, s)$ *is also imposed in the definition of the set* $\mathcal{K}(\Psi)$.

In the case that congruential constraints are imposed only on the denominators of the approximants (which corresponds to the case $a = 1$ and $r = 0$ in our setup), Theorem 5.1.3 follows without much difficulty from the theorem of Duffin and Schaeffer in Diophantine approximation (cf. [Prolegomena, Theorem 0.2.6, p.17]) as noticed by S. Hartman and Szüsz in [116]. On the other hand, in the case that both the numerators and the denominators belong to pre–assigned arithmetic progressions, the question was studied by G. Harman in [109] from the perspective of counting the number of solutions to Diophantine inequalities. Therefore, the main novelty in Theorem 5.1.3 is the fact that the result holds with the extra condition $\gcd(am + r, bn + s) = \gcd(a, b, r, s)$, which was a question left unanswered in [109]. It should be noted that the main feature of the proof of Theorem 5.1.3 consists of establishing the optimal regularity of the set $\{(am + r)/(bn + s)\}_{n,m \in \mathbb{Z}}$ in $\mathbb{R}$. While this is a result interesting in its own right that can be used to simplify a great deal of G. Harman's proof, it does not follow in the same way as the optimal regularity of the rationals in $\mathbb{R}$ as soon as $r \neq 0$ or $s \neq 0$ (see subsection 5.2.2 for definitions and details).

An application of the Mass Transference Principle (see [Prolegomena, Theorem 0.1.10, p.12]) allows one to translate Theorem 5.1.3 into a result on the Hausdorff measures and dimension of the set $\mathcal{K}(\Psi)$. Here, $\mathcal{H}^t$ stands for the $t$–dimensional Hausdorff measure and dim for the Hausdorff dimension.

**Corollary 5.1.4.** *Let $t \in (0,1)$. Then, under the assumptions of Theorem 5.1.3,*

$$\mathcal{H}^t\left(\mathcal{K}(\Psi)\right) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} n\Psi(bn+s)^t < \infty, \\ \\ \infty & \text{if } \sum_{n=1}^{\infty} n\Psi(bn+s)^t = \infty. \end{cases}$$

*In particular,* $\dim\left(\mathcal{K}(\Psi)\right) = \inf\left\{t > 0 \ : \ \sum_{n=1}^{\infty} n\Psi(bn+s)^t < \infty\right\}$.

*This result still holds with the additional condition* $\gcd(am+r, bn+s) = \gcd(a,b,r,s)$ *in the definition of the set $\mathcal{K}(\Psi)$.*

### 5.1.2 The theory of uniform approximation

Even though the introduction of the concept of *hat exponent* (see [Prolegomena, subsection 0.2.1]) has made the distinction between uniform problems and asymptotic problems more systematic in Diophantine approximation, results on uniform approximation under constraints remain quite rare in the literature : one can for instance mention the recent work of Chan in [60] on uniform approximation by sums of two rationals or the work of Dodson, Rynne and Vickers showing in [79] that if $\mathcal{M} \subset \mathbb{R}^k$ $(k \geq 3)$ belongs to a general class of smooth manifolds then, for almost all points lying on $\mathcal{M}$ (with respect to the induced measure), Dirichlet's Theorem cannot be infinitely improved in some sense made precise in the paper.

However, in the case that the numerators and the denominators of the approximants are subject to congruential constraints as those under consideration so far, a reasonably complete theory of uniform approximation can be established, both from a metrical and a non–metrical point of view. This is the subject of this subsection. To this end, a few definitions are first introduced.

**Definition 5.1.5.** *Given a function $\Psi : [1,\infty) \to (0,\infty)$, a real number $\xi$ is said to admit a* $\Psi$–uniform $(a,b,r,s)$–approximation *if there exists $Q_0 \geq 1$ such that, for any integer $Q \geq Q_0$, there are integers $m$ and $n$ such that*

$$\left|\xi - \frac{am+r}{bn+s}\right| \leq \frac{\Psi(Q)}{bn+s} \quad \textit{and} \quad 1 \leq bn+s \leq Q.$$

*The set of real numbers admitting a $\Psi$–uniform $(a,b,r,s)$–approximation will be denoted by $\mathcal{U}(\Psi)$.*

*Furthermore, $\xi \in \mathbb{R}$ will be said to admit a uniform $(a,b,r,s)$–approximation with exponent $\mu \in [0,1]$ if there exists $c > 0$ such that $\xi \in \mathcal{U}\left(Q \mapsto cQ^{-\mu}\right)$.*

From a non–metrical point of view, a necessary and sufficient condition, explicit in terms of the continued fraction expansion, can be given for an irrational $\xi$ to be uniformly approximable at order $\Psi$ up to an explicit constant depending on $\xi$ (that is, for there to exist $c := c(\xi)$

such that $\xi \in \mathcal{U}(c\Psi)$). In what follows, the sequence of the partial quotients of $\xi$ (resp. of its convergents) will be denoted by $(a_k(\xi))_{k\geq 0}$ or by $(a_k)_{k\geq 0}$ for the sake of simplicity (resp. by $(p_k(\xi)/q_k(\xi))_{k\geq 0}$ or by $(p_k/q_k)_{k\geq 0}$), with $a_0 \in \mathbb{Z}$ and $a_k \geq 1$ for $k \geq 1$. The necessary and sufficient condition is technical by nature and is concerned with the indices $k \geq 1$ for which the relations

$$\gcd(p_{k-1}, a) \,|\, r, \quad \gcd(q_{k-1}, b) \,|\, s \quad \text{and} \quad \gcd(bp_{k-1}, aq_{k-1}) \,|\, (sp_{k-1} - rq_{k-1}) \tag{5.6}$$

are not simultaneously satisfied.

**Theorem 5.1.6.** *Let $\xi$ be an irrational number given by its continued fraction expansion $\xi = [a_0; a_1, \dots]$. Let $\Psi : [1, \infty) \to (0, \infty)$ be a continuous non–increasing function. Set*

$$\widetilde{\Psi} : Q \in [1, \infty) \mapsto Q\Psi(Q) \in (0, \infty) \tag{5.7}$$

*and assume that there exist $\gamma > 0$, $\kappa \geq 1$ and $\eta \geq 1$ satisfying*

$$\inf_{Q\geq 1} \widetilde{\Psi}(Q) \geq \gamma, \quad \widetilde{\Psi}(Q) \leq \kappa\widetilde{\Psi}(2Q) \quad \text{and} \quad \widetilde{\Psi}(Q) \leq \eta\widetilde{\Psi}(ab(Q+1)) \quad \text{for all } Q \geq 1. \tag{5.8}$$

*Then there exists a constant $c := c(\xi) > 0$ such that $\xi \in \mathcal{U}(c\Psi)$ if, and only if, there exists an integer $M \geq 1$ such that for all indices $k \geq 1$ for which conditions (5.6) are* **not** *met, one has $a_k \leq M\widetilde{\Psi}(q_k)$.*

*Furthermore,*

$$c(\xi) = 8(ab)^2\kappa\eta \max\{4M, \gamma^{-1}\} \quad \text{and} \quad Q_0 = ab \tag{5.9}$$

*are admissible values, where $Q_0$ is the parameter introduced in Definition 5.1.5.*

**Remark 5.1.7.**

- The existence of $\kappa$ together with the assumption of the monotonicity of $\Psi$ actually implies the existence of $\eta$ in (5.8). However, the explicit presence of these two constants makes the definition of $c(\xi)$ in (5.9) more effective.

- Conditions (5.8) should be seen as an attempt to remove any assumption of monotonicity on the function $\widetilde{\Psi}$ : indeed, it is easily checked that they are automatically satisfied if $\widetilde{\Psi}$ is assumed to be non–decreasing (with $\gamma = \widetilde{\Psi}(1)$ and $\kappa = \eta = 1$).

- The existence of the constants $\kappa$ and $\eta$ (which is ensured for a fairly large class of functions — for instance any function rational in $\log Q$ and $Q$) means that the function $\widetilde{\Psi}$ does not admit abrupt variations. It is a weaker assumption than the usual one when trying to remove the assumption of monotonicity : transposed in this context, the latter would ask that, for every $c > 1$, $\widetilde{\Psi}(cQ) < c\widetilde{\Psi}(Q)$ for all $Q \geq 1$ (see, e.g., [57] and [71, §4.1]).

- The existence of the constant $\gamma$ is a relatively mild restriction. Indeed, it is well–known that if a real number $\alpha$ satisfies Dirichlet's theorem with $(2Q)^{-1}$ as the approximating function instead of $Q^{-1}$ (that is, if the right–hand side of the first inequality in (5.1) is replaced with $(2qQ)^{-1}$), then $\alpha$ has to be rational (see the discussion held after [Prolegomena, Theorem 0.0.1, p.2]). This implies in particular that the function $\widetilde{\Psi}$ in Theorem 5.1.6 cannot tend to zero.

- Condition (5.6) obviously holds in the "homogeneous case" $r = s = 0$, in which case one finds again the aforementioned result on uniform approximation with exponent 1 where the constant $c(\xi) = ab$ was proved to be admissible for all $\xi \in \mathbb{R}\backslash\mathbb{Q}$.
- Any badly approximable number has uniformly bounded partial quotients regardless of whether condition (5.6) is met or not. Therefore, all badly approximable numbers admit a uniform $(a, b, r, s)$–approximation with exponent 1. This shows in particular that the set of real numbers for which Dirichlet's theorem holds up to a constant in the context of $(a, b, r, s)$–approximation has full Hausdorff dimension. In the case of badly approximable numbers, the existence of a uniform $(a, b, r, s)$–approximation with exponent 1 will be shown to be a direct consequence of the Three Distance Theorem in subsection 5.3.2.
- It will be clear that the proof of Theorem 5.1.6 can be adapted to show that, given any $\mu \in (0, 1]$, there always exists an irrational $\xi$ such that $\xi$ does not admit a uniform $(a, b, r, s)$–approximation with exponent $\mu$ as soon as $r \neq 0$ or $s \neq 0$.

From a metrical point of view, the only known result in the context of uniform $(a, b, r, s)$–approximation seems to be that of S. Hartman who proved in [115] that almost no real number satisfies Dirichlet's theorem if the denominators of the approximants are prescribed to be odd. The following corollary of Theorem 5.1.6, which is very much the main result of this chapter, provides a reasonably complete answer to this problem. It constitutes the first example of a Khintchine type result in the context of uniform approximation. The reader should note the differences with respect to a standard Khintchine type result such as Theorem 5.1.3.

**Corollary 5.1.8.** *Let $\Psi : [1, \infty) \to (0, \infty)$ be a continuous non–increasing function such that the function $\widetilde{\Psi}$ as defined by (5.7) is non–decreasing.*

*If $r \neq 0$ or $s \neq 0$, then*

$$\lambda\left(\mathcal{U}\left(\Psi\right)\right) = \begin{cases} \texttt{ZERO} & \text{if } \sum_{Q=1}^{\infty} \frac{1}{Q^2\Psi(Q)} = \infty \\ \\ \texttt{FULL} & \text{if } \sum_{Q=1}^{\infty} \frac{1}{Q^2\Psi(Q)} < \infty. \end{cases}$$

Thus, as soon as $r \neq 0$ or $s \neq 0$, almost no real number admits a uniform $(a, b, r, s)$–approximation with exponent 1. This also holds true if one takes $\Psi(Q) = \log Q/Q$ as the approximating function. On the other hand, almost all real numbers belong to the set $\bigcap_{n=1}^{\infty} \mathcal{U}\left(Q \mapsto (\log Q)^{1+1/n} Q^{-1}\right)$.

The chapter is organized as follows : the results on asymptotic approximation (Theorems 5.1.1 and 5.1.3 and Corollary 5.1.4) will first be proved in section 5.2. Then proofs for Theorem 5.1.6 and Corollary 5.1.8, dealing with uniform approximation, will be provided in section 5.3. Finally, various applications of Diophantine approximation with congruential constraints on both the numerator and the denominator of the approximants will be mentioned in section 5.4. In particular, applications to the estimate of some trigonometrical functions and to so–called visibility problems in geometry will be considered.

## 5.2 Proofs of the results related to asymptotic approximation

Theorem 5.1.1, Theorem 5.1.3 and Corollary 5.1.4 are proved in this section.

### 5.2.1 Non–metrical point of view

We first begin with a proof of Theorem 5.1.1. This can actually be seen as a consequence of Minkowski's theorem on the product of two linear forms (see, e.g., [57, Theorem 1 p.46]).

*Proof of Theorem 5.1.1.* Let $\xi \in \mathbb{R}\backslash\mathbb{Q}$. Consider the linear forms $L_1(x, y) = by$ and $L_2(x, y) = b\xi y - ax$ with determinant $\Delta = -ab$ and set $\eta := s$ and $\nu := s\xi - r$. From Minkowski's theorem on the product of two linear forms, there exist integers $m$ and $n$ such that

$$|L_1(m, n) + \eta| \, . \, |L_2(m, n) + \nu| \; = \; |bn + s| \, . \, |\xi(bn + s) - (am + r)| \; \leq \; \frac{ab}{4} \cdot$$

As $\xi$ is irrational, given $\epsilon > 0$, one can furthermore add the constraint that

$$|L_2(m, n) + \nu| \; := \; |(b\xi n - am) + (s\xi - r)| \; < \; \epsilon$$

(see, e.g., [57, Theorem 1, p.46] for details). Since $\nu := s\xi - r \notin b\xi\mathbb{Z} + a\mathbb{Z}$, one gets infinitely many pairs of integers $(m, n) \in \mathbb{Z}^2$ satisfying (5.5) by letting $\epsilon$ tend to zero. **Q.E.D.**

**Remark 5.2.1.** Theorem 5.1.1 can be generalized to the case of inhomogeneous approximation in the following way : for any $\xi \in \mathbb{R}\backslash\mathbb{Q}$ and any $\alpha \in \mathbb{R}$, there exist infinitely many pairs $(m, n) \in \mathbb{Z}^2$ such that the inequality

$$|\xi(bn + s) - (am + r) + \alpha| \; \leq \; \frac{ab}{4\,|bn + s|}$$

holds if $s\xi + r + \alpha \notin b\xi\mathbb{Z} + a\mathbb{Z}$ (this follows readily from the previous proof). If, however, $s\xi + r + \alpha \in b\xi\mathbb{Z} + a\mathbb{Z}$, the situation is essentially the same as the "homogeneous" case $r = s = 0$ and it is easily seen, using for instance (5.4), that the result still holds upon choosing some constant bigger than $ab/4$ depending on $\alpha$ in the right–hand side of the inequality.

### 5.2.2 Metrical point of view

A proof is now provided for Theorem 5.1.3. The notation from this theorem is kept in this subsection. Since the set $\mathcal{K}\,(\Psi)$ is clearly invariant by translation by a multiple of the integer $a$, it suffices to establish the Khintchine type result for the set $\mathcal{K}\,(\Psi) \cap (0, a)$ which, for the sake of simplicity of notation, will still be denoted by $\mathcal{K}\,(\Psi)$ in what follows.

The convergence part of Theorem 5.1.3 can be obtained in a classical way as a consequence of the Borel–Cantelli lemma : details are left to the reader (see, e.g., [51, p.13]). In order to prove the divergence part, the concept of an *optimal regular system* is introduced. Recall that $\lambda$ denotes the one–dimensional Lebesgue measure.

**Definition 5.2.2.** *Let $E \subset \mathbb{R}$ be a bounded open interval and let $\mathcal{S} := (\alpha_j)_{j\geq 1}$ denote a sequence of distinct real numbers.*

*The sequence $\mathcal{S}$ is an* optimal regular system of points in $E$ *if there exist positive constants $c_1$ and $c_2$ depending only on $\mathcal{S}$ and, for any interval $I$ contained in $E$, a number $K_0$ depending on $\mathcal{S}$ and $I$ such that the following property holds : for any $K \geq K_0$, there exist integers $1 \leq i_1 < \cdots < i_t \leq K$ with $\alpha_{i_h} \in I$ for $h = 1, \ldots, t$ satisfying*

$$|\alpha_{i_h} - \alpha_{i_l}| \geq \frac{c_1}{K} \quad \text{for} \quad 1 \leq h \neq l \leq t \quad \text{and} \quad t \geq c_2 \lambda(I) K.$$

The next theorem, due to Beresnevich in [13, 14] (see also [51, Chap. 6]), shows that the set of real numbers close to infinitely many points in an optimal regular system satisfies the divergent part of a Khintchine type statement.

**Theorem 5.2.3** (Beresnevich)**.** *Let $E$ be a bounded interval and let $\mathcal{S} := (\alpha_j)_{j\geq 1}$ denote an optimal regular system in $E$. Given a non–increasing continuous function $\Psi : [1, \infty) \to (0, \infty)$, define the set $\mathcal{K}_{\mathcal{S}}(\Psi)$ as*

$$\mathcal{K}_{\mathcal{S}}(\Psi) := \limsup_{j\to\infty} \left\{ \xi \in E \ : \ |\xi - \alpha_j| < \Psi(j) \right\}.$$

*Then the set $\mathcal{K}_{\mathcal{S}}(\Psi)$ has full Lebesgue measure if the sum $\sum_{j\geq 1} \Psi(j)$ diverges.*

**Remark 5.2.4.** This divergence statement holds even if the set $\mathcal{S}$ is *regular* without being *optimal*. See [20] for further details[1].

Let

$$\mathcal{S} := (0, a) \cap \left\{ \frac{am + r}{bn + s} \right\}_{m,n\geq 0}. \tag{5.10}$$

The goal is to prove that $\mathcal{S}$ is an optimal regular system in the interval $E := (0, a)$. Here, the elements of $\mathcal{S}$ are ordered by increasing denominator and, for two elements of $\mathcal{S}$ with the same denominator, by increasing numerator in such a way that the divergence part of Theorem 5.1.3 will follow at once from Theorem 5.2.3.

It does not seem to be straightforward that the optimal regularity of $\mathcal{S}$ in $E$ can be obtained in the same way as the optimal regularity of the rationals in the unit interval as established by Bugeaud in [51, Proposition 5.3] : indeed, Bugeaud's argument strongly rests on considerations of length combined with Dirichlet's theorem applied to each irrational in the unit interval. In this case however, it follows from Corollary 5.1.8 that a Dirichlet type result is satisfied by almost no irrational if $r \neq 0$ or $s \neq 0$.

In order to establish the optimal regularity of $\mathcal{S}$ with respect to $E$, two preliminary lemmata are first required. For the classical results related to some arithmetical functions mentioned in the proofs, see, e.g., [108].

[1]This remark is due to an anonymous referee who gave valuable comments on the paper [A5] referenced in the List of Publications p.xi.

**Lemma 5.2.5.** *Let $q \geq 1$ be an integer such that* $\gcd(a,r,q) = 1$.

*Then*

$$\sum_{\substack{0 \leq am+r \leq x \\ \gcd(am+r,q)=1}} 1 \;=\; x\,\frac{\gcd(q,a)}{qa}\,\varphi\!\left(\frac{q}{\gcd(q,a)}\right) + O\left(2^{\omega(q)}\right),$$

*where $\varphi$ denotes Euler's totient function and $\omega(q)$ the number of distinct prime divisors of $q$ and where the implicit constant depends only on $a$.*

*Proof.* Let $\mu(\,.\,)$ denote the Möbius function. Since for any integer $n \geq 1$, $\sum_{d|n} \mu(d)$ equals 1 if $n = 1$ and 0 otherwise, one gets, for $x \geq a$, denoting by $\lfloor\,.\,\rfloor$ the floor function,

$$\begin{aligned}
\sum_{\substack{0 \leq am+r \leq x \\ \gcd(am+r,q)=1}} 1 &= \sum_{0 \leq am+r \leq x} \; \sum_{d|\gcd(am+r,q)} \mu(d) \\
&= \sum_{d|q} \mu(d) \sum_{\substack{0 \leq m \leq (x-r)/a \\ am \equiv -r \pmod{d}}} 1 \;=\; \sum_{\substack{d|q \\ \gcd(d,a)|r}} \mu(d) \sum_{\substack{0 \leq m \leq (x-r)/a \\ am \equiv -r \pmod{d}}} 1 \\
&= \sum_{\substack{d|q \\ \gcd(d,a)|r}} \mu(d) \left(\frac{1}{d}\left\lfloor \frac{x-r}{a} \right\rfloor \gcd(d,a) + O\left(\gcd(d,a)\right)\right) \\
&= \frac{x}{a} \sum_{\substack{d|q \\ \gcd(d,a)|r}} \frac{\mu(d)\gcd(d,a)}{d} + O\left(\sum_{\substack{d|q \\ \gcd(d,a)|r}} \mu(d)\gcd(d,a)\right).
\end{aligned}$$

Now, on the one hand,

$$\left|\sum_{\substack{d|q \\ \gcd(d,a)|r}} \mu(d)\gcd(d,a)\right| \;\leq\; a \sum_{d|q} |\mu(d)| \;=\; a2^{\omega(q)},$$

which provides the error term in the conclusion of the lemma. On the other, any integer $d$ dividing $q$ can be written in a unique way in the form $d = kl$ with $k|\frac{q}{\gcd(q,a)}$ and $l|\gcd(q,a)$ with $\gcd(k,l) = 1$. Therefore, from the multiplicativity of the Möbius function,

$$\begin{aligned}
\sum_{\substack{d|q \\ \gcd(d,a)|r}} \frac{\mu(d)\gcd(d,a)}{d} &= \sum_{\substack{k|\frac{q}{\gcd(q,a)} \\ l|\gcd(q,a),\, l|r}} \frac{\mu(k)\mu(l)\gcd(l,a)}{kl} \\
&= \left(\sum_{k|\frac{q}{\gcd(q,a)}} \frac{\mu(k)}{k}\right) \cdot \left(\sum_{\substack{l|\gcd(q,a) \\ l|r}} \frac{\mu(l)\gcd(l,a)}{l}\right) \\
&= \frac{\varphi\left(\frac{q}{\gcd(q,a)}\right)}{\frac{q}{\gcd(q,a)}} \left(\sum_{l|\gcd(q,a,r)} \mu(l)\right) \;=\; \varphi\left(\frac{q}{\gcd(q,a)}\right)\frac{\gcd(q,a)}{q},
\end{aligned}$$

where the second last equation follows from the well–known fact that

$$\sum_{k|q'} \frac{\mu(k)}{k} = \frac{\varphi(q')}{q'} \tag{5.11}$$

for all $q' \geq 1$ and the last equation from the assumption that $\gcd(q,a,r) = 1$. This completes the proof. **Q.E.D.**

The second lemma generalizes the classical estimate

$$\sum_{k=1}^{Q} \varphi(k) \ = \ \frac{Q^2}{2\zeta(2)} + O(Q \log Q),$$

a proof of which can for instance be found in [108, Theorem 330].

**Lemma 5.2.6.** *Let $u \geq 1$ and $v \geq 0$ be integers and let $Q \geq u$ be a real number. Then*

$$\sum_{1 \leq uk+v \leq Q} \varphi(uk+v) \ = \ C(u,v)Q^2 + O(Q \log Q),$$

*where the implicit constant depends only on $u$ and $v$ and where*

$$C(u,v) = \frac{\varphi(\gcd(u,v))}{\gcd(u,v)} \left( 2u\zeta(2) \prod_{\substack{\pi \ prime \\ \pi | u}} \left(1 - \frac{1}{\pi^2}\right) \right)^{-1}.$$

*Proof.* Assume that $Q = uk + v$ for some integer $k \geq 1$. It is clearly sufficient to establish the result in this case. Then, if $v \neq 0$,

$$\sum_{1 \leq ul+v \leq Q} \varphi(ul+v) \ = \ \sum_{l=0}^{k} \varphi(ul+v) \underset{(5.11)}{=} \sum_{l=0}^{k} (ul+v) \sum_{d|(ul+v)} \frac{\mu(d)}{d} \cdot \tag{5.12}$$

If $v = 0$, the last two sums should start with $l = 1$. To avoid cumbersome notation, the proof will be given in the case $v \neq 0$ and the reader can easily check that it remains valid if $v = 0$ with very little modification.

The relation $d|(ul+v)$ means that there exists $d' \in \mathbb{Z}$ such that $dd' - ul = v$. This last Diophantine equation is solvable in $(d', l) \in \mathbb{Z}^2$ if, and only if, $\delta := \gcd(d,u)|v$, in which case any solution is of the form

$$(d', l) \ = \ \left(d'_0 + \frac{u}{\delta}t, -l_0 + \frac{d}{\delta}t\right),$$

where $t \in \mathbb{Z}$ and $(d'_0, -l_0)$ is a particular solution. Then the constraint $0 \leq l \leq k$ amounts to the following one : $\frac{\delta}{d} l_0 \leq t \leq (k + l_0)\frac{\delta}{d}$. Thus, (5.12) becomes :

$$\sum_{l=0}^{k} \varphi(ul+v) \ = \ \sum_{\substack{1 \leq dd' \leq uk+v \\ dd' \equiv v \pmod u}} d'\mu(d)$$

$$= \sum_{\delta \mid \gcd(u,v)} \sum_{\substack{1 \le d \le uk+v \\ \gcd(d,u)=\delta}} \sum_{\frac{\delta l_0}{d} \le t \le (k+l_0)\frac{\delta}{d}} \left( d_0' + \frac{ut}{\delta} \right) \mu(d)$$

$$= \sum_{\delta \mid \gcd(u,v)} \sum_{\substack{1 \le d \le uk+v \\ \gcd(d,u)=\delta}} \mu(d) \left( \sum_{\frac{\delta l_0}{d} \le t \le (k+l_0)\frac{\delta}{d}} \left( d_0' + \frac{ut}{\delta} \right) \right)$$

$$= \sum_{\delta \mid \gcd(u,v)} \sum_{\substack{1 \le d \le uk+v \\ \gcd(d,u)=\delta}} \mu(d) \left( \frac{u\delta}{2} \left( \frac{k}{d} \right)^2 + O\left( \frac{k\delta}{d} \right) \right)$$

$$= \sum_{\delta \mid \gcd(u,v)} \left( \sum_{\substack{1 \le d \le uk+v \\ \gcd(d,u)=\delta}} \frac{\mu(d)}{d^2} \right) \frac{u\delta k^2}{2} + O\left( \sum_{\delta \mid \gcd(u,v)} k\delta \sum_{\substack{1 \le d \le uk+v \\ \gcd(d,u)=\delta}} \frac{1}{d} \right), \tag{5.13}$$

where the error term in this last equation is clearly $O\left(k \log k\right)$. Now, on the one hand,

$$\sum_{\substack{1 \le d \le uk+v \\ \gcd(d,u)=\delta}} \frac{\mu(d)}{d^2} = \sum_{\substack{d=1 \\ \gcd(d,u)=\delta}}^{\infty} \frac{\mu(d)}{d^2} - \sum_{\substack{d=uk+v+1 \\ \gcd(d,u)=\delta}}^{\infty} \frac{\mu(d)}{d^2}$$

and, on the other,

$$\left| \sum_{\substack{d=uk+v+1 \\ \gcd(d,u)=\delta}}^{\infty} \frac{\mu(d)}{d^2} \right| \le \sum_{d=uk+v+1}^{\infty} \frac{1}{d^2} = O\left( \frac{1}{k} \right),$$

hence, from (5.13),

$$\sum_{l=0}^{k} \varphi(ul+v) = \sum_{\delta \mid \gcd(u,v)} \left( \sum_{\substack{d=1 \\ \gcd(d,u)=\delta}}^{\infty} \frac{\mu(d)}{d^2} \right) \frac{u\delta k^2}{2} + O\left(k \log k\right). \tag{5.14}$$

Since $\frac{\mu(d)}{d^2}$ is a multiplicative function, the series appearing on the right–hand side of this equation can be simplified. Indeed, assume first that $\delta = 1$. Then the expansion in the Euler product of the series under consideration gives

$$\sum_{\substack{d=1 \\ \gcd(d,u)=1}}^{\infty} \frac{\mu(d)}{d^2} = \prod_{\substack{\pi \text{ prime} \\ \gcd(\pi,u)=1}} \left( 1 + \sum_{l=1}^{\infty} \frac{\mu\left(\pi^l\right)}{\pi^{2l}} \right)$$

$$= \prod_{\substack{\pi \text{ prime} \\ \gcd(\pi,u)=1}} \left( 1 - \frac{1}{\pi^2} \right) = \left( \zeta(2) \prod_{\substack{\pi \text{ prime} \\ \pi \mid u}} \left( 1 - \frac{1}{\pi^2} \right) \right)^{-1}.$$

If, now, $\delta \ge 2$ is a divisor of $\gcd(u,v)$, let $d$ be a square–free integer such that $\gcd(d,u) = \delta$. Write $d = \delta d'$ in such a way that $d'$ is a square–free integer satisfying $\gcd(d',\delta) = 1$ and therefore $\gcd(d', u) = 1$. Then,

$$\sum_{\substack{d=1 \\ \gcd(d,u)=\delta}}^{\infty} \frac{\mu(d)}{d^2} = \sum_{\substack{d'=1 \\ \gcd(d',u)=1}}^{\infty} \frac{\mu(\delta d')}{(\delta d')^2} = \frac{\mu(\delta)}{\delta^2} \sum_{\substack{d'=1 \\ \gcd(d',u)=1}}^{\infty} \frac{\mu(d')}{(d')^2} = \frac{\mu(\delta)}{\delta^2} \left( \zeta(2) \prod_{\substack{\pi \text{ prime} \\ \pi | u}} \left(1 - \frac{1}{\pi^2}\right) \right)^{-1} .$$

Setting $\mu(1) = 1$ and combining this with (5.14), one gets, in the case that $Q = uk + v$ for some $k \geq 1$,

$$\sum_{0 \leq ul+v \leq Q} \varphi(ul + v) = (Q - v)^2 \left( 2u\zeta(2) \prod_{\substack{\pi \text{ prime} \\ \pi | u}} \left(1 - \frac{1}{\pi^2}\right) \right)^{-1} \sum_{\delta | \gcd(u,v)} \frac{\mu(\delta)}{\delta} + O\left(Q \log Q\right),$$

which completes the proof from (5.11). **Q.E.D.**

*Completion of the proof of Theorem 5.1.3.* The optimal regularity of the subset $\mathcal{S}'$ of $\mathcal{S}$ (defined by (5.10)) made up of fractions of the form $(am + r)/(bn + s)$ satisfying $\gcd(am + r, bn + s) = \gcd(a, b, r, s)$ will now be established. It should be clear that it may be assumed, without loss of generality, that $\gcd(a, b, r, s) = 1$.

Let us first prove the existence of a subsequence of the sequence $(bn + s)_{n\geq 0}$ of the form $(un+v)_{n\geq 0}$ ($u, v \geq 0$ integers) such that $\gcd(un+v, a, r) = 1$ for all $n \geq 1$ if $\delta := \gcd(a, r) > 1$. Under the assumption that $\gcd(\delta, b, s) = 1$, a prime divisor $\pi$ of $\delta$ cannot divide both $b$ and $s$. It is therefore possible to fix an integer $n_\pi$ defined modulo $\pi$ such that $bn_\pi + s \not\equiv 0 \pmod{\pi}$ (set for example $n_\pi \equiv (1 - s)b^{-1} \pmod{\pi}$ if $\gcd(\pi, b) = 1$ and $n_\pi \equiv 1 \pmod{\pi}$ otherwise). From the Chinese remainder theorem, there exists an integer $n_0$, defined uniquely modulo $\prod_{\pi|\delta} \pi$ (the product is taken over prime numbers), such that $n_0 \equiv n_\pi \pmod{\pi}$ for all primes $\pi$ dividing $\delta$. Then, set $u := b\prod_{\pi|\delta} \pi$ and $v := bn_0 + s$ in such a way that $(uk + v)_{k\geq 0} = \left(b\left(n_0 + k\prod_{\pi|\delta} \pi\right) + s\right)_{k\geq 0}$. It is then clear that for any element $N$ of the sequence $(uk+v)_{k\geq 0}$, $\gcd(N, \delta) = 1$ since for all prime divisor $\pi$ of $\delta$,

$$N \equiv bn_0 + s \equiv bn_\pi + s \not\equiv 0 \pmod{\pi}.$$

Let $I = (\alpha, \beta)$ (with $\alpha < \beta$) denote an open interval contained in $(0, a)$. Consider the set of all elements of the sequence $(un + v)_{n\geq 0}$ which lie in the interval $[Q/2, Q]$, where $Q \geq u$ is a real number. It follows from Lemma 5.2.5 that, for a fixed $n \geq 1$, the number of integers $m \geq 0$ such that $\gcd(am + r, un + v) = 1$ and $(am + r)/(un + v) \in I$ is

$$\sum_{\substack{\alpha q < am+r < \beta q \\ \gcd(am+r,q)=1}} 1 = \lambda(I)\frac{\gcd(q, a)}{a}\varphi\left(\frac{q}{\gcd(q, a)}\right) + O(2^{\omega(q)}),$$

where $q := un + v$. From the well–known estimate $2^{\omega(q)} = o\left(q^{\epsilon}\right)$ valid for all $\epsilon > 0$, for $Q$ (and therefore for $q \geq Q/2$) large enough depending only on $a$, $r$ and $\lambda(I)$, this last quantity is such that

$$\sum_{\substack{\alpha q < am+r < \beta q \\ \gcd(am+r,q)=1}} 1 \;\geq\; \lambda(I)\frac{\varphi(q)}{2a}, \tag{5.15}$$

where we used the fact that $\varphi\left(\frac{q}{\gcd(q,a)}\right) \geq \frac{\varphi(q)}{\gcd(q,a)}$.

Define now ${\mathcal{S}'}_Q(I)$ as the subset of $\mathcal{S}'$ made up of all those irreducible fractions in $I$ of the form $(am+r)/(un+v)$ and such that $Q/2 \leq un+v \leq Q$ : the distance between two distinct elements $(am+r)/(un+v)$ and $(am'+r)/(un'+v)$ of ${\mathcal{S}'}_Q(I)$ satisfies the inequality

$$\left|\frac{am+r}{un+v} - \frac{am'+r}{un'+v}\right| \;\geq\; \frac{1}{(un+v)(un'+v)} \;\geq\; \frac{1}{Q^2}.$$

Moreover, it follows from (5.15) that the cardinality $\#{\mathcal{S}'}_Q(I)$ of the set ${\mathcal{S}'}_Q(I)$ satisfies the estimate

$$\#{\mathcal{S}'}_Q(I) \;\geq\; \frac{\lambda(I)}{2a}\sum_{Q/2\leq un+v\leq Q}\varphi(un+v).$$

Therefore, from Lemma 5.2.6, for $Q$ large enough depending only on $a$, $u$, $v$ and $\lambda(I)$,

$$\#{\mathcal{S}'}_Q(I) \;\geq\; \frac{C(u,v)}{4a}\lambda(I)Q^2.$$

Up to constants, $\lambda(I)Q^2$ elements of ${\mathcal{S}'}_Q(I) \subset \mathcal{S}'$ have been found in $I$ such that the gap between any two of them is $Q^{-2}$. Furthermore, from the indexing adopted for $\mathcal{S}$ (which is also used for $\mathcal{S}'$), it should be clear that the largest index of an element of ${\mathcal{S}'}_Q(I)$ is at most $aQ^2$. Since this holds for all $Q$ large enough (depending only on $\mathcal{S}'$ and $I$), it is easy to see that Definition 5.2.2 applies.

This completes the proof of the optimal regularity of the subset of $\mathcal{S}'$ and therefore of Theorem 5.1.3. **Q.E.D.**

The Mass Transference Principle, due to S. Velani and V. Beresnevich, allows one to deduce Corollary 5.1.4 from Theorem 5.1.3 without much difficulty. Here, the result of [Prolegomena, Theorem 0.1.10, p.12] is not given in full generality but adapted to our purpose.

**Theorem 5.2.7** (Mass Transference Principle)**.** *Let $\Omega$ be a compact interval in $\mathbb{R}$ with non–empty interior and let $t \in (0,1)$. Denote by $(J_i)_{i\geq 0}$ a sequence of intervals in $\Omega$ whose lengths tend to zero as $i$ tends to infinity. For any interval $J$ centered at $x \in \Omega$ with half–length $r$, denote by $J^t$ the interval centered at $x$ with half–length $r^t$. Assume furthermore that*

$$\lambda\left(\limsup_{i\to\infty} J_i^t\right) \;=\; \lambda\left(\Omega\right). \tag{5.16}$$

*Then*

$$\mathcal{H}^t\left(\limsup_{i\to\infty} J_i\right) \;=\; \mathcal{H}^t\left(\Omega\right) \;=\; \infty.$$

*Deduction of Corollary 5.1.4 from Theorem 5.2.7.* Let $t \in (0,1)$. If the sum $\sum_{n=1}^{\infty} n\Psi(bn+s)^t$ converges, a standard covering argument shows that $\mathcal{H}^t\left(\mathcal{K}\left(\Psi\right)\right) = 0$ : here again, details are left to the reader.

Assume now that the sum $\sum_{n=1}^{\infty} n\Psi(bn+s)^t$ diverges and recall that the set $\mathcal{K}(\Psi)$ has been restricted without loss of generality to the interval $(0,a)$. Set $\Omega = [0,a]$ in the assumptions of Theorem 5.2.7 and chose $(J_i)_{i\geq 0}$ as being the sequence of all those intervals contained in $\Omega$ of length $2\Psi(bn+s)$ and centered at rationals of the form $(am+r)/(bn+s)$ with $\gcd(am+r, bn+s) = \gcd(a,b,r,s)$. These intervals are indexed in the usual way (see after (5.10)). Then, condition (5.16) is met from the divergence part of Theorem 5.1.3 so that applying Theorem 5.2.7 completes the proof. **Q.E.D.**

## 5.3 Proofs of the results related to uniform approximation

This section is devoted to the proofs of Theorem 5.1.6 and Corollary 5.1.8. Throughout, conditions (5.2) will be strengthened in assuming, without loss of generality from the discussion held in the introduction, that

$$r \neq 0 \quad \text{or} \quad s \neq 0. \tag{5.17}$$

First, some auxiliary results, dealing mainly with properties of continued fraction expansions, are recalled.

### 5.3.1 Some properties of the continued fraction expansion of an irrational

The next lemma collects some well–known properties of the continued fraction expansion of an irrational.

**Lemma 5.3.1.** *Let $\xi$ be an irrational number with partial quotients $(a_k)_{k\geq 0}$ and convergents $(p_k/q_k)_{k\geq 0}$. Set conventionally $p_{-1}=1$, $q_{-1}=0$, $p_0=a_0$ and $q_0=1$.*

*Then :*

1. *For any $k \geq 0$.*

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k. \tag{5.18}$$

   *In particular, $p_k$ and $q_k$ are coprime.*

2. *The numerators and the denominators of the convergents of $\xi$ satisfy the recurrence relations*

$$p_k = a_k p_{k-1} + p_{k-2} \quad \textit{and} \quad q_k = a_k q_{k-1} + q_{k-2} \tag{5.19}$$

   *valid for all $k \geq 1$.*

3. *For any $k \geq 0$,*

$$\frac{1}{a_{k+1}+1} < \frac{q_k}{q_{k+1}} = [0; a_{k+1}, a_{k+2}, \dots] < \frac{1}{a_{k+1}}. \tag{5.20}$$

4. *For any $k \geq 1$, set*

$$\phi_k := \frac{q_k\xi - p_k}{q_{k-1}\xi - p_{k-1}}.$$

*Then,* $\phi_k < 0$,

$$1 + a_{k+1}\phi_k = \phi_k\phi_{k+1} \quad \textit{and} \quad |\phi_k| < \frac{1}{a_{k+1}}\cdot \tag{5.21}$$

5. *For any integer* $k \geq -1$, *set*

$$\eta_k := (-1)^k\left(q_k\xi - p_k\right).$$

*Then, the sequence* $(\eta_k)_{k\geq -1}$ *is positive and decreasing and, furthermore,*

$$\frac{1}{2} \leq \frac{q_{k+1}}{q_k + q_{k+1}} \leq \eta_k q_{k+1} \leq 1 \tag{5.22}$$

*for all* $k \geq -1$.

6. *Let* $k \geq 1$, $a_0 \in \mathbb{Z}$ *and* $a_1, \dots, a_k \geq 1$ *be integers. Let furthermore* $E(a_0, a_1, \dots, a_k)$ *denote the set of real numbers whose* $k+1$ *first partial quotients are* $a_0, a_1, \dots, a_k$. *Then,*

$$E(a_0, a_1, \dots, a_k) = \begin{cases} \left[\frac{p_k}{q_k}, \frac{p_k+p_{k-1}}{q_k+q_{k-1}}\right) & \textit{if } k \textit{ is even} \\ \left(\frac{p_k+p_{k-1}}{q_k+q_{k-1}}, \frac{p_k}{q_k}\right] & \textit{if } k \textit{ is odd,} \end{cases}$$

*where* $p_{k-1}/q_{k-1} = [a_0; a_1, \dots, a_{k-1}]$ *and* $p_k/q_k = [a_0; a_1, \dots, a_k]$. *In particular,*

$$\frac{1}{2q_k^2} \leq \lambda\left(E(a_0, a_1, \dots, a_k)\right) = \frac{1}{q_k(q_k + q_{k-1})} \leq \frac{1}{q_k^2}\cdot \tag{5.23}$$

7. *For any* $k \geq 1$,

$$\prod_{j=1}^{k} a_j \leq q_k \leq \prod_{j=1}^{k}(a_j + 1) \leq 2^k \prod_{j=1}^{k} a_j, \tag{5.24}$$

$$(1 + a_0a_1)\prod_{j=2}^{k} a_j \leq p_k \leq (1 + a_0a_1)\prod_{j=2}^{k}(a_j + 1) \leq 2^{k-1}(1 + a_0a_1)\prod_{j=2}^{k} a_j. \tag{5.25}$$

8. *Let* $d, t, u \geq 1$ *be integers and let* $\boldsymbol{k} \in (\mathbb{N}^*)^d$. *Denote by* $E_{\boldsymbol{k}}^{(t)}$ *the set of all those irrationals* $\xi$ *in the interval* $[t, t+1]$ *such that* $(a_0(\xi), a_1(\xi), \dots, a_d(\xi)) = (t, \boldsymbol{k})$. *Let* $E_{\boldsymbol{k},u}^{(t)}$ *denote the subset of* $E_{\boldsymbol{k}}^{(t)}$ *made up of all those irrationals* $\xi$ *such that* $a_{d+1}(\xi) = u$. *Then*

$$\frac{1}{3u^2} < \frac{\lambda\left(E_{\boldsymbol{k},u}^{(t)}\right)}{\lambda\left(E_{\boldsymbol{k}}^{(t)}\right)} < \frac{2}{u^2}\cdot \tag{5.26}$$

*If* $t = 0$, *the set* $E_{\boldsymbol{k}}^{(t)}$ *(resp.* $E_{\boldsymbol{k},u}^{(t)}$*) will be more conveniently denoted by* $E_{\boldsymbol{k}}$ *(resp. by* $E_{\boldsymbol{k},u}$*).*

*Proof.* Inequalities (5.24) and (5.25) can easily be obtained by induction from relations (5.19). All the other results are standard. See, e.g., [51, Chap. 1] for proofs or [57, Chap. 1]. **Q.E.D.**

The following generalizes a result well–known in the case $a_1 = a_2 = 1$. The proof, which is elementary, is left to the reader.

**Lemma 5.3.2.** *For $i = 1, 2$, let $a_i \geq 0, b_i \geq 0$ and $m_i \geq 1$ denote natural integers. Then, the system of equations*

$$\begin{cases} a_1 x \equiv b_1 \pmod{m_1} \\ a_2 x \equiv b_2 \pmod{m_2} \end{cases}$$

*admits a solution $x \in \mathbb{Z}$ if, and only if,*

$$\gcd(m_1, a_1) \,|\, b_1, \quad \gcd(m_2, a_2) \,|\, b_2 \quad \text{and} \quad \gcd(a_1 m_2, a_2 m_1) \,|\, a_1 b_2 - a_2 b_1.$$

### 5.3.2 Non–metrical point of view

It is remarkable that, in the case where $\xi$ is a badly approximable irrational number, the result of Theorem 5.1.6 can be generalized by proving that $\xi$ admits an inhomogeneous uniform $(a, b, r, s)$–approximation with exponent 1. In what follows, **Bad** denotes the set of badly approximable irrationals.

**Proposition 5.3.3.** *Let $\xi \in$ **Bad** and $\alpha \in \mathbb{R}$.*

*Then, there exists a constant $c(\xi) > 0$ such that, for all real numbers $Q \geq 2b$, there are integers $m$ and $n$ satisfying*

$$|(bn + s)\xi - (am + r) + \alpha| \leq \frac{c(\xi)}{Q} \quad \text{and} \quad 0 \leq bn + s \leq Q.$$

*Furthermore, $c(\xi) = 2ab(M + 2)$ is an admissible value, where $M$ is an upper bound for the partial quotients of $b\xi/a$.*

Proposition 5.3.3 will follow without much difficulty from the *Three Distance Theorem*, also referred to in the literature as *the Steinhaus Theorem*, the *Three Length*, *Three Gap* or *Three Step Theorem*. The latter states that, for any positive integer $Q$ and for any irrational $\xi$, the points $(\{i\xi\})_{0 \leq i \leq Q}$ partition the unit interval into $Q+1$ subintervals, the lengths of which take at most three values, one being the sum of the other two (here, $\{x\}$ denotes the fractional part of a real number $x$). The reader is referred to [2] for a complete survey on the topic and to the references therein for various proofs of the precise statement of the result given below. The latter uses the fact that, for any integer $Q \geq 1$, there exist unique integers $k \geq 1$, $p$ and $w$ such that

$$Q = pq_{k-1} + q_{k-2} + w \quad \text{with} \quad 1 \leq p \leq a_k \quad \text{and} \quad 0 \leq w < q_{k-1}, \tag{5.27}$$

where $(q_k)_{k \geq 0}$ is the sequence of the denominators of the convergents of a given irrational $\xi$. Such a decomposition can be obtained thanks to the greedy algorithm. The notation introduced in Lemma 5.3.1 is kept in the statement of the Three Distance Theorem, in particular see (5.22) for the definition of $\eta_k$.

**Theorem 5.3.4** (The Three Distance Theorem)**.** *Let $\xi$ be an irrational and let $Q \geq 1$ be a positive integer given in the form (5.27).*

*Then, the unit interval is divided by the points $0$, $\{\xi\}$, $\{2\xi\}, \ldots, \{Q\xi\}$ into $Q+1$ subintervals which satisfy the following conditions :*

- $Q+1-q_{k-1}$ *of them have length* $\eta_{k-1}$,
- $w+1$ *have length* $\eta_{k-2}-p\eta_{k-1}$,
- $q_{k-1}-(w+1)$ *have length* $\eta_{k-2}-(p-1)\eta_{k-1}$.

**Remark 5.3.5.** As $\xi$ is irrational, the three lengths are distinct. The third length, which is the largest since it is the sum of the other two, does not always appear. The other two do always appear.

*Proof of Proposition 5.3.3 from Theorem 5.3.4.* In this proof, $(a_k)_{k\geq 0}$ (resp. $(p_k/q_k)_{k\geq 0}$) refers to the sequence of the partial quotients (resp. of the convergents) of the irrational $b\xi/a$, where $\xi \in \mathbf{Bad}$. The integer $M$ denotes an upper bound for the sequence $(a_k)_{k\geq 0}$.

Let $Q \geq 1$ be an integer and let $k \geq 1$, $p$ and $w$ be integers as given by (5.27). From Theorem 5.3.4, the unit interval is partitioned by the numbers $(\{ib\xi/a\})_{0\leq i\leq Q}$ into $Q+1$ subintervals of lengths at most $\eta_{k-2}$. Modulo $a$ this is saying that the point $s\xi - r + \alpha$ lies within a distance $a\eta_{k-2}/2$ from $b\xi n$ for some integer $n$ in the interval $[\![0, Q]\!]$. In other words, there exist $m \in \mathbb{Z}$ and $n \in [\![0, Q]\!]$ such that

$$\left|(b\xi n - am) + (s\xi - r + \alpha)\right| \;\leq\; \frac{a\eta_{k-2}}{2} \qquad \text{and} \qquad 0 \;\leq\; bn+s \;\leq\; bQ+s \;\leq\; 2bQ,$$

whence

$$\begin{aligned} Q\left|\xi\,(bn+s) - (am+r) + \alpha\right| \;&\leq\; \frac{a}{2}Q\eta_{k-2} \\ &\underset{(5.22)\&(5.27)}{\leq} \frac{a}{2}\,(a_k+2) \\ &\leq\; \frac{a}{2}(M+2). \end{aligned}$$

Assume now that $Q \geq 2b$ is a real number and set $Q' = \lfloor Q/(2b)\rfloor \geq 1$ : from what precedes, there exist integers $m$ and $n$ such that $0 \leq bn+s \leq 2bQ' \leq Q$ and

$$Q\left|\xi\,(bn+s) - (am+r) + \alpha\right| \;\leq\; \frac{Q}{Q'}\frac{a}{2}(M+2) \;\leq\; \left(1+\frac{1}{Q'}\right)ab(M+2) \;\leq\; 2ab(M+2).$$

This completes the proof of Proposition 5.3.3. **Q.E.D.**

The rest of this subsection is devoted to the proof of Theorem 5.1.6, where the notation introduced in Lemma 5.3.1 will be systematically used with respect to a fixed $\xi \in \mathbb{R}\backslash\mathbb{Q}$. To this end, first note that, given $m, n \in \mathbb{Z}$ and $k \geq 1$, it follows from (5.18) that there exists a unique pair $(u, v) \in \mathbb{Z}^2$, given by

$$u \;=\; (-1)^{k-1}\left[(am+r)q_{k-1} - (bn+s)p_{k-1}\right] \tag{5.28}$$

and

$$v \;=\; (-1)^{k-1}\left[(bn+s)p_{k-2} - (am+r)q_{k-2}\right], \tag{5.29}$$

such that

$$am + r \;=\; up_{k-2} + vp_{k-1}, \tag{5.30}$$

and

$$bn + s \;=\; uq_{k-2} + vq_{k-1}. \tag{5.31}$$

Furthermore, in this case, on noticing that

$$\frac{|\xi(bn+s) - (am+r)|}{q_{k-1}\,|q_{k-2}\xi - p_{k-2}|} \;=\; \frac{1}{q_{k-1}} \cdot \frac{|\xi\,(uq_{k-2} + vq_{k-1}) - (up_{k-2} + vp_{k-1})|}{|q_{k-2}\xi - p_{k-2}|} \;=\; \frac{1}{q_{k-1}}\,|u + v\phi_{k-1}|\,,$$

where $\phi_{k-1}$ has been defined in (5.21), inequalities (5.22) imply that

$$\frac{1}{2q_{k-1}}\,|u + v\phi_{k-1}| \;<\; |\xi(bn+s) - (am+r)| \;<\; \frac{1}{q_{k-1}}\,|u + v\phi_{k-1}|\,. \tag{5.32}$$

*Proof of the necessary part of Theorem 5.1.6.* Assume that there exists a strictly increasing sequence $(k_l)_{l\geq 0}$ of natural integers such that conditions (5.6) are not met for the index $k_l$ and such that

$$\lim_{l\to\infty} \frac{a_{k_l}}{\widetilde{\Psi}\,(q_{k_l})} \;=\; \infty. \tag{5.33}$$

For a contradiction, assume that there is a $Q_0 \geq 1$ such that for each integer $Q \geq Q_0$, there exist $m, n \in \mathbb{Z}$ satisfying

$$0 \leq bn + s \leq Q \quad \text{ and } \quad \Psi(Q)^{-1}\,|\xi(bn+s) - (am+r)| \;\leq\; c(\xi) \tag{5.34}$$

for some constant $c(\xi) > 0$. Assuming without loss of generality that $k_0$ has been chosen in such a way that $q_{k_0}/2 \geq Q_0$, set furthermore, for all $l \geq 0$,

$$Q_{k_l} := \left\lceil \frac{q_{k_l}}{2} \right\rceil,$$

where $\lceil\,.\,\rceil$ denotes the ceiling function. Let $l \geq 0$ and $m$ and $n$ be integers verifying (5.34) for the integer $Q_{k_l}$. It then follows from (5.29) that

$$\begin{aligned}
|v| \;=\; q_{k_l-2}\left|(am+r) - (bn+s)\frac{p_{k_l-2}}{q_{k_l-2}}\right| \underset{(5.34)}{\leq}\; & q_{k_l-2}\,|\xi(bn+s) - (am+r)| \\
& + q_{k_l-2}\left|\xi - \frac{p_{k_l-2}}{q_{k_l-2}}\right| Q_{k_l} \\
\underset{(5.22)\&(5.34)}{\leq}\; & c(\xi) q_{k_l-2}\Psi\left(\left\lceil\frac{q_{k_l}}{2}\right\rceil\right) + \frac{\lceil q_{k_l}/2\rceil}{q_{k_l-1}} \\
\underset{(\Psi\text{ decreases})}{\leq}\; & c(\xi) q_{k_l-2}\Psi\left(\frac{q_{k_l}}{2}\right) + \frac{q_{k_l}/2 + 1}{q_{k_l-1}} \\
\underset{(5.20)}{\leq}\; & c(\xi)\frac{q_{k_l-2}}{q_{k_l}/2}\widetilde{\Psi}\left(\frac{q_{k_l}}{2}\right) + \frac{1}{2}(a_{k_l} + 1) + \frac{1}{q_{k_l-1}} \\
\underset{(5.8)}{=}\; & O\left(\widetilde{\Psi}\,(q_{k_l})\right) + \frac{a_{k_l}}{2} + O\,(1)\,.
\end{aligned}$$

Therefore,

$$|v\phi_{k_l-1}| \underset{(5.21)}{\leq} \frac{O\left(\widetilde{\Psi}\left(q_{k_l}\right)\right) + a_{k_l}/2 + O\left(1\right)}{a_{k_l}} \underset{(5.33)}{=} 1/2 + o(1).$$

On the other hand, the integer $u$ cannot equal zero in the representations (5.30) and (5.31) : indeed, this would otherwise contradict the fact that conditions (5.6) are not met for the index $k_l$ from Lemma 5.3.2.

Thus, since $|u| \geq 1$, one gets :

$$\begin{aligned}
\Psi\left(Q_{k_l}\right)^{-1}|\xi(bn+s)-(am+r)| &\underset{(\Psi \text{ decreases})}{\geq} \Psi\left(\frac{q_{k_l}}{2}\right)^{-1}|\xi(bn+s)-(am+r)| \\
&\underset{(5.32)}{\geq} \widetilde{\Psi}\left(\frac{q_{k_l}}{2}\right)^{-1}\frac{q_{k_l}/2}{2q_{k_l-1}}\left(|u|-|v\phi_{k_l-1}|\right) \\
&\underset{(5.8)}{\geq} \left(\kappa\widetilde{\Psi}\left(q_{k_l}\right)\right)^{-1}\frac{q_{k_l}}{4q_{k_l-1}}\left(1-\frac{1}{2}+o(1)\right) \\
&\underset{(5.20)}{\geq} \frac{1}{4\kappa}\widetilde{\Psi}\left(q_{k_l}\right)^{-1}a_{k_l}\left(\frac{1}{2}+o(1)\right),
\end{aligned}$$

which, from (5.33), contradicts (5.34) for $l$ large enough and completes the proof. **Q.E.D.**

The proof of the sufficiency of the conditions attached to (5.6) in Theorem 5.1.6 is more involved.

*Proof of the sufficient part of Theorem 5.1.6.* Assume that $Q \geq 1$ is an integer written in the form (5.27) for some integers $k \geq 1$, $p$ and $w$. From (5.30), (5.31) and (5.32), the problem comes down to proving the existence of integers $u$ and $v$ (and therefore $m$ and $n$) such that an upper bound depending only on $a$,$b$, $\xi$ and $\Psi$ might be found for the quantity

$$\widetilde{\Psi}(Q)^{-1}\frac{Q}{q_{k-1}}|u+v\phi_{k-1}|$$

under the constraint $0 \leq bn+s \leq Q$.

To this end, set $d := \gcd(bp_{k-1}, aq_{k-1})$ and consider the unique integer $u$ lying in the interval $[\![0, d-1]\!]$ which satisfies the congruence

$$u \equiv (-1)^{k-1}\left(rp_{k-1} - sq_{k-1}\right) \pmod{d}. \tag{5.35}$$

Since $p_{k-1}$ and $q_{k-1}$ are coprime, one has in fact

$$0 \leq u \leq ab. \tag{5.36}$$

Furthermore, under these assumptions, the equation

$$u - (-1)^{k-1}\left(rp_{k-1} - sq_{k-1}\right) = (-1)^{k-1}\left(aq_{k-1}m - bp_{k-1}n\right) \tag{5.37}$$

is solvable in $(m,n)\in\mathbb{Z}^2$ and the set of all solutions can be written in the form

$$\left(m_0-(-1)^{k-1}\frac{bp_{k-1}}{d}h\,;\,n_0-(-1)^{k-1}\frac{aq_{k-1}}{d}h\right),$$

where $h\in\mathbb{Z}$ and $(m_0,n_0)\in\mathbb{Z}^2$ is a particular solution. This implies that there exists a unique pair $(m,n)\in\mathbb{Z}^2$ satisfying (5.37) with the additional constraint $0\leq n<aq_{k-1}/d$. For such a pair, it should be clear that

$$0\;\leq\;bn+s\;\leq\;baq_{k-1}+s\underset{(5.27)}{\leq}\min\{abQ,2abq_{k-1}\}. \tag{5.38}$$

On the other hand, eliminating $am+r$ in equations (5.28) and (5.29) gives

$$v=(-1)^{k-1}\left(\frac{p_{k-2}}{q_{k-2}}-\frac{p_{k-1}}{q_{k-1}}\right)q_{k-2}(bn+s)-u\frac{q_{k-2}}{q_{k-1}},$$

whence

$$|u+v\phi_{k-1}|\;<\;|u|\,.\left|1-\phi_{k-1}\frac{q_{k-2}}{q_{k-1}}\right|+|bn+s|\,.\,|\phi_{k-1}|\,.\left|\frac{p_{k-2}}{q_{k-2}}-\frac{p_{k-1}}{q_{k-1}}\right|q_{k-2}.$$

Taking into account (5.18), (5.21), (5.36) and (5.38), this leads to the inequality

$$|u+v\phi_{k-1}|\;<\;2ab+2ab\;=\;4ab. \tag{5.39}$$

Since the function $\Psi$ is non–increasing and since $Q\leq q_k+q_{k-1}\leq 2q_k$ from (5.27), for such a choice of the integers $u$ and $v$ (and therefore, of the integers $m$ and $n$), one has :

$$\begin{aligned}\Psi(Q)^{-1}\left|\xi(bn+s)-(am+r)\right|\;&\leq\;\Psi(2q_k)^{-1}\left|\xi(bn+s)-(am+r)\right|\\&=\;2q_k\widetilde{\Psi}(2q_k)^{-1}\left|\xi(bn+s)-(am+r)\right|\\&\underset{(5.32)}{\leq}\;\frac{2q_k}{q_{k-1}}\widetilde{\Psi}(2q_k)^{-1}\left|u+v\phi_{k-1}\right|\\&\underset{(5.20)}{\leq}\;2\left(a_k+1\right)\widetilde{\Psi}(2q_k)^{-1}\left|u+v\phi_{k-1}\right|\\&\underset{(5.39)}{\leq}\;16ab\,a_k\widetilde{\Psi}(2q_k)^{-1}\underset{(5.8)}{\leq}\;16\kappa ab\,a_k\widetilde{\Psi}(q_k)^{-1}.\end{aligned} \tag{5.40}$$

Now, if conditions (5.6) are not satisfied, this last quantity is less than $16\kappa abM$ for some integer $M\geq 1$. If, however, conditions (5.6) are met, instead of choosing $u$ according to the constraints (5.35) and (5.36), set $u=0$. Then, from Lemma 5.3.2, there exist $v\in[\![0,ab]\!]$ and $(m,n)\in\mathbb{Z}^2$ such that

$$vp_{k-1}=am+r\quad\text{ and }\quad vq_{k-1}=bn+s,$$

in which case $0\leq bn+s\leq abQ$ and, repeating the above calculations,

$$\begin{aligned}\Psi(Q)^{-1}\left|\xi(bn+s)-(am+r)\right|\;&\leq\;2\left(a_k+\frac{1}{q_{k-1}}\right)\widetilde{\Psi}(2q_k)^{-1}\left|v\phi_{k-1}\right|\\&\underset{(5.8)}{\leq}\;4\kappa ab\,a_k\widetilde{\Psi}(q_k)^{-1}\left|\phi_{k-1}\right|\end{aligned}$$

$$\underset{(5.8)\&(5.21)}{\leq} 4\kappa ab\gamma^{-1}.$$

Thus, it has been proved that for all integers $Q \geq 1$, there exists $(m,n) \in \mathbb{Z}^2$ such that $\Psi(Q)^{-1}\left|\xi(bn+s)-(am+r)\right| \leq 4\kappa ab \max\{4M, \gamma^{-1}\}$ under the constraint $0 \leq bn+s \leq abQ$. Assume now that $Q \geq ab$ is any real number and set $Q' := \lfloor Q/(ab)\rfloor \geq 1$. Then there exist integers $m$ and $n$ such that $0 \leq bn+s \leq abQ' \leq Q$ and

$$\begin{aligned}
\Psi(Q)^{-1}\left|\xi(bn+s)-(am+r)\right| &\leq 4\kappa ab \max\{4M, \gamma^{-1}\}\frac{\Psi(Q')}{\Psi(Q)}\\
&\underset{(\Psi \text{ decreases})}{\leq} 4\kappa ab \max\{4M, \gamma^{-1}\}\frac{\Psi(Q')}{\Psi(ab(Q'+1))}\\
&= 4\kappa(ab)^2 \max\{4M, \gamma^{-1}\}\left(1+\frac{1}{Q'}\right)\frac{\widetilde{\Psi}(Q')}{\widetilde{\Psi}(ab(Q'+1))}\\
&\underset{(5.8)}{\leq} 8\kappa\eta(ab)^2 \max\{4M, \gamma^{-1}\}.
\end{aligned}$$

This completes the proof of Theorem 5.1.6. **Q.E.D.**

**Remark 5.3.6.** If there exist integers $M \geq 1$ and $k_0 \geq 1$ such that, for all $k \geq k_0$, the inequality $a_k \leq M\widetilde{\Psi}(q_k)$ holds whenever conditions (5.6) are not met, then the conclusion of Theorem 5.1.6 remains true upon choosing $Q_0 = ab(q_{k_0-1}+q_{k_0-2})$ (which quantity equals $ab$ when $k_0 = 1$).

Indeed, the previous proof applies with the exception that the upper bound $16\kappa abM$ used for the right–hand side of (5.40) when conditions (5.6) are not satisfied is only valid when $k \geq k_0$. From the uniqueness of the decomposition (5.27), this imposes the condition $Q \geq q_{k_0-1}+q_{k_0-2}$. Therefore, in the last step of the proof, the integer $Q' := \lfloor Q/(ab)\rfloor$ will be asked to be bigger than $q_{k_0-1}+q_{k_0-2}$, hence the choice of $Q_0$ in this case.

### 5.3.3 Metrical point of view

This subsection is devoted to the proof of Corollary 5.1.8. Throughout, the result will be established in the case when $s \neq 0$ : it is not difficult to verify that the reasoning below can easily be modified to obtain the same result in the case when $s = 0$ and $r \neq 0$ working with the numerators of the convergents rather than with the denominators.

Consider a function $\Psi$ satisfying the assumptions of Corollary 5.1.8. Since $\widetilde{\Psi}$ is non–decreasing, it is clear that conditions (5.8) are satisfied, so that the conclusions of Theorem 5.1.6 hold. In what follows, the metrical result of Corollary 5.1.8 will be proved for the set $\mathcal{U}(\Psi)\cap[0,a]$ which, for the sake of simplicity, will still be denoted by $\mathcal{U}(\Psi)$ : it should be clear that this suffices to establish Corollary 5.1.8 in full generality.

More precisely, it will be shown that :

a) if the sum $\sum_{Q\geq 1}\frac{1}{Q^2\Psi(Q)}$ converges, then, for almost all $\xi \in [0,a]\backslash\mathbb{Q}$, there exists an integer $k_0(\xi) \geq 1$ such that, for all $k \geq k_0(\xi)$, $a_k(\xi) \leq \tau\widetilde{\Psi}\left(q_k(\xi)\right)$. The fact that $\lambda\left(\mathcal{U}\left(\Psi\right)\right) = a$ will then follow from Theorem 5.1.6 and Remark 5.3.6 for a suitable choice of the parameter $\tau > 0$.

b) if the sum $\sum_{Q\geq 1}\frac{1}{Q^2\Psi(Q)}$ diverges, then the set of $\xi\in[0,1]\backslash\mathbb{Q}$ such that, for all integers $M\geq 1$, there exist infinitely many indices $k\geq 1$ such that $b|q_{k-1}(\xi)$ and $a_k(\xi)\geq M\widetilde{\Psi}\left(q_k(\xi)\right)$ has strictly positive measure. By virtue of (5.9) in Theorem 5.1.6, an element $\xi$ belonging to the latter set cannot belong to the set

$$\mathcal{V}\left(\Psi\right):=\bigcup_{c\geq 1}\mathcal{U}\left(c\Psi\right) \tag{5.41}$$

whose complement has therefore strictly positive measure. Showing that $\mathcal{V}\left(\Psi\right)$ has either zero or full measure will then complete the proof in this case also.

The proof of Corollary 5.1.8 requires a Borel–Berstein type technical lemma on continued fractions.

#### 5.3.3.1 A Borel–Berstein type technical lemma on continued fractions

The classical theorem of Borel–Bernstein on continued fractions states that, given a sequence $(u_k)_{k\geq 1}$ of positive integers, if the sum $\sum_{k\geq 1}u_k^{-1}$ diverges, then, for almost all $\xi:=[0;a_1,a_2,\dots]$ in $[0,1)$, there exist infinitely many integers $k\geq 1$ such that $a_k\geq u_k$. Further, if the sum converges, then, for almost all $\xi:=[0;a_1,a_2,\dots]$ in $[0,1)$, there exist only a finite number of integers $k\geq 1$ such that $a_k\geq u_k$ (see, e.g., [51, Theorem 1.11] for a proof). The following generalizes the Borel–Bernstein Theorem and is the key step in proving Corollary 5.1.8.

**Lemma 5.3.7.** *Let $A\geq 1$ and $d\geq 1$ be integers. Denote by $\boldsymbol{f}:=(\boldsymbol{f}_k)_{k\geq 1}$ a sequence of functions such that, for every $k\geq 1$, the function*

$$\boldsymbol{f}_k\ :\xi\in[0,1]\backslash\mathbb{Q}\ \mapsto\ \boldsymbol{f}_k(\xi)\in[\![1,A]\!]^d$$

*is measurable. Assume furthermore that $\varphi:=(\varphi_k)_{k\geq 1}$ is a sequence of positive integers for which there exists an integer $c\geq d+1$ such that the two series $\sum_{k=0}^{\infty}\varphi_k$ and $\sum_{k=0}^{\infty}\varphi_{ck}$ converge (resp. diverge) simultaneously. For any $k\geq 1$, define the sets*

$$E_k^d\left(\boldsymbol{f}_k,\varphi_k\right)\ :=\ \left\{\xi\in[0,1]\backslash\mathbb{Q}\ :\ \left(a_k,a_{k+1},\dots,a_{k+d-1}\right)=\boldsymbol{f}_k\left(\xi\right)\ \text{ and }\ a_{k+d}\geq\varphi_k\right\}$$

*and*

$$\mathcal{S}^d\left(\boldsymbol{f},\varphi\right)\ :=\ \limsup_{k\to\infty}E_k^d\left(\boldsymbol{f}_k,\varphi_k\right).$$

*Then*

$$\lambda\left(\mathcal{S}^d\left(\boldsymbol{f},\varphi\right)\right)\begin{cases}=0 & \text{if } \sum_{k=0}^{\infty}\varphi_k^{-1}\ <\ \infty,\\ \geq\frac{\log 2}{4(2(2A)^d)^4} & \text{if } \sum_{k=0}^{\infty}\varphi_k^{-1}\ =\ \infty.\end{cases}$$

**Remark 5.3.8.** The assumption of the existence of the constant $c$ is a restriction of a technical nature : as will be clear from the proof, it plays no role but to ensure that for an element $x$ lying in the intersection $E_{ck}^d\left(\boldsymbol{f}_{ck},\varphi_{ck}\right)\cap E_{cl}^d\left(\boldsymbol{f}_{cl},\varphi_{cl}\right)$, where $k$ and $l$ are two distinct positive integers, the two blocks $(a_{ck}(x),\dots,a_{ck+d}(x))$ and $(a_{cl}(x),\dots,a_{cl+d}(x))$ do not overlap.

**Notation.** In order to prove Lemma 5.3.7, the notation introduced in the statement of the result is kept. Two additional sets are defined as follows : given positive integers $k$, $d$ and $\beta$, given $\boldsymbol{\alpha} \in (\mathbb{N}^*)^d$, let

$$E_k^d(\boldsymbol{\alpha}, \beta) := \{\xi \in [0,1]\backslash\mathbb{Q} \,:\, (a_k, a_{k+1}, \ldots, a_{k+d-1}) = \boldsymbol{\alpha} \text{ and } a_{k+d} \geq \beta\}$$

and

$$\widetilde{E_k^d}(\boldsymbol{\alpha}, \beta) := \{\xi \in [0,1]\backslash\mathbb{Q} \,:\, (a_k, a_{k+1}, \ldots, a_{k+d-1}) = \boldsymbol{\alpha} \text{ and } a_{k+d} = \beta\}.$$

*Proof of the convergent part of Lemma 5.3.7.* The convergent part of Lemma 5.3.7 follows in the same way as the convergent part of the theorem of Borel–Bernstein, which in turn is nothing but a consequence of the Borel–Cantelli Lemma. Details are provided here for the sake of completeness.

Suppose $\sum_{k=0}^{\infty} \varphi_k^{-1} < \infty$ and let $E(\varphi_k) := \{\xi \in [0,1]\backslash\mathbb{Q} \,:\, a_k \geq \varphi_k\}$ $(k \geq 1)$. From the uniqueness of the continued fraction expansion of an irrational, one gets for all $k \geq 1$, using point (5.26) from Lemma 5.3.1,

$$\lambda(E(\varphi_{k+1})) = \sum_{\boldsymbol{\alpha}\in(\mathbb{N}^*)^k} \sum_{u \geq \varphi_{k+1}} \lambda(E_{\boldsymbol{\alpha},u}) \underset{(5.26)}{\leq} \sum_{\boldsymbol{\alpha}\in(\mathbb{N}^*)^k} \sum_{u \geq \varphi_{k+1}} \frac{2}{u^2}\lambda(E_{\boldsymbol{\alpha}}) = \sum_{u \geq \varphi_{k+1}} \frac{2}{u^2} \leq \frac{2}{\varphi_{k+1}}\cdot$$

Thus, the series $\sum_{k\geq 1} \lambda(E(\varphi_k))$ converges. Since $\mathcal{S}^d(\boldsymbol{f}, \varphi) \subset \limsup_{k\geq 1} E(\varphi_k)$, the result follows from the Borel–Cantelli lemma. **Q.E.D.**

**Remark 5.3.9.** The proof of the convergent part of Lemma 5.3.7 is also valid if $d = 0$ (the only condition defining the set $E_k^0(\boldsymbol{f}_k, \varphi_k)$ is then that $a_k \geq \varphi_k$), in which case the integer $c$ in the assumptions can be taken as equal to 1.

The proof of the divergence half of Lemma 5.3.7 is more involved. The use of the Gauss measure $\mu$ will make it simpler. The latter is defined for any element $E$ of the Borel $\sigma$–algebra $\mathcal{B}_{[0,1]}$ of $[0,1]$ by the formula

$$\mu(E) := \frac{1}{\log 2}\int_E \frac{\mathrm{d}x}{1+x}\cdot$$

It should be clear that

$$\frac{\lambda}{2\log 2} \leq \mu \leq \frac{\lambda}{\log 2}\cdot \tag{5.42}$$

In particular, the Lebesgue measure $\lambda$ restricted to $[0,1]$ and the Gauss measure $\mu$ are mutually absolutely continuous and therefore have the same sets of full and null measures. Define furthermore the Gauss map $T$ as follows :

$$T \,: x = [0; a_1, a_2, \ldots] \in [0,1]\backslash\mathbb{Q} \mapsto \left\{\frac{1}{x}\right\} = [0; a_2, a_3, \ldots] \in [0,1]\backslash\mathbb{Q},$$

where $\{x\}$ denotes the fractional part of a real number $x$. It is a well–known fact (see, e.g., [87, Theorem 3.7]) that the system $\left(T, \mu, \mathcal{B}_{[0,1]}\right)$ is ergodic in $[0,1]$ and hence that $\mu$ is $T$ invariant.

Two classical lemmata, which will be used in the proof of the divergent part of Lemma 5.3.7, are now introduced. The first one is essentially due to Khintchine (see, e.g., [134] or [135]).

**Lemma 5.3.10.** *Let $\boldsymbol{\alpha} \in (\mathbb{N}^*)^d$, where $d \geq 1$. Denote by $E_{\boldsymbol{\alpha}}$ the set*

$$E_{\boldsymbol{\alpha}} := \{\xi \in [0,1) \ : \ (a_1, \dots, a_d) = \boldsymbol{\alpha}\}.$$

*Let $F$ be a $\mu$–measurable set in $[0,1]$.*

*Then, there exists an absolute constant $\theta \in (0,1)$ such that for any $k \geq 0$,*

$$\mu\left(E_{\boldsymbol{\alpha}} \cap T^{-k-d}(F)\right) \ = \ \mu\left(E_{\boldsymbol{\alpha}}\right)\mu\left(F\right)\left(1 + O\left(\theta^{\sqrt{k}}\right)\right).$$

*The implicit constant in this last equation is also absolute.*

*Proof.* See [159] for an explicit proof. **Q.E.D.**

The second lemma provides a partial converse to the Borel–Cantelli lemma.

**Lemma 5.3.11.** *Let $(E_i)_{i\geq 0}$ be a sequence of $\mu$–measurable sets in $[0,1]$ such that $\sum_{i=0}^{\infty}\mu\left(E_i\right) = \infty$.*

*Then,*

$$\mu\left(\limsup_{i\to\infty} E_i\right) \ \geq \ \limsup_{i\to\infty}\left(\frac{\left(\sum_{k=1}^{i}\mu\left(E_k\right)\right)^2}{\sum_{1\leq k,l\leq i}\mu\left(E_k \cap E_l\right)}\right).$$

*Proof.* See, e.g., [51, p.125]. **Q.E.D.**

*Proof of the divergent part of Lemma 5.3.7.* Suppose $\sum_{k=0}^{\infty}\varphi_k^{-1} = \infty$. The result will be established in four steps.

**Step 1.** Given $k \geq 0$, the first step consists of finding a lower and an upper bound for $\mu\left(E_0^d\left(\boldsymbol{\alpha}, \varphi_k\right)\right)$ independently of $\boldsymbol{\alpha} \in [\![1, A]\!]^d$. To this end, first notice that, from the uniqueness of the continued fraction expansion of an irrational,

$$\mu\left(E_0^d\left(\boldsymbol{\alpha}, \varphi_k\right)\right) \ = \ \sum_{\alpha_{d+1}=\varphi_k}^{\infty}\mu\left(\widetilde{E_0^d}\left(\boldsymbol{\alpha}, \alpha_{d+1}\right)\right).$$

Now, it follows from (5.42) that, given $\alpha_{d+1} \geq \varphi_k$,

$$\frac{\lambda\left(\widetilde{E_0^d}\left(\boldsymbol{\alpha}, \alpha_{d+1}\right)\right)}{2\log 2} \ \leq \ \mu\left(\widetilde{E_0^d}\left(\boldsymbol{\alpha}, \alpha_{d+1}\right)\right) \ \leq \ \frac{\lambda\left(\widetilde{E_0^d}\left(\boldsymbol{\alpha}, \alpha_{d+1}\right)\right)}{\log 2}.$$

Furthermore, denoting $\boldsymbol{\alpha} \in [\![1, A]\!]^d$ by $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$, (5.23) and (5.24) imply that

$$\frac{1}{2\,(2A)^{2d}\,\alpha_{d+1}^2} \ \leq \ \frac{1}{2^{2d+1}\prod_{k=1}^{d+1}\alpha_k^2} \ \leq \ \lambda\left(\widetilde{E_0^d}\left(\boldsymbol{\alpha}, \alpha_{d+1}\right)\right) \ \leq \ \frac{1}{\prod_{k=1}^{d+1}\alpha_k^2} \ \leq \ \frac{1}{\alpha_{d+1}^2},$$

hence, on the one hand,

$$\mu\left(E_0^d\left(\boldsymbol{\alpha}, \varphi_k\right)\right) \ \leq \ \frac{1}{\log 2}\sum_{\alpha_{d+1}=\varphi_k}^{\infty}\frac{1}{\alpha_{d+1}^2} \ \leq \ \frac{1}{(\log 2)\,\varphi_k}$$

and, on the other,

$$\mu\left(E_0^d(\boldsymbol{\alpha},\varphi_k)\right) \;\geq\; \frac{1}{4\,(2A)^{2d}\,\log 2}\sum_{\alpha_{d+1}=\varphi_k}^{\infty}\frac{1}{\alpha_{d+1}^2} \;\geq\; \frac{1}{4\,(2A)^{2d}\,\log 2\,(\varphi_k+1)}\cdot$$

Thus, it has been proved that, for any $k \geq 0$ and any $\boldsymbol{\alpha} \in [\![1, A]\!]^d$,

$$\frac{1}{4\,(2A)^{2d}\,\log 2\,(\varphi_k+1)} \;\leq\; \mu\left(E_0^d(\boldsymbol{\alpha},\varphi_k)\right) \;\leq\; \frac{1}{(\log 2)\,\varphi_k}\cdot \tag{5.43}$$

**Step 2.** The second step consists of finding a lower bound for $\mu\left(E_{ck}^d(\boldsymbol{f}_{ck},\varphi_{ck})\right)$ for $k$ large enough depending on a fixed parameter $\epsilon \in (0,1)$.

Let $k \geq 1$ : it should be clear that

$$\begin{aligned}\mu\left(E_{ck}^d(\boldsymbol{f}_{ck},\varphi_{ck})\right) &= \sum_{\boldsymbol{\alpha}\in[\![1,A]\!]^d}\mu\left(E_{ck}^d(\boldsymbol{f}_{ck},\varphi_{ck})\cap \boldsymbol{f}_{ck}^{-1}(\{\boldsymbol{\alpha}\})\right)\\ &= \sum_{\boldsymbol{\alpha}\in[\![1,A]\!]^d}\mu\left(E_{ck}^d(\boldsymbol{\alpha},\varphi_{ck})\cap \boldsymbol{f}_{ck}^{-1}(\{\boldsymbol{\alpha}\})\right),\end{aligned}$$

whence, using the $T$ invariance of the measure $\mu$ and Lemma 5.3.10,

$$\begin{aligned}\mu\left(E_{ck}^d(\boldsymbol{f}_{ck},\varphi_{ck})\right) &= \sum_{\boldsymbol{\alpha}\in[\![1,A]\!]^d}\mu\left(E_0^d(\boldsymbol{\alpha},\varphi_{ck})\cap T^{-ck}\left(\boldsymbol{f}_{ck}^{-1}(\{\boldsymbol{\alpha}\})\right)\right)\\ &= \sum_{\boldsymbol{\alpha}\in[\![1,A]\!]^d}\mu\left(E_0^d(\boldsymbol{\alpha},\varphi_{ck})\right)\,\mu\left(\boldsymbol{f}_{ck}^{-1}(\{\boldsymbol{\alpha}\})\right)\,\left(1+O\left(\theta^{\sqrt{ck-d}}\right)\right).\end{aligned}$$

Let $\epsilon \in (0,1)$. Choose $k_0 \geq 1$ large enough so that for all $k \geq k_0$, $\left(1+O\left(\theta^{\sqrt{ck-d}}\right)\right) \geq 1-\epsilon$. It then follows from (5.43) that

$$\begin{aligned}\mu\left(E_{ck}^d(\boldsymbol{f}_{ck},\varphi_{ck})\right) &\geq \frac{1-\epsilon}{4\,(2A)^{2d}\,\log 2\,(\varphi_{ck}+1)}\sum_{\boldsymbol{\alpha}\in[\![1,A]\!]^d}\mu\left(\boldsymbol{f}_{ck}^{-1}(\{\boldsymbol{\alpha}\})\right)\\ &= \frac{1-\epsilon}{4\,(2A)^{2d}\,\log 2\,(\varphi_{ck}+1)}\cdot\end{aligned} \tag{5.44}$$

**Step 3.** The third step consists of finding an upper bound for $\mu\left(E_{ck}^d(\boldsymbol{f}_{ck},\varphi_{ck})\cap E_{cl}^d(\boldsymbol{f}_{cl},\varphi_{cl})\right)$ for $k$ and $l$ large enough depending on a fixed parameter $\epsilon \in (0,1)$.

Let $l \geq k \geq 1$ : it should be clear that

$$\begin{aligned}&\mu\left(E_{ck}^d(\boldsymbol{f}_{ck},\varphi_{ck})\cap E_{cl}^d(\boldsymbol{f}_{cl},\varphi_{cl})\right)\\ &\quad= \sum_{\boldsymbol{\alpha_1},\boldsymbol{\alpha_2}\in[\![1,A]\!]^d}\mu\left(E_{ck}^d(\boldsymbol{f}_{ck},\varphi_{ck})\cap E_{cl}^d(\boldsymbol{f}_{cl},\varphi_{cl})\cap \boldsymbol{f}_{ck}^{-1}(\{\boldsymbol{\alpha_1}\})\cap \boldsymbol{f}_{cl}^{-1}(\{\boldsymbol{\alpha_2}\})\right)\\ &\quad= \sum_{\boldsymbol{\alpha_1},\boldsymbol{\alpha_2}\in[\![1,A]\!]^d}\mu\left(E_{ck}^d(\boldsymbol{\alpha_1},\varphi_{ck})\cap E_{cl}^d(\boldsymbol{\alpha_2},\varphi_{cl})\cap \boldsymbol{f}_{ck}^{-1}(\{\boldsymbol{\alpha_1}\})\cap \boldsymbol{f}_{cl}^{-1}(\{\boldsymbol{\alpha_2}\})\right),\end{aligned}$$

whence, using the $T$ invariance of the measure $\mu$ and Lemma 5.3.10,

$$\mu\left(E_{ck}^d(\boldsymbol{f}_{ck},\varphi_{ck})\cap E_{cl}^d(\boldsymbol{f}_{cl},\varphi_{cl})\right)$$

$$
\begin{aligned}
&= \sum_{\boldsymbol{\alpha_1},\boldsymbol{\alpha_2}\in[\![1,A]\!]^d} \mu\left(E_0^d\left(\boldsymbol{\alpha_1},\varphi_{ck}\right)\cap T^{-ck}\left(E_{cl}^d\left(\boldsymbol{\alpha_2},\varphi_{cl}\right)\cap \boldsymbol{f}_{ck}^{-1}\left(\{\boldsymbol{\alpha_1}\}\right)\cap \boldsymbol{f}_{cl}^{-1}\left(\{\boldsymbol{\alpha_2}\}\right)\right)\right)\\
&= \left(1+O\left(\theta^{\sqrt{ck-d}}\right)\right)\times\\
&\quad\left(\sum_{\boldsymbol{\alpha_1},\boldsymbol{\alpha_2}\in[\![1,A]\!]^d} \mu\left(E_0^d\left(\boldsymbol{\alpha_1},\varphi_{ck}\right)\right)\,\mu\left(E_{cl}^d\left(\boldsymbol{\alpha_2},\varphi_{cl}\right)\cap \boldsymbol{f}_{ck}^{-1}\left(\{\boldsymbol{\alpha_1}\}\right)\cap \boldsymbol{f}_{cl}^{-1}\left(\{\boldsymbol{\alpha_2}\}\right)\right)\right)\\
&= \left(1+O\left(\theta^{\sqrt{ck-d}}\right)\right)\times\\
&\quad\left(\sum_{\boldsymbol{\alpha_1},\boldsymbol{\alpha_2}\in[\![1,A]\!]^d} \mu\left(E_0^d\left(\boldsymbol{\alpha_1},\varphi_{ck}\right)\right)\,\mu\left(E_0^d\left(\boldsymbol{\alpha_2},\varphi_{cl}\right)\cap T^{-cl}\left(\boldsymbol{f}_{ck}^{-1}\left(\{\boldsymbol{\alpha_1}\}\right)\cap \boldsymbol{f}_{cl}^{-1}\left(\{\boldsymbol{\alpha_2}\}\right)\right)\right)\right)\\
&= \left(1+O\left(\theta^{\sqrt{ck-d}}\right)\right)\cdot\left(1+O\left(\theta^{\sqrt{cl-d}}\right)\right)\times\\
&\quad\left(\sum_{\boldsymbol{\alpha_1},\boldsymbol{\alpha_2}\in[\![1,A]\!]^d} \mu\left(E_0^d\left(\boldsymbol{\alpha_1},\varphi_{ck}\right)\right)\,\mu\left(E_0^d\left(\boldsymbol{\alpha_2},\varphi_{cl}\right)\right)\,\mu\left(\boldsymbol{f}_{ck}^{-1}\left(\{\boldsymbol{\alpha_1}\}\right)\cap \boldsymbol{f}_{cl}^{-1}\left(\{\boldsymbol{\alpha_2}\}\right)\right)\right).
\end{aligned}
$$

Given $\epsilon\in(0,1)$, choose $k_0\geq 1$ large enough so that for all $k\geq k_0$, $\left(1+O\left(\theta^{\sqrt{ck-d}}\right)\right)\leq 1+\epsilon$. It then follows from (5.43) that, for all $l\geq k\geq k_0$,

$$
\begin{aligned}
\mu\left(E_{ck}^d\left(\boldsymbol{f}_{ck},\varphi_{ck}\right)\cap E_{cl}^d\left(\boldsymbol{f}_{cl},\varphi_{cl}\right)\right) &\leq \frac{(1+\epsilon)^2}{(\log 2)^2}\cdot\frac{1}{\varphi_{ck}\varphi_{cl}}\sum_{\boldsymbol{\alpha_1},\boldsymbol{\alpha_2}\in[\![1,A]\!]^d}\mu\left(\boldsymbol{f}_{ck}^{-1}\left(\{\boldsymbol{\alpha_1}\}\right)\cap \boldsymbol{f}_{cl}^{-1}\left(\{\boldsymbol{\alpha_2}\}\right)\right)\\
&= \frac{(1+\epsilon)^2}{(\log 2)^2\,\varphi_{ck}\varphi_{cl}}\cdot \qquad (5.45)
\end{aligned}
$$

**Step 4.** Let $\epsilon\in(0,1)$. Choose $k_0\geq 1$ large enough so that the conclusions of steps 2 and 3 hold for all $l\geq k\geq k_0$. By assumption on the integer $c\geq d+1$ in Lemma 5.3.7 and from (5.44), the series $\sum_{k\geq 1}\mu\left(E_{ck}^d\left(\boldsymbol{f}_{ck},\varphi_{ck}\right)\right)$ diverges. Therefore, Lemma 5.3.11 applies and, on using (5.44) and (5.45) and on noticing that $\limsup_{k\to\infty}E_{ck}^d\left(\boldsymbol{f}_{ck},\varphi_{ck}\right)\subset\mathcal{S}^d\left(\boldsymbol{f},\varphi\right)$, one gets

$$
\begin{aligned}
\mu\left(\mathcal{S}^d\left(\boldsymbol{f},\varphi\right)\right) &\geq \limsup_{i\to\infty}\left(\left(\frac{\log 2\,(1-\epsilon)}{4\,(2A)^{2d}\,\log 2\,(1+\epsilon)}\right)^2\frac{\left(\sum_{k=k_0}^{i}(\varphi_{ck}+1)^{-1}\right)^2}{\sum_{k_0\leq k,l\leq i}(\varphi_{ck}\varphi_{cl})^{-1}}\right)\\
&\geq \left(\frac{1-\epsilon}{1+\epsilon}\right)^2\frac{1}{\left(8\,(2A)^{2d}\right)^2}\cdot
\end{aligned}
$$

The result then follows from (5.42) on letting $\epsilon$ tend to 0. **Q.E.D.**

#### 5.3.3.2 Completion of the proof of Corollary 5.1.8

The completion of the proof of Corollary 5.1.8 requires the introduction of a final two lemmata. The first one is well–known and the second one is elementary.

**Lemma 5.3.12.** *Let $l\geq 2$ and $q\geq 0$ be integers. Consider the map*

$$
\widetilde{T}\;:x\pmod{a}\;\mapsto\; lx+\frac{q}{l}\pmod{a}.
$$

*Let $A \subset [0,a)$ be such that $\widetilde{T}(A) \subset A$.*

*Then $\lambda(A) \in \{0,a\}$.*

*Proof.* See, e.g., [183, Lemma 7]. **Q.E.D.**

**Lemma 5.3.13.** *Let $\alpha, \beta \in \mathbb{Z}/b\mathbb{Z}$.*

*Then, there exist $i_1(\alpha,\beta)$ and $i_2(\alpha,\beta)$ in $\mathbb{Z}/b\mathbb{Z}$ such that, defining*

$$u_{-1} = \alpha, \quad u_0 = \beta, \quad u_1 = i_1(\alpha,\beta)u_0 + u_{-1} \quad \text{and} \quad u_2 = i_2(\alpha,\beta)u_1 + u_0,$$

*one has $u_2 = 0$ in $\mathbb{Z}/b\mathbb{Z}$.*

*Proof.* All equations in this proof must be read in $\mathbb{Z}/b\mathbb{Z}$.

If $\alpha = 0$ (resp. $\beta = 0$), the choices $i_1(0,\beta) = 1$ and $i_2(0,\beta) = -1$ (resp. $i_1(\alpha,0) = i_2(\alpha,0) = 0$) independently of $\beta \in \mathbb{Z}/b\mathbb{Z}$ (resp. of $\alpha \in \mathbb{Z}/b\mathbb{Z}$) are easily seen to satisfy the conclusion of the lemma.

Assume therefore that $\alpha \neq 0$ and $\beta \neq 0$. Viewing $\alpha$ and $\beta$ as integers in the interval $[\![0, b-1]\!]$, one can then write $u_k = \gcd(\alpha,\beta)v_k$ for $k = -1, \dots, 2$, where the finite sequence $(v_k)_{-1\leq k\leq 2}$, well–defined in $\mathbb{Z}/\gcd(\alpha,\beta,b)\mathbb{Z}$, satisfies in this ring a recurrence relation similar to that of $(u_k)_{-1\leq k\leq 2}$ in $\mathbb{Z}/b\mathbb{Z}$. Even if it means proving the result for the lift to $\mathbb{Z}/b\mathbb{Z}$ of the sequence $(v_k)_{-1\leq k\leq 2}$ which satisfies the conditions $v_{-1} \equiv \alpha/\gcd(\alpha,\beta) \pmod b$ and $v_0 \equiv \beta/\gcd(\alpha,\beta) \pmod b$, it may be assumed without loss of generality that $\gcd(\alpha,\beta) = 1$.

From Dirichlet's theorem on arithmetic progressions, the sequence of integers $(\alpha + i\beta)_{i\geq 0}$ contains infinitely many primes. Therefore, there exists $i \in \mathbb{Z}/b\mathbb{Z}$ such that $\alpha + i\beta$ is invertible in $\mathbb{Z}/b\mathbb{Z}$. Setting $i_1(\alpha,\beta) = i$ and $i_2(\alpha,\beta) = -u_0u_1^{-1}$ yields the result. **Q.E.D.**

*Completion of the proof of Corollary 5.1.8.* It is well–known that, for almost all $\xi \in [0,1]\backslash\mathbb{Q}$,

$$\lim_{k\to\infty} \sqrt[k]{q_k(\xi)} \;=\; \exp\left(\frac{\pi^2}{12\log 2}\right).$$

This follows for instance from Birkhoff's pointwise ergodic theorem applied to the ergodic system $\left(T, \mu, \mathcal{B}_{[0,1]}\right)$ introduced in the preceding subsection — see [87, Corollary 3.8] for details.

In particular, there exist two positive constants $B$ and $B'$ such that, for almost all $\xi \in [0,1]\backslash\mathbb{Q}$, there exists an integer $k_0$ depending on $B, B'$ and $\xi$ such that, for all $k \geq k_0$,

$$\exp\left(B'k\right) \;\leq\; q_k(\xi) \;\leq\; \exp\left(Bk\right).$$

Set $\tau = \left(8(ab)^2 \max\{4, \widetilde{\Psi}(1)^{-1}\}\right)^{-1}$, which corresponds to the inverse of the constant given in (5.9) (with $M = 1$) for natural choices of the parameters $\kappa, \eta$ and $\gamma$ under the assumption of the monotonicity of $\widetilde{\Psi}$. Then, by virtue of Theorem 5.1.6 and Remark 5.3.6, one gets on the one hand that, almost surely,

$$\left\{\xi \in [0,1]\backslash\mathbb{Q} \;:\; \exists k_0\left(B',\xi\right) \geq 0,\ \forall k \geq k_0\left(B',\xi\right),\ a_k(\xi) \leq \tau\widetilde{\Psi}\left(e^{B'k}\right)\right\} \subset \mathcal{U}(\Psi). \tag{5.46}$$

On the other, from (5.9), it should be clear that, almost surely,

$$\bigcap_{M=1}^{\infty}\left\{\xi\in[0,1]\backslash\mathbb{Q}\;:\;a_k\left(\xi\right)\geq M\widetilde{\Psi}\left(e^{Bk}\right)\;\text{ and }\;b|q_{k-1}\left(\xi\right)\;\text{i.o.}\right\}$$
$$\subset\bigcap_{c=1}^{\infty}\left([0,1]\backslash\mathcal{U}\left(c\Psi\right)\right)\;=\;[0,1]\backslash\mathcal{V}\left(\Psi\right),\tag{5.47}$$

where $\mathcal{V}\left(\Psi\right)$ has been defined in (5.41).

Notice also that for any $C>0$, the two series $\sum_{Q\geq1}\left(\widetilde{\Psi}\left(e^{CQ}\right)\right)^{-1}$ and $\sum_{Q\geq1}\left(Q^2\Psi(Q)\right)^{-1}$ converge (resp. diverge) simultaneously. This follows from the change of variable $y=e^{Cx}$ in the corresponding integral $\displaystyle\int\frac{\mathrm{d}x}{\widetilde{\Psi}\left(e^{Cx}\right)}$ under the assumption of the monotonicity of $\widetilde{\Psi}$.

Assume first that the series $\sum_{Q\geq1}\left(Q^2\Psi(Q)\right)^{-1}$ converges. It then follows from Lemma 5.3.7 and Remark 5.3.9 that, for almost all $\xi\in[0,1]\backslash\mathbb{Q}$, there exist only finitely many indices $k\geq0$ such that $a_k\left(\xi\right)\geq\tau\widetilde{\Psi}\left(e^{B'k}\right)$. Therefore, the set in the left–hand side of (5.46) has full measure. The same holds true for any of its translates by an integer $k\in[\![0,a-1]\!]$ by the invariance of the Lebesgue measure and by elementary properties of the continued fraction expansion of an irrational. This completes the proof in this case.

Assume now that the series $\sum_{Q\geq1}\left(Q^2\Psi(Q)\right)^{-1}$ diverges. From Lemma 5.3.13 and formulae (5.19), for any pair $(\alpha,\beta)\in[\![1,b]\!]^2$ and any integer $k\geq4$ , there exists $(a_{k-2},a_{k-1})=\left(i_1\left(\alpha,\beta\right),i_2\left(\alpha,\beta\right)\right)\in[\![1,b]\!]^2$ such that, if $q_{k-4}\equiv\alpha\pmod{b}$ and $q_{k-3}\equiv\beta\pmod{b}$, then $q_{k-1}\equiv0\pmod{b}$. Apply then Lemma 5.3.7 with $A=b$, $d=2$, $c=3$, $\varphi=\left(\widetilde{\Psi}\left(e^{Bk}\right)\right)_{k\geq0}$ and, for $k\geq4$,

$$\boldsymbol{f}_k\;:\xi\in[0,1]\backslash\mathbb{Q}\;\mapsto\;\left(i_1\left(\alpha,\beta\right),i_2\left(\alpha,\beta\right)\right)\in[\![1,b]\!]^2\;\text{ if }\left(q_{k-4}\left(\xi\right),q_{k-3}\left(\xi\right)\right)\equiv(\alpha,\beta)\pmod{b}.$$

For such a choice of $\varphi$ and of $\boldsymbol{f}:=\left(\boldsymbol{f}_k\right)_{k\geq4}$, consider the sequence of sets $\left(\mathcal{S}^2\left(\boldsymbol{f},M\varphi\right)\right)_{M\geq1}$ as defined in Lemma 5.3.7. It should be clear that this is a sequence decreasing for inclusion and that, for any $M\geq1$,

$$\mathcal{S}^2\left(\boldsymbol{f},M\varphi\right)\subset\left\{\xi\in[0,1]\backslash\mathbb{Q}\;:\;a_k\left(\xi\right)\geq M\widetilde{\Psi}\left(e^{Bk}\right)\;\text{ and }\;b|q_{k-1}\left(\xi\right)\;\text{i.o.}\right\}.\tag{5.48}$$

Furthermore, from the Monotone Convergence Theorem and Lemma 5.3.7,

$$\lambda\left(\bigcap_{M=1}^{\infty}\mathcal{S}^2\left(\boldsymbol{f},M\varphi\right)\right)\;=\;\lim_{M\to\infty}\lambda\left(\bigcap_{M'=1}^{M}\mathcal{S}^2\left(\boldsymbol{f},M'\varphi\right)\right)\;=\;\lim_{M\to\infty}\lambda\left(\mathcal{S}^2\left(\boldsymbol{f},M\varphi\right)\right)\;\geq\;\frac{\log2}{2^{14}b^8}.$$

Combining this last inequality with (5.47) and (5.48) shows that the complement of the set $\mathcal{V}\left(\Psi\right)$ has strictly positive measure. Now, it should be clear from its definition in (5.41) that the set $\mathcal{V}\left(\Psi\right)$ is invariant under the map $x\pmod{a}\mapsto tx\pmod{a}$, where $t\geq a$ is any integer congruent to 1 modulo $a$. From Lemma 5.3.12, this implies that the complement of $\mathcal{V}\left(\Psi\right)$ in $[0,a]$ has full measure, that is, that

$$\lambda\left(\mathcal{U}\left(\Psi\right)\right)\;\leq\;\lambda\left(\mathcal{V}\left(\Psi\right)\right)\;=\;0.$$

This completes the proof of Corollary 5.1.8. **Q.E.D.**

## 5.4 Some applications

Some of the applications of the theory developed in this chapter are mentioned in this section.

A Dirichlet type result can always be used to obtain bounds for certain types of exponential sums. In this respect, Theorem 5.1.6 may help to improve or specify some exponential sums when the numerators and the denominators of the rational approximants are restricted to prescribed arithmetic progressions — see, e.g., [93] or [186, p.172]. On the other hand, Walfisz proved in [195] a very particular case of the Khintchine type result stated in Theorem 5.1.3 in order to study the behaviour of the elliptic function

$$\vartheta(z)=\sum_{n=-\infty}^{+\infty} z^{n^2}$$

near its circle of convergence : indeed, he established that, for almost all $\alpha\in\mathbb{R}$,

$$\vartheta\left(re^{2i\pi\alpha}\right) \underset{r\to 1^-}{=} \Omega\left(\sqrt[4]{\frac{1}{1-r}\log\left(\frac{1}{1-r}\right)}\right).$$

This complements a result of Hardy and Littlewood who had previously proved in [106] that, if $\alpha$ is badly approximable, then the very accurate estimates

$$\vartheta\left(re^{2i\pi\alpha}\right) \underset{r\to 1^-}{=} O\left(\sqrt[4]{\frac{1}{1-r}}\right) \quad\text{and}\quad \vartheta\left(re^{2i\pi\alpha}\right) \underset{r\to 1^-}{=} \Omega\left(\sqrt[4]{\frac{1}{1-r}}\right)$$

hold.

Two specific applications of Theorem 5.1.1 and Corollary 5.1.8 will now be developed. The first one is mainly due to S. Hartman who was the first to notice in [114] that a result such as Theorem 5.1.1 enables one to determine the value of $\liminf_{n\to\infty}(\sin n)^n$. This can be generalized thanks to the inhomogeneous version of Theorem 5.1.1 mentioned in Remark 5.2.1.

**Proposition 5.4.1.** *Let $\xi$ be an irrational which is not a rational multiple of $\pi$. Let also $\alpha\in\mathbb{R}$. Then,*

$$\liminf_{n\to\infty}\left(\sin(n\xi+\alpha)\right)^n=\liminf_{n\to\infty}\left(\cos(n\xi+\alpha)\right)^n=-1$$

*and*

$$\limsup_{n\to\infty}\left(\sin(n\xi+\alpha)\right)^n=\limsup_{n\to\infty}\left(\cos(n\xi+\alpha)\right)^n=1.$$

*Proof.* It will be proved that $\liminf_{n\to\infty}\left(\sin(n\xi+\alpha)\right)^n=-1$. All the other equations can be established in a similar fashion.

From Remark 5.2.1, there exist two sequences of integers $(c_n)_{n\geq 1}$ and $(d_n)_{n\geq 1}$ with $d_n\geq 1$ and $\lim_{n\to\infty} d_n=\infty$ such that, for all $n\geq 1$,

$$\left|d_n\frac{\pi}{2\xi}-c_n-\frac{\alpha}{\xi}\right|\leq\frac{2}{d_n}, \tag{5.49}$$

$$d_n \equiv 3 \pmod 4\,, \tag{5.50}$$

and

$$c_n \equiv 1 \pmod 2\,. \tag{5.51}$$

Therefore, for all $n \geq 1$,

$$\left| d_n \frac{\pi}{2\xi} - c_n - \frac{\alpha}{\xi} \right| = O\left(\frac{1}{d_n}\right) \underset{(5.49)}{=} O\left(\frac{1}{c_n}\right).$$

With the help of a Taylor expansion, this implies that

$$\sin(c_n\xi + \alpha) = \sin\left(d_n \frac{\pi}{2} + O\left(\frac{1}{c_n}\right)\right) \underset{(5.50)}{=} -1 + O\left(\frac{1}{c_n^2}\right),$$

whence

$$(\sin(c_n\xi + \alpha))^{c_n} = \left(-1 + o\left(\frac{1}{c_n}\right)\right)^{c_n} \underset{n\to\infty}{\overset{(5.51)}{\longrightarrow}} -1.$$

**Q.E.D.**

The second application is of a geometrical nature and exploits the link between approximation by rationals with numerators and denominators in given arithmetic progressions and pseudo–lattices in dimension 2. More precisely, a natural analogue of Pólya's orchard problem is now discussed. The latter is formulated in [161, Problem 239] in this form :

**Problem 5.4.2** (Pólya's orchard problem)**.** *"How thick must be the trunks of the trees in a regularly spaced circular forest grow if they are to block completely the view from the center?"*

Assume that the forest (or the orchard) is situated in a disk of integer radius $N \geq ab$ and that each point of the lattice $b\mathbb{Z} \times a\mathbb{Z}$ different from the origin and lying in this disk is the center of a tree of radius $r > 0$ (here, $a, b \geq 1$). Minkowski's Convex Body Theorem can then be used to solve the visibility problem above and to obtain that the choice of $r = ab/N$ blocks the view from the center (see, e.g., [146, Lemma 3]). Allen in [3] computed the infimum of all radii of trees preventing an observer situated at the origin from seeing a point outside the forest and Kruskal generalized this result to more general configurations of trees (see [146]).

In what follows, the horizon will be said to be visible from the origin in the direction given by a line $\Delta$ passing through the origin if, given a forest of a prescribed type lying in the half–plane $\{x \geq 0\}$, the line $\Delta$ does not intersect any of the trees in the forest.

For the forests under consideration, the latter will be planted in a subset of $\Lambda \cap \{x \geq 0\}$, where $\Lambda$ is the pseudo–lattice $\Lambda = (b\mathbb{Z} + s) \times (a\mathbb{Z} + r)$, with $a, b, r$ and $s$ satisfying (5.2) and $r \neq 0$ or $s \neq 0$.

The connection between Diophantine approximation and the problem of visibility is then given by this simple fact : an inequality of the type $|(bn + s)\xi - (am + r)| \leq c$, where $c \geq 0$ is real, $\xi$ is irrational and $(m, n) \in \mathbb{Z}^2$, precisely means that the vertical segment joining the point $(bn + s, am + r) \in \Lambda$ to the line $\Delta\ : y = \xi x$ has a length less than $c$. Therefore, the intersection

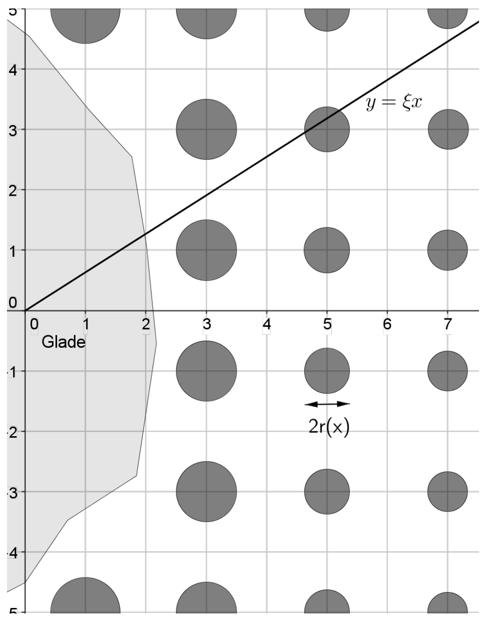



Figure 5.1: Geometric interpretation of Theorem 5.1.1 in the case of the pseudo–lattice $\Lambda = (2\mathbb{Z}+1)^2$. Each tree centered at $(2n+1, 2m+1)$ has radius $r(2n+1) = 1/|2n+1|$.

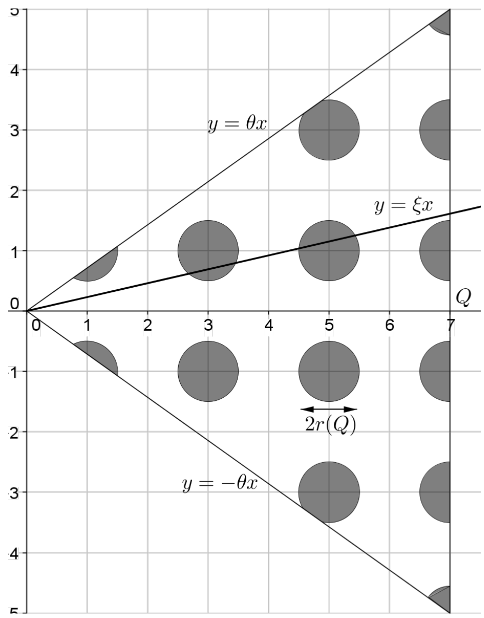



Figure 5.2: Geometric interpretation of Corollary 5.1.8 in the case of the pseudo–lattice $\Lambda = (2\mathbb{Z}+1)^2$. The orchard has depth $Q$ and all the trees have the same radius $r(Q) > 0$.

between $\Delta$ and the closed ball centered at $(bn+s, am+r)$ with radius $c$ is non–empty : if the latter ball represents a tree in a forest, the horizon is not visible in the direction given by $\Delta$.

For the sake of simplicity, the results will be stated from a qualitative point of view : although possible, none of the constants mentioned below will be made effective.

**Geometrical interpretation of Theorem 5.1.1.** The forest is defined this way : a tree of radius $(ab)/(4(bn+s))$ is planted at each point $(bn+s, am+r) \in \Lambda \cap \{x > 0\}$. The observer is situated at the origin in a glade of any shape but with bounded diameter (cf. Figure 5.1).

From Theorem 5.1.1, for any line of sight with irrational slope, the observer will never see the horizon, no matter how big the glade is. From Remark 5.1.2, it is however possible to see the horizon along a direction given by a line with rational slope if, for instance, the glade contains a disk centered at the origin with sufficiently large radius. On the other hand, if the constant $ab/4$ appearing in Theorem 5.1.1 was to be optimal uniformly in $\xi \in \mathbb{R}\backslash\mathbb{Q}$ (see Conjecture 5.5.1 in section 5.5 below), this would imply that there exist angles of sight with irrational slope if the constant $ab/4$ were to be replaced by another one small enough in the value of the radii of the trees (again, provided that the glade at the origin is *big enough*).

**Geometrical interpretation of Corollary 5.1.8.** Given $\theta > 0$ and $Q \geq 1$, the forest — which will more conveniently be referred to as an orchard — is defined this way (see also

Figure 5.2) : a tree of radius $r(Q) > 0$ is planted at each element of the set $\Lambda \cap \mathcal{L}$, where

$$\mathcal{L} := \left\{(x, y) \in \mathbb{R}^2 \ : \ 0 \leq x \leq Q \ \text{ and } \ -\theta x \leq y \leq \theta x\right\}.$$

From Corollary 5.1.8, if the radius of the trees is chosen in such a way that $r(Q) = \log Q/Q$, then, for almost all $\xi \in (-\theta, \theta)$, there exist arbitrarily large values of $Q$ such that the horizon is visible in the direction $y = \xi x$. On the other hand, if $r(Q) = (\log Q)^{1+\epsilon}/Q$ for some $\epsilon > 0$, then, provided that the depth $Q$ of the orchard is large enough (depending on $\xi$), the horizon is never visible in the direction $y = \xi x$ for almost all $\xi \in (-\theta, \theta)$.

## 5.5 Notes for the chapter

- A natural question related to Theorem 5.1.1 is whether the constant $(ab)/4$ appearing on the right–hand side of (5.5) is optimal. This has been proved by Eggan in [86, Theorem 3.2] (following ideas due to Cassels — see the proof of [57, Theorem II B p.49]) in the case when the parity of the numerators and the denominators of the rational approximants are prescribed in a non–trivial way (that is, when $a = b = 2$ and $r \neq 0$ or $s \neq 0$). It is therefore tempting to set the following conjecture :

  **Conjecture 5.5.1.** *If $r \neq 0$ or $s \neq 0$, the constant $(ab)/4$ appearing on the right–hand side of (5.5) cannot be improved uniformly in $\xi \in \mathbb{R}\backslash\mathbb{Q}$.*

- A question related to Theorem 5.1.6 is the study of the size of the set of well–approximable numbers admitting a Dirichlet type approximation in the context of $(a, b, r, s)$–approximation. In this respect, the following conjecture seems of relevance.

  **Conjecture 5.5.2.** *The set of real numbers which are not in* **Bad** *and which admit a uniform $(a, b, r, s)$–approximation with exponent 1 has full Hausdorff dimension.*

  This conjecture is trivially true if $r = s = 0$ from the discussion held in the introduction. On the other hand, the construction of a Cantor set to prove the conjecture seems easier in the case when $\gcd(a, b) = 1$ : this is because, if one can ensure that the denominators $q_k$ (resp. the numerators $p_k$) of the convergents of an irrational $\xi$ are all coprime to $b$ (resp. to $a$), then conditions (5.6) always hold true. However, if $\gcd(a, b) > 1$, the third condition in (5.6) turns out to be more delicate to deal with.

- The geometrical interpretations of Theorem 5.1.1 and Corollary 5.1.8 fall within the category of obstruction (or view) problems about which the literature is abundant — see [3, 146] and the references therein. They are made simpler by the fact that the pseudo–lattices under consideration are obtained from standard lattices by rational shifts (that is, the vectors of translation have rational coordinates). As pointed out by Professor Barak Weiss, a much more general formulation of the problem has been given by C. Bishop in [46] and reads as follows :

  **Problem 5.5.3.** *Suppose we stand in a forest with tree trunks of radius $\epsilon > 0$ and no two trees centered closer than unit distance apart. Can the trees be arranged so that we* *can never see further than some distance $L < \infty$, no matter where we stand and what direction we look in? What is the size of $L$ in terms of $\epsilon$?*

It is not difficult to see that in a forest such as the one described above, the estimate $L = o\left(\epsilon^{-1}\right)$ can *never* hold. In the other direction, Y. Peres proved in [46] that it is possible to construct a forest with $L = O\left(\epsilon^{-4}\right)$. In ongoing work, we are able to prove that one can obtain $L = O\left(\epsilon^{-32/11}\right)$ unconditionally (note that $32/11 = 2.\overline{90}$) and $L = O\left(\epsilon^{-2+\delta}\right)$ for any $\delta > 0$ under the exponent pair conjecture. The methods involved are from analytic number theory as they rely on Fourier analysis and very sharp estimates for exponential sums. A natural extension of this work is to construct "forests" in higher dimensions with minimal visibility.

It should also be noted that the ideas in this problem of Diophantine geometry are related to the Danzer problem, which asks for the existence of a subset $S \subset \mathbb{R}^n$ $(n \geq 1)$ with finite density which has the property that every closed convex body of volume one in $\mathbb{R}^n$ contains a point of $S$. This question remains open — see [181] and the references therein for further details.

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