Linear Algebra and its Applications 435 (2011) 1008–1018

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

P^1_{max} and S_{max} properties and asymptotic stability in the max algebra ☆

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ARTICLE INFO

Article history:
Received 9 July 2010
Accepted 24 February 2011
Available online 27 March 2011

Submitted by H. Schneider

AMS classification:
15A80
15B48
39A30

Keywords:
P-matrix
Max algebra
Asymptotic stability
Convex hull
Generalised spectral radius
Discrete inclusions

ABSTRACT

In this work, we introduce the class of P^1_{max}-matrices for the max algebra and derive some properties of these that echo similar results for P-matrices in the conventional algebra. In analogy with Song et al. (1999) [1], we define the row-P^1_{max}-property and the S_{max}-property of a finite set of nonnegative matrices. Moreover, we relate these concepts to stability questions for sets of matrices and difference inclusions in the max algebra.

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1. Introduction

The class of P-matrices has been extensively studied due to its importance in fields such as statistics, optimisation and dynamical systems [2–5]. A matrix $A \in \mathbb{R}^{n \times n}$ is a P-matrix if all of its principal minors are positive [6]. The relevance of such matrices to the linear complementarity problem is well documented and details can be found in [2]. P-matrices are also intimately connected with the stability theory of positive linear systems and with the long-term behaviour of Lotka–Volterra systems

☆ Supported by the Irish Higher Educational Authority (HEA) PRTLI Network Mathematics Grant.
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doi:10.1016/j.laa.2011.02.054

in ecological modelling [3]. Yet another context in which P-matrices play a role is in the study of globally univalent functions, motivated by applications in Economics and Biology [4,5].

The results to be presented here relate most directly to characterizations of P-matrices within the class of so-called Z-matrices. Recall that a Z-matrix A is one for which $a_{ij} \leqslant 0$ for $i \neq j$. For a Z-matrix A, the following conditions are equivalent [7,8]:

(i) A is a P-matrix;
(ii) A is positive-stable;
(iii) for every non-zero $x \in \mathbb{R}^n$ there is some i with $x_i(Ax)_i > 0$;
(iv) every principal submatrix of A is positive-stable;
(v) there exists some $v > 0$ with $Av > 0$.

Property (v) above is usually referred to as the S-property [6].

The authors of [1] investigated extending the P-property of a single matrix to sets of matrices; specifically, they introduced the *row-P*-property for a set of matrices and demonstrated that property (iii) above holds uniformly for all matrices in a set if and only if the set possesses the *row-P*-property. Furthermore, they also showed that the *row-P*-property was equivalent to the S-property for a compact set of Z-matrices.

We shall be concerned with extending results such as those described above concerning P-properties of single matrices and sets of matrices to the setting of the max algebra. In keeping with [9], we define the max algebra to consist of the nonnegative real numbers equipped with the two operations $a \oplus b = \max(a, b)$ and $a \otimes b = ab$. These operations extend to nonnegative matrices and vectors in the standard way. We shall explore the connection between matrix stability in the max algebra and concepts analogous to P-matrices in this setting. The specific notion of matrix stability considered here is that explored in [10] for a single matrix and corresponds to asymptotic stability of the discrete-time system

$$x(k+1) = A \otimes x(k).$$

In Section 3, we introduce the concept of a P^1_{max}-matrix and show that equivalences analogous to (i)–(v) given above also hold in the max algebra. In Section 4, in analogy with the work of [1], we introduce the row-P^1_{max}-property and the S_{max}-property for sets of matrices. We show that the results of [1] extend naturally to this setting and relate the S_{max}-property for a set of matrices to the stability of its max-convex hull.

Moreover, we study difference equations and inclusions with delay over the max algebra and investigate the role played by the P^1_{max}-property in the stability of these. In particular, we show in Section 3 that the result of [11] on harmless off-diagonal delays also holds for difference equations in the max algebra. In Section 4, we present a further extension of this result to difference inclusions over the max algebra.

2. Preliminaries and notation

The set of all nonnegative real numbers is denoted by $\mathbb{R}_+$; the set of all n-tuples of nonnegative real numbers is denoted by $\mathbb{R}^n_+$ and the set of all $n \times n$ matrices with nonnegative real entries is denoted by $\mathbb{R}^{n\times n}_+$. For $v \in \mathbb{R}^n_+$ and $1 \leq i \leq n$, v_i denotes the ith component of v. For $A \in \mathbb{R}^{n\times n}_+$ and $1 \leq i, j \leq n$, a_{ij} refers to the (i, j)th entry of A. For $1 \leq j \leq n$, $A^{(j)}$ denotes the jth row of A. The matrix $A = [a_{ij}]$ is nonnegative (positive) if $a_{ij} \geq 0$ ($a_{ij} > 0$) for $1 \leq i, j \leq n$. This is denoted by $A \in \mathbb{R}^{n\times n}_+$ ($A > 0$). Similarly, for $v \in \mathbb{R}^n$, we say v is nonnegative (positive) and write $v \in \mathbb{R}^n_+$ or $v \geqslant 0$ ($v > 0$) if $v_i \geq 0$ ($v_i > 0$) for $1 \leq i \leq n$.

The weighted directed graph of A is denoted by $D(A)$. It is an ordered pair (V, E) where V is a finite set of vertices $\{1, 2, \ldots, n\}$ and E is a set of directed edges, with an edge (i, j) from i to j if and only if $a_{ij} > 0$. A walk is a sequence of vertices $i = i_1, i_2, \ldots, i_k = j$ between any two vertices i, j in $D(A)$, where (i_p, i_{p+1}) is an edge for $p = 1, \ldots, k-1$. The weight of the walk $i_1, i_2, \ldots, i_k$ of length $k-1$ is given by $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k}$. A path is a walk all of whose vertices are distinct.

A cycle Γ of length k is a closed path of the form $i_1, i_2, \ldots, i_k, i_1$ where $i_1, i_2, \ldots, i_k$ are in V and distinct. We use the notation $\pi(\Gamma) = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$ for the weight and $l(\Gamma)$ for the length of the cycle Γ. The kth root of its weight is called its cycle geometric mean. For a matrix $A \in \mathbb{R}_+^{n\times n}$, the maximal cycle geometric mean over all possible cycles in $D(A)$ is denoted by $\mu(A)$. A cycle with maximum cycle geometric mean is called a critical cycle. Vertices that lie on some critical cycle are known as critical vertices. The critical matrix of A [12–14], A^C, is formed from the submatrix of A consisting of the rows and columns corresponding to critical vertices as follows. Set $a_{ij}^C = a_{ij}$ if (i,j) lies on a critical cycle and $a_{ij}^C = 0$ otherwise.

As previously mentioned, the max algebra consists of the set of nonnegative numbers together with two binary operations: $a \oplus b = \max(a,b)$, $a \otimes b = ab$ where $a, b \in \mathbb{R}_+$; these operations extend to nonnegative matrices and vectors in the obvious fashion. Standard references on the properties of the max (and max-plus) algebra include [15–17]. We denote by $A_\otimes^k = \underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ times}}$ the kth power of A in the max algebra.

$\lambda \in \mathbb{R}_+$ is said to be a max eigenvalue of A if there is some $v \in \mathbb{R}_+^n$ with

$$(A \otimes v)_i = \max_{1 \le j \le n} a_{ij} v_j = \lambda v_i, \quad i = 1, 2, \ldots, n.$$

v is then said to be a max eigenvector. The maximum cycle geometric mean in $D(A)$, $\mu(A)$ (if $D(A)$ is acyclic then we define $\mu(A) = 0$), can be characterised in the following equivalent ways:

(i) $\max\{\lambda \in \mathbb{R}_+ : \exists v \in \mathbb{R}_+^n, v \neq 0 \text{ such that } A \otimes v = \lambda v\}$ [10].
(ii) $\lim_{k\to\infty} \mu(A_\otimes^k)^{\frac{1}{k}}$ [12].

If $A \in \mathbb{R}_+^{n\times n}$ is an irreducible matrix, then $\mu(A)$ is the unique max eigenvalue of A and there is a positive max eigenvector $v > 0$ corresponding to it [9,15].

In keeping with [10] the matrix A is said to be asymptotically stable if $\lim_{k\to\infty} A_\otimes^k = 0$. As shown in [10, Theorem 2, 18], this is equivalent to $\mu(A) < 1$.

In the conventional algebra, a matrix is a P-matrix if all of its principal submatrices have positive determinant [6]. In defining the notion of P_{max}^1-matrices in the next section, we shall make use of a definition of matrix permanent in the max algebra [19,20]. Formally, the max permanent is given by

$$per_{\max}(A) = \max_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)} \tag{1}$$

where S_n denotes the set of all permutations of the numbers $1, 2, \ldots, n$.

3. P_{max}^1-matrices and asymptotic stability

In this section, we define the class of P_{max}^1-matrices. Further, we demonstrate the relationship between these matrices and the stability properties of matrices and difference equations in the max algebra. The results presented here echo similar facts established for the conventional algebra.

Formally, $A \in \mathbb{R}_+^{n\times n}$ is said to be a P_{max}^1-matrix if $per_{\max}(B) < 1$ for all principal submatrices B of A. The following theorem presents some equivalent conditions for $A \in \mathbb{R}_+^{n\times n}$ to be a P_{max}^1-matrix.

Theorem 3.1. *Let $A \in \mathbb{R}_+^{n\times n}$. Then the following are equivalent:*

(i) A is a P_{max}^1-matrix;
(ii) A is asymptotically stable, that is, $\mu(A) < 1$ [10];
(iii) for each $x \neq 0$ in $\mathbb{R}_+^n$, there exists an $i \in \{1, 2, \ldots, n\}$ such that $(A \otimes x)_i < x_i$;

(iv) for all principal submatrices B of A, $\mu(B) < 1$*;*
(v) there exists a vector $v > 0$ *such that* $A \otimes v < v$*.*

Proof. $(i) \iff (ii)$ Assume that we have $per_{\max}(B) < 1$ for all principal submatrices B of A. Let $(i_1, i_2, \ldots, i_k, i_1)$ be a critical cycle in $D(A)$. (If there is no cycle in $D(A)$ then $\mu(A) = 0$ and we are done.) Further, let $B \in \mathbb{R}_+^{k\times k}$ be the principal submatrix of A corresponding to $i_1, i_2, \ldots, i_k$. Then we have

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} = b_{1,\sigma(1)} b_{2,\sigma(2)} \cdots b_{k,\sigma(k)} \le per_{\max}(B)$$

for some permutation $\sigma \in S_k$. It follows immediately that $\mu(A) < 1$.

For the converse, assume $\mu(A) < 1$. So, all cycle products of any length in $D(A)$ are less than 1. Let a principal submatrix $B \in \mathbb{R}_+^{k\times k}$ of A be given with $per_{\max}(B)$ equal to $b_{1,\sigma(1)} b_{2,\sigma(2)} \cdots b_{k,\sigma(k)}$. Since $\sigma \in S_k$ is a permutation and can be written as a product of cyclic permutations, it follows that $per_{\max}(B)$ can be decomposed into cycle products. It is immediate that $per_{\max}(B) < 1$.

$(ii) \iff (iii)$ Let $\mu(A) < 1$. Suppose that there exists $x \neq 0$ in $\mathbb{R}_+^n$ such that $(A \otimes x)_i \ge x_i$ for each $i \in \{1, 2, \ldots, n\}$. Then $A \otimes x \ge x$. This implies that $A_\otimes^k \otimes x \ge x$ for some $x \neq 0$ in $\mathbb{R}_+^n$. Thus, as $k \to \infty$, the kth power of A does not converge to zero which contradicts $\mu(A) < 1$.

Conversely, assume (iii) and let $i_1, i_2, \ldots, i_k, i_{k+1} = i_1$ be a cycle of length k with the cycle product $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$ in $D(A)$ for $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$. (If $D(A)$ contains no cycles, then $\mu(A) = 0$ and we are done.) Define $x \in \mathbb{R}_+^n$ as follows:

$$x_{i_2} = 1$$

$$x_{i_j} = \frac{x_{i_{j-1}}}{a_{i_{j-1} i_j}}, \quad j = 3, \ldots, k$$

$$x_{i_1} = \frac{x_{i_k}}{a_{i_k i_1}}$$

$$x_p = 0, \quad p \neq \{i_1, i_2, \ldots, i_k\}.$$

By assumption there exists some index i with $(A \otimes x)_i < x_i$. Clearly i must be in $\{i_1, i_2, \ldots, i_k\}$. Consider the following two cases.

- $i = i_1 \Rightarrow a_{i_1 i_1} x_{i_1} \oplus a_{i_1 i_2} x_{i_2} \oplus \cdots \oplus a_{i_1 i_k} x_{i_k} < x_{i_1}$. Since $x_{i_1} = \frac{x_{i_k}}{a_{i_k i_1}} \neq 0$, it easily follows from the second term in the left side that $a_{i_1 i_2} x_{i_2} < x_{i_1}$. Hence,

$$a_{i_1 i_2} x_{i_2} < \frac{x_{i_k}}{a_{i_k i_1}} = \frac{x_{i_2}}{a_{i_2 i_3} a_{i_3 i_4} \cdots a_{i_k i_1}} \Rightarrow a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} < 1.$$

- $i = i_j (1 < j \le k) \Rightarrow a_{i_j i_1} x_{i_1} \oplus a_{i_j i_2} x_{i_2} \oplus \cdots \oplus a_{i_j i_k} x_{i_k} < x_{i_j}$. Similarly, it follows from the $(j+1)$th term that

$$a_{i_j i_{j+1}} x_{i_{j+1}} < x_{i_j} \Rightarrow a_{i_j i_{j+1}} \frac{x_{i_j}}{a_{i_j i_{j+1}}} < x_{i_j} \Rightarrow 1 < 1.$$

The second condition is not possible. As a result, we have $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} < 1$. As this is true for any cycle in $D(A)$, it follows that $\mu(A) < 1$.

$(ii) \iff (iv)$ First, let $\mu(A) < 1$. Then, all cycle products in $D(A)$ are less than one. Let a principal submatrix B^* of A be given and let Γ be a critical cycle in $D(B^*)$. Since Γ also defines a cycle in $D(A)$, $\pi(\Gamma) < 1$. As Γ was arbitrary, $\mu(B^*) < 1$. The converse is immediate.

$(ii) \iff (v)$ First, suppose $\mu(A) < 1$. Let $\mathbf{1}_n \in \mathbb{R}_+^n$ denote the vector of all ones. We can choose $\epsilon > 0$ so that $\mu(A + \epsilon \mathbf{1}_n \mathbf{1}_n^T) < 1$. Since $A + \epsilon \mathbf{1}_n \mathbf{1}_n^T$ is an irreducible matrix, it follows from the

Perron–Frobenius theorem for the max algebra [9] that there is some $v > 0$ with $(A + \epsilon \mathbf{1}_n \mathbf{1}_n^T) \otimes v = \mu(A + \epsilon \mathbf{1}_n \mathbf{1}_n^T) v < v$. It follows immediately that

$$A \otimes v \leq \left(A + \epsilon \mathbf{1}_n \mathbf{1}_n^T \right) \otimes v < v.$$

For the converse, assume that there exists $v > 0$ satisfying $A \otimes v < v$. As above, choose $\epsilon > 0$ so that

$$\left(A + \epsilon \mathbf{1}_n \mathbf{1}_n^T \right) \otimes v < v.$$

As $(A + \epsilon \mathbf{1}_n \mathbf{1}_n^T)$ is irreducible, $A + \epsilon \mathbf{1}_n \mathbf{1}_n^T$ has a positive left max eigenvector $w > 0$ [9]. Multiplying both sides of the above equation with w^T from the left, we see that

$$w^T \otimes \left(A + \epsilon \mathbf{1}_n \mathbf{1}_n^T \right) \otimes v < w^T \otimes v.$$

Since w is the left max eigenvector of $A + \epsilon \mathbf{1}_n \mathbf{1}_n^T$, it follows that $\mu(A + \epsilon \mathbf{1}_n \mathbf{1}_n^T) w^T \otimes v < w^T \otimes v$.

But $w^T \otimes v > 0$ which implies directly that

$$\mu(A) \leq \mu \left(A + \epsilon \mathbf{1}_n \mathbf{1}_n^T \right) < 1.$$

This completes the proof. □

The final result of this section is concerned with the relation of the P^1_{max}-property to the stability of delayed difference equations over the max algebra. In [11], it was shown for conventional algebra that off-diagonal delays had no effect on the stability of a differential equation if and only if $-A$ is a P-matrix where A is the system matrix. We shall prove a corresponding fact for difference equations in the max algebra without restricting diagonal delays to be zero.

Consider the delayed system of difference equations given by

$$x_i(k+1) = \bigoplus_{j=1}^{n} a_{ij} x_j(k - \tau_{ij}), \quad i = 1, 2, \ldots, n \tag{2}$$

where $A \in \mathbb{R}_+^{n \times n}$ and $\tau_{ij} \geqslant 0$ are nonnegative integers for all $1 \leq i, j \leq n$.

Theorem 3.2. *Consider the system of delayed difference equations (2) where $\tau_{ij} \geq 0$ for all i, j. The following are equivalent:*

(i) A is a P^1_{max}-matrix;
(ii) (2) is asymptotically stable for all $\tau_{ij} \geqslant 0$;
(iii) (2) is asymptotically stable for some $\tau_{ij} \geqslant 0$.

Proof. We shall prove that (i) implies (ii) and that (iii) implies (i). The implication (ii) ⇒ (iii) is trivial.

Assume that A is a P^1_{max}-matrix and let $\tau_{ij} \geqslant 0$ be any set of nonnegative integer delays. Define the state vector by $x(k) = (x_1(k), x_2(k), \ldots, x_n(k))^T \in \mathbb{R}_+^n$ and suppose that the delays τ_{ij} take values in the set $\{0, 1, \ldots, \tau_{\max}\}$ for all $1 \leq i, j \leq n$, where $\tau_{\max} = \max_{i,j} \tau_{ij}$.

As all delays are nonnegative integers less than or equal to $\tau_{\max}$, we can write the delayed system in (2) in the following form

$$x(k+1) = A_0 \otimes x(k) \oplus A_1 \otimes x(k-1) \oplus \cdots \oplus A_{\tau_{\max}} \otimes x(k - \tau_{\max}) \tag{3}$$

where $A_w (w = 0, 1, \ldots, \tau_{\max})$ in $\mathbb{R}_+^{n \times n}$ are defined as follows. The (i, j)th entry of A_w is equal to a_{ij} if $\tau_{ij} = w$ and all other entries of A_w are zero. Note that

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{\tau_{\max}}.$$

By setting $\hat{x}(k) = (x(k - \tau_{\max}), x(k - \tau_{\max} + 1), \ldots, x(k))^T \in \mathbb{R}_+^{n(\tau_{\max}+1)}$, we see that the stability of (2) is equivalent to the stability of

$$\underbrace{\begin{bmatrix} x(k-\tau_{\max}+1) \\ x(k-\tau_{\max}+2) \\ \vdots \\ x(k) \\ x(k+1) \end{bmatrix}}_{\hat{x}(k+1)} = \underbrace{\begin{bmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & I \\ A_{\tau_{\max}} & \dots & \dots & \dots & A_1 & A_0 \end{bmatrix}}_{C} \otimes \underbrace{\begin{bmatrix} x(k-\tau_{\max}) \\ x(k-\tau_{\max}+1) \\ \vdots \\ x(k-1) \\ x(k) \end{bmatrix}}_{\hat{x}(k)}$$

where $C \in \mathbb{R}_+^{n(\tau_{\max}+1)\times n(\tau_{\max}+1)}$ is the companion matrix associated to (2).

It follows from Theorem 3.1 that A is a P^1_{max}-matrix if and only if $\mu(A) < 1$. Since $A = A_0 \oplus A_1 \oplus \cdots \oplus A_{\tau_{\max}}$, it follows from Theorem 5.1 in [21] that $\mu(C) < 1$. Thus, the system (2) is asymptotically stable.

Now assume that for some integer values of $\tau_{ij} \geqslant 0$, the system (2) is asymptotically stable. Then we can proceed as above to write the system in the form (3). By assumption the companion matrix C associated with the system will have $\mu(C) < 1$. It then follows from Theorem 5.1 in [21] that $\mu(A) < 1$ and hence that A is a P^1_{max}-matrix by Theorem 3.1.

This completes the proof. □

4. The row-$\mathsf{P}^1_{\max}$-property and $\mathsf{S}_{\max}$-property for sets of matrices and generalised spectral radius

In this section, in the spirit of [1] we extend the P^1_{max}-property to sets of matrices and derive analogous results to the equivalence of (i), (ii), (iii) and (v) established in Theorem 3.1. Further, we are concerned with the relation between the row-P^1_{max}-property for sets of matrices, the S_{max}-property and the stability of discrete inclusions in the max algebra.

Throughout this section, $\Psi \subset \mathbb{R}_+^{n\times n}$ denotes a finite set of $n \times n$ nonnegative matrices:

$$\Psi := \{A_1, A_2, \dots, A_p : p > 0\}. \tag{4}$$

We define the row representative set of Ψ as follows

$$\mathsf{R} = \left\{ M \in \mathbb{R}_+^{n\times n} : \text{for } 1 \leqslant j \leqslant n \text{ there exists } A_{i_j} \in \Psi \text{ with } M^{(j)} = A_{i_j}^{(j)} \right\}. \tag{5}$$

Briefly, the matrices $M \in \mathsf{R}$ are formed by choosing corresponding rows from some $A_{i_j} \in \Psi$ where $1 \le i_j \le p$. The following two definitions play a central role in what follows and are inspired by the work of [1] for the conventional algebra.

(i) Ψ has the *row-P^1_{max}-property* if every matrix $M \in \mathsf{R}$ is a P^1_{max}-matrix.
(ii) Ψ has the *S_{max}-property* if there is $v > 0$ such that $A_i \otimes v < v$ for all $i \in \{1, 2, \dots, p\}$.

Note that if Ψ has the row-P^1_{max}-property, then each $A_i \in \Psi$ is also a P^1_{max}-matrix. Hence, Ψ is called a P^1_{max}-matrix set over the max algebra.

Given the set Ψ, we define the matrix $S \in \mathbb{R}_+^{n\times n}$ by

$$S = A_1 \oplus A_2 \oplus \cdots \oplus A_p. \tag{6}$$

4.1. Max-convex hull

The max-convex hull of Ψ is given by

$$CO_{\max}(\Psi) = \left\{ \bigoplus_{i=1}^{p} \alpha_i A_i : A_i \in \Psi, \alpha_i \ge 0, 1 \le i \le p \text{ and } \bigoplus_{i=1}^{p} \alpha_i = 1 \right\}. \tag{7}$$

$CO_{\max}(\Psi)$ is said to be asymptotically stable if $\mu(A) < 1$ for all $A \in CO_{\max}(\Psi)$.

4.2. Generalised spectral radius

In our main result, Theorem 4.1 below, we shall present some facts relating P^1_{max}-matrix sets and the stability of discrete inclusions in the max algebra. We first recall the definition of the generalised spectral radius for the max algebra, which will play a key role in what follows.

Formally, we consider the inclusion:

$$x(k+1) \in A_w \otimes x(k), \quad w \in \{1, 2, \ldots, p\} \tag{8}$$

associated with the set of matrices Ψ. We say that (8) is asymptotically stable if all solutions $x(k)$ converge to zero as k tends to ∞.

As with discrete linear inclusions in the conventional algebra, the generalised spectral radius is intimately related to the asymptotic stability of (8). The max-algebraic version of this concept was introduced in [22] and a version of the so- called Generalised Spectral Radius Theorem was presented there. Subsequent work showing the connection between the max version of the generalised spectral radius and the conventional spectral radius of Hadamard powers was presented in [23].

Before proceeding, we need to introduce some notation. For the set Ψ, let $\Psi^m_\otimes$ denote the set of all products of matrices from Ψ of length $m \geq 1$ in the max algebra

$$\Psi^m_\otimes := \{A_{j_1} \otimes \cdots \otimes A_{j_m} : 1 \leqslant j_i \leqslant p \text{ for } 1 \leqslant i \leqslant m\}. \tag{9}$$

The max version of the generalised spectral radius, $\mu(\Psi)$ is defined by

$$\mu(\Psi) = \limsup_{m\to\infty} (\max_{\psi \in \Psi^m_\otimes} \mu(\psi))^{\frac{1}{m}}. \tag{10}$$

As shown in [22], $\mu(\Psi) < 1$ is equivalent to the asymptotic stability of (8).

The next result is the main contribution of this section. In it, we show the relationship between the row-P^1_{max}-property, the S_{max}-property and the stability of discrete inclusions with delay for the max algebra.

In statement (v) of the theorem, for $w \in \{1, 2, \ldots, p\}$ the notation a^w_{ij} denotes the (i, j)th entry of the matrix $A_w \in \Psi$.

Theorem 4.1. *Let Ψ be a set of $n \times n$ nonnegative matrices given by (4). Then the following are all equivalent:*

(i) Ψ has the row-P^1_{max}-property;
(ii) the generalised spectral radius $\mu(\Psi) < 1$;
(iii) Ψ has the S_{max}-property;
(iv) $CO_{\max}(\Psi)$ is asymptotically stable;
(v) the delayed difference inclusion given by

$$x_i(k+1) \in \bigoplus_{j=1}^{n} a^w_{ij} x_j(k - \tau_{ij}), \quad i = 1, 2, \ldots, n, w \in \{1, 2, \ldots, p\} \tag{11}$$

is asymptotically stable for all $\tau_{ij} \geq 0$, $1 \leqslant i, j \leqslant n$.

Before proving this result, we shall state two key propositions. First, we relate the stability of the matrix S given by (6) to the S_{max}-property of the set R .

Proposition 4.1. *Let S be the matrix given by (6) and $v > 0$ be given. Then, $S \otimes v < v$ is equivalent to $M \otimes v < v$ for all $M \in$ R .*

Proof. Let $v > 0$ be given and let M be a matrix in R . From the definition of R , for each $j \in \{1, 2, \ldots, n\}$ there exists some $A_{i_j} \in \Psi$ with $1 \leqslant i_j \leqslant p$ such that $M^{(j)} = A^{(j)}_{i_j}$. It is explicit that for all j, if $S \otimes v < v$, then

$$M^{(j)} \otimes v = A^{(j)}_{i_j} \otimes v \leqslant S^{(j)} \otimes v < v_j.$$

Hence, $M \otimes v < v$ for all $M \in \mathrm{R}$. For the converse, if $M \otimes v < v$ for all $M \in \mathrm{R}$, $A_i \otimes v < v$ for all $A_i \in \Psi$ since every matrix is also a row representative of itself. Thus, we observe that

$$\bigoplus_{i=1}^{p} A_i \otimes v < \bigoplus_{i=1}^{p} v \Rightarrow S \otimes v < v. \quad \square$$

The next proposition is a restatement of a result of [25] for the max-plus algebra, which was phrased in the language of discrete event systems. In the interests of clarity and completeness we provide a direct max-algebraic proof here.

Proposition 4.2. *Let Ψ be a set of $n \times n$ nonnegative matrices given by (4). Let S be the matrix given by (6). Then, $\mu(S) = \mu(\Psi)$.*

Proof. We shall first show that $\mu(\Psi) \leq \mu(S)$. Consider some $\psi \in \Psi^m_\otimes$. It is explicit that $\psi \leq S^m_\otimes$. Then, we have $\mu(\psi) \leq \mu(S^m_\otimes)$. Since this is true for any ψ, we can write

$$\max_{\psi \in \Psi^m_\otimes} \mu(\psi) \leq \mu\left(S^m_\otimes\right).$$

Taking mth root and $\limsup_{m\to\infty}$ of both sides, we obtain

$$\limsup_{m\to\infty} \left(\max_{\psi \in \Psi^m_\otimes} \mu(\psi)\right)^{\frac{1}{m}} \leq \limsup_{m\to\infty} \mu\left(S^m_\otimes\right)^{\frac{1}{m}} = \mu(S),$$

where the final equality follows from the remarks in Section 4 of [12]. Thus, we have $\mu(\Psi) \leq \mu(S)$.

To complete the proof, we show that $\mu(S) \leq \mu(\Psi)$. Let Γ be a critical cycle of length k in $D(S)$ with product $\pi(\Gamma) = s_{i_1 i_2} s_{i_2 i_3} \cdots s_{i_k i_1}$ $(i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\})$. Since $S = A_1 \oplus A_2 \oplus \cdots \oplus A_p$, it follows that there are indices $j_1, j_2, \ldots, j_k \in \{1, 2, \ldots, p\}$ such that

$$\mu(S)^k = \pi(\Gamma) = a^{j_1}_{i_1 i_2} a^{j_2}_{i_2 i_3} \cdots a^{j_k}_{i_k i_1} \leq (A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_k})_{i_1 i_1}.$$

Write $M = A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_k}$. Then, $M \in \Psi^k_\otimes$. For all $r \geq 1$,

$$(M^r_\otimes)_{i_1 i_1} \geq \mu(S)^{kr}.$$

Note that $M^r_\otimes \in \Psi^{kr}_\otimes$ and the above relation implies that $\max_{\psi \in \Psi^{kr}_\otimes} \mu(\psi)^{\frac{1}{kr}} \geq \mu(S)$. If we take $\limsup_{m\to\infty}$ of both sides, we obtain

$$\limsup_{m\to\infty} (\max_{\psi \in \Psi^m_\otimes} \mu(\psi))^{\frac{1}{m}} \geq \mu(S).$$

Thus, we have $\mu(S) \leq \mu(\Psi)$.

So, $\mu(S) = \mu(\Psi)$ as claimed. $\square$

Proof of Theorem 4.1. We will show that each of the conditions from (i) to (v) is equivalent to $\mu(S) < 1$.

(i): First, denote the multigraph associated with the set Ψ by $D(\Psi)$. This consists of the vertices $\{1, 2, \ldots, n\}$ with an edge of weight a^k_{ij} from i to j for every $A_k \in \Psi$ with $1 \leq k \leq p$ for which $a^k_{ij} > 0$. With analogous definitions to the case of a simple graph, $\mu(D)$ denotes the maximal cycle geometric mean of $D(\Psi)$.

Now, assume that Ψ has the row-P^1_{max}-property. Then, $\mu(M) < 1$ for all $M \in \mathrm{R}$. This implies that all cycle products in $D(\Psi)$ are less than one. It follows from Lemma 5.1 in [21] that $\mu(D) = \mu(S)$. So, we obtain that $\mu(S) < 1$.

For the converse, assume $\mu(S) < 1$. Then, from Theorem 3.1 there exists a vector $v > 0$ such that $S \otimes v < v$. It automatically follows from Proposition 4.1 that $\mu(M) < 1$ for all $M \in \mathcal{R}$. So, every $M \in \mathcal{R}$ is a P^1_{max}-matrix. Thus, Ψ has the row-P^1_{max}-property.

(*ii*): It is immediate from Proposition 4.2 that $\mu(\Psi) < 1$ if and only if $\mu(S) < 1$.

(*iii*): First, assume Ψ has the S_{max}-property. Then, there exists a vector $v > 0$ such that $A_i \otimes v < v$ for $1 \leqslant i \leqslant p$. As in the proof of Proposition 4.1 if we add both sides from 1 to p such that $\bigoplus_{i=1}^{p} A_i \otimes v < \bigoplus_{i=1}^{p} v$, we obtain $S \otimes v < v$. Thus, $\mu(S) < 1$.

The converse is trivial.

(*iv*): Let $CO_{\max}(\Psi)$ be asymptotically stable. Notice that $S \in CO_{\max}(\Psi)$. We immediately see that $\mu(S) < 1$.

Now, let $\mu(S) < 1$. Since $A \leq S$ for all $A \in CO_{\max}(\Psi)$, $CO_{\max}(\Psi)$ is asymptotically stable.

(*v*): Following the same procedure as in Theorem 3.2, we can define $\tau_{\max} = \max_{i,j} \tau_{ij}$, $\hat{x}(k) = (x(k-\tau_{\max}), x(k-\tau_{\max}+1), \ldots, x(k))^T$ and companion matrices $C_1, C_2, \ldots, C_p$ where

$$C_w = \begin{bmatrix} 0 & I & 0 & \ldots & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 & I \\ B^w_{\tau_{\max}} & \ldots & \ldots & \ldots & B^w_1 & B^w_0 \end{bmatrix}$$

for $1 \leqslant w \leqslant p$. Note that $A_w = \bigoplus_{i=0}^{\tau_{\max}} B^w_i$.

Then the inclusion (11) is equivalent to the inclusion

$$\hat{x}(k+1) \in C_w \otimes \hat{x}(k),\ w = 1, 2, \ldots, p. \tag{12}$$

By Proposition 4.2, (12) is asymptotically stable if and only if $\mu(C_1 \oplus C_2 \oplus \cdots \oplus C_p) < 1$.

Define $\bar{C} = C_1 \oplus C_2 \oplus \cdots \oplus C_p$ and write

$$\bar{C} = \begin{bmatrix} 0 & I & 0 & \ldots & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 & I \\ \bar{B}_{\tau_{\max}} & \ldots & \ldots & \ldots & \bar{B}_1 & \bar{B}_0 \end{bmatrix}.$$

Then for $i = 0, \ldots, \tau_{\max}$, $\bar{B}_i = \bigoplus_{w=1}^{p} B^w_i$. It follows from Theorem 5.1 in [21] that $\mu(\bar{C}) < 1$ if and only if

$$\mu\left(\bigoplus_{i=0}^{\tau_{\max}} \bar{B}_i\right) < 1.$$

However

$$\bigoplus_{i=0}^{\tau_{\max}} \bar{B}_i = \bigoplus_{i=0}^{\tau_{\max}} \bigoplus_{w=1}^{p} B^w_i = \bigoplus_{w=1}^{p} \bigoplus_{i=0}^{\tau_{\max}} B^w_i = \bigoplus_{w=1}^{p} A_w = S.$$

Thus we have shown that (12) is asymptotically stable if and only if $\mu(S) < 1$. This completes the proof.

Comments on Theorem 4.1. The above result establishes that Ψ has the S_{max}-property if and only if $\mu(M) < 1$ for all M in $\mathcal{R}$. This echoes Theorem 11 of [1] and the result of [24] on linear copositive

Lyapunov functions in the conventional algebra. Note that as in Theorem 3.2, point (v) above is also equivalent to the asymptotic stability of (12) for some $\tau_{ij} \geqslant 0$.

Finally, we present the following result, which is a max-algebra version of Theorem 2 in [1]. As before, the notation a_{ij}^{w} is used to denote the (i, j) entry of the matrix A_w, while $A^{(j)}$ is used to denote the jth row of A.

Proposition 4.3. *Let Ψ be a set of $n \times n$ nonnegative matrices given by (4). Ψ has the row-P^1_{max}-property if and only if for any $x \neq 0$ in $\mathbb{R}^n_+$, there exists an index $k (1 \leq k \leq n)$ such that $(A_i \otimes x)_k < x_k$ for every matrix $A_i \in \Psi$ $(1 \leq i \leq p)$.*

Proof. Let Ψ have the row-P^1_{max}-property. Assume that there exists an $x^* \neq 0$ in $\mathbb{R}^n_+$ such that for every index j with $1 \leqslant j \leqslant n$ there is $A_{i_j} \in \Psi$ satisfying $(A_{i_j} \otimes x^*)_j \geq x^*_j$. It is obvious that $(S \otimes x^*)_j \geq x^*_j$. For each j, there exists an index $k \in \{1, 2, \ldots, n\}$ such that $s_{jk} x^*_k \geq x^*_j$. Since $s_{jk} = a^{i_j}_{jk}$ for some $i_j \in \{1, 2, \ldots, p\}$, we have $A^{(j)}_{i_j} \otimes x^* \geq x^*_j$. We can then construct $M \in \mathrm{R}$ by setting $M^{(j)} = A^{(j)}_{i_j}$ and it is clear that $M \otimes x^* \geq x^*$. This contradicts the assumption that every matrix in R is a P^1_{max}-matrix.

Conversely, let $M \in \mathrm{R}$ be given and let $x \neq 0$ be in $\mathbb{R}^n_+$. Then, there is some k such that $(A_i \otimes x)_k < x_k$, $\forall i \in \{1, 2, \ldots, p\}$. Since it is true for all $A_i \in \Psi$, we also have $(S \otimes x)_k < x_k$. It implies that $(M \otimes x)_k < x_k$. Hence, M is a P^1_{max}-matrix. Thus, Ψ has the row-P^1_{max}-property. This completes the proof. □

5. Conclusions

We have defined P^1_{max}-matrices over the max algebra and shown how some basic properties of P-matrices extend to this class. Further, the relation between the P^1_{max}-property, the S_{max}-property and stability of delayed difference equations has been described. In the spirit of [1] we have also extended the P^1_{max}-property to sets of matrices and shown that the relation between P-matrix sets and the S-property for Z-matrices in the conventional algebra extends to this new setting. The implications of the row-P^1_{max}-property for the stability of max-convex hulls, as well as delayed and undelayed difference inclusions have also been explored.

Acknowledgement

The authors gratefully acknowledge the very helpful comments of the anonymous reviewer, which have considerably improved the manuscript.

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