The Modular and Renormalisation Groups in the Quantum Hall Effect *

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Abstract

An analytic form for the crossover of the conductivity tensor between two Hall plateaux, as a function of the external magnetic field, is proposed. The form of the crossover is obtained from the action of a symmetry group, a particular subgroup of the modular group, on the upper-half complex conductivity plane, by assuming that the β-function describing the crossover is a holomorphic function of the conductivity. The group action also leads to a selection rule, $|p_1 q_2 - p_2 q_1| = 1$, for allowed transitions between Hall plateaux with filling factors $\nu_1 = p_1/q_1$ and $\nu_2 = p_2/q_2$, where q_1 and q_2 are odd.

1   The Quantum Hall Effect

The classical Hall effect occurs when an electric current is passed through a two dimensional slab of conducting or semi-conducting material in a perpendicular magnetic field $\underline{B}$. If the slab lies in the $x - y$ plane, with its longer axis aligned with the x-direction and the magnetic field in the z-direction, then a transverse voltage, the Hall voltage V_{xy}, is generated across the short axis of the sample, when a current I is passed through the sample by applying a longitudinal voltage, V_{xx}, across the long axis. Thus we can define two resistances, a longitudinal resistance R_{xx} and a transverse resistance R_{xy} using Ohm's law,

$$V_{xx} = I \ R_{xx}, \qquad V_{xy} = I \ R_{xy}. \tag{1}$$

In two dimensions resistivity and resistance have the same dimensions and the classical analysis of the Hall effect gives the transverse resistivity ρ_{xy}

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as proportional to the external field $R_{xy} = \rho_{xy} = \frac{B}{ne}$ where n is the density of charge carriers with charge e. For homogeneous conditions the longitudinal resistivity, ρ_{xx}, is related to R_{xx} by a dimensionless geometrical ratio. Classically a graph of ρ_{xy} against B is a straight line whose slope is inversely proportional to n - giving a useful technique for measuring n and the sign of e in a semi-conductor.

Quantum mechanically it was discovered by von Klitzing et al [1] that, for low enough temperatures and high enough magnetic fields, $\rho_{xy}(B)$ increases in a series of steps with very stable plateaux at multiples $\frac{1}{\nu}$ of a fundamental unit of resistance $R_H = h/e^2 = 25.8128k\Omega$, where ν is an integer, h is Planck's constant and e the electric charge. At the same time, $\rho_{xx} = 0$ for values of B where $\rho_{xy} = \frac{1}{\nu}R_H$. This is the integer Quantum Hall Effect (QHE). To achieve some insight into the physics of the integer QHE consider a quantum mechanical two dimensional gas of free electrons. In an external magnetic field this is equivalent to a harmonic oscillator with energy levels Landau levels

$$E_N = \hbar w_c \left(N + \frac{1}{2} \right)$$

where $w_c = \frac{eB}{m}$ is the cyclotron frequency and m is the electron mass — see for example Landau and Lifschitz, [2]. Each level has a degeneracy, g, proportional to the external magnetic field and the area of the sample, A,

$$g = \frac{eBA}{h}.$$

Defining $n_B = \frac{eB}{h}$ as the degeneracy/unit area we see that

$$n_B = \frac{eB}{h} = \frac{ne^2}{h}\rho_{xy}, \tag{2}$$

and the filling factor, ν, for the Landau levels is

$$\nu = n/n_B = \frac{h}{e^2}\frac{1}{\rho_{xy}} \tag{3}$$

where we had used the 'classical' expression $\rho_{xy} = B/ne$ to eliminate B. Thus $\rho_{xy} = \frac{1}{\nu}\frac{h}{e^2} = \frac{1}{\nu}R_H$ and the Hall plateau observed by von Klitzing correspond to stable states in which ν Landau levels are exactly filled. The stability of these states is analogous to the chemical stability of the noble gases, when electron shells are exactly filled. In order to observe the effect, one must work at temperatures low enough that

$$kT \ll \hbar w_c.$$

For $B \sim 1 Tesla$, this requires $T \ll 1K$. This simple analysis omits many ingredients, such as the crucial rôle played by impurities in the QHE, but serves to give some insight into the phenomenon.

Later, in 1982, Tsui et al [3], observed Hall plateaux at fractional filling factors $\nu = p/q$ with p and q small integers, but only with odd q. Again $\rho_{xx} = 0$ at the plateaux. This is the fractional QHE. A good review is Prange and Girvin's book [4]. Note that the QHE can be described equally well in terms of conductivities, rather than resistivities where $\sigma = \rho^{-1}$, $\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$ and $\rho = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix}$.

2 Scaling and Crossover in the Quantum Hall Effect

In order to understand how the resistivity changes as B is varied, one can define β-functions for ρ_{xx} and ρ_{xy}. Khmel'nitskii [5] defined these as

$$\beta_{xx} = L\frac{d\rho_{xx}}{dL} \qquad \beta_{xy} = L\frac{d\rho_{xy}}{dL} \qquad (4)$$

where L is a characteristic length for the electron dynamics — it would be the mean free path in the absence of an external magnetic field but in general will depend on the field. The idea is that different Hall plateaux can be interpreted as different phases of the 2-D electron gas and the plateaux should be fixed points at which the β-functions vanish. Khmel'nitskii suggested a phase diagram for the conductivities, based on topological considerations for the flow, similar to the figure below. There is a further fixed point at a critical field, B_c, between any two Hall plateaux — where the dotted lines cross the solid lines in the figure.
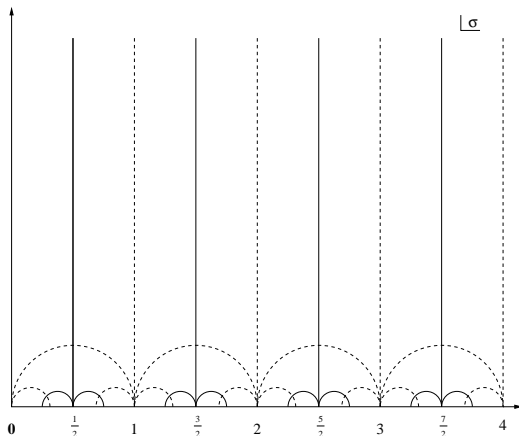

Figure 1. The phase diagram for quantum Hall transitions.
Solid lines represent phase boundaries.

In fact the transition between two hall plateaux $\nu : p_1/q_1 \rightarrow p_2/q_2$ is a quantum phase transition. (Fisher [6]). This differs from the usual phase transitions of statistical mechanics in that the long range order of the ordered phase is destroyed, not by thermal fluctuations as the temperature is varied, but by quantum fluctuations as the external magnetic field is varied. Thus there is a correlation length, ξ, which diverges (in an infinite sample) as the external field approaches a critical value B_c with a critical exponent $\nu_\xi > 0$,

$$\xi \sim |B - B_c|^{-\nu_\xi}.$$

Pruisken [7] has argued that only the field B and the temperature T are relevant near B_c, so L should be a function of B and T only, $L(B,T)$, and similarly for ρ. Since, in units of h/e^2, $\rho_{xy}(B,T)$ is dimensionless it should

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only depend on a dimensionless ratio $\tilde{b} = \Delta B / T^u$ where $\Delta B = B - B_c$ and μ is an anomalous dimension. At various temperatures T we can plot $\rho_{xy}\left(\frac{\Delta B}{T^\mu}\right)$ as a function of ΔB and it should have the same value $\rho_{xy}^c = \rho_{xy}(0)$ at $\Delta B = 0$ for all temperatures, though its slope at $\Delta B = 0$ will be different for different temperatures. Experimentally this is how the critical field, B_c, for the transition between two plateaux is measured, see Shahar et al. [8], figures 1 and 2. There is also experimental evidence, that μ is the same for every quantum Hall transition, giving rise to the concept of super-universality.

Note that, since $\nu \propto 1/B$ classically, we could equally well use the dimensionless parameter, $v = \frac{ne}{T^\mu}\Delta\nu$ to describe the crossover between two Hall plateau, instead of $\tilde{b}$.

3   The Modular Group and the Quantum Hall Effect.

The modular group, $\Gamma(1)$ consisting of 2×2 matrices with integer entries and determinant one, $Sl(2, \mathbf{Z})$, has a natural action on the upper-half complex conductivity plane $\sigma = \sigma_{xy} + i\sigma_{xx}(\sigma_{xx} \geq 0)$, originally noted by Lütken and Ross [9],

$$\sigma \rightarrow \frac{a\sigma + b}{c\sigma + d}, \qquad ad - bc = 1. \qquad (5)$$

Lütken and Ross suggested that this should be a symmetry of the partition function. Fixed points of $\Gamma(1)$ should then be fixed points of the renormalisation group flow. In a further development, [10], Lütken and Ross observed that actually $\Gamma(1)$ is too large — it can map filling factors $p/q \rightarrow p'/q'$ with q odd and q' even, which is incompatible with the experimental observation that only odd denominators are observed in the QHE. A better group to use is a sub-group $\Gamma_0(2) \subset \Gamma(1)$, where $\gamma \in \Gamma_0(2)$ if

$$\gamma = \begin{pmatrix} a & b \\ 2c & d \end{pmatrix}, \qquad ad - 2bd = 1.$$

$\Gamma_0(2)$ preserves the parity of the denominator. Using $\Gamma_0(2)$ one can make predictions about the critical conductivities, since they should correspond to fixed points of $\Gamma_0(2)$. Thus for the transition $\nu : 1 \rightarrow 2$ one expects $\sigma_c = \frac{3+i}{2} \Rightarrow \rho_c = \frac{3+i}{5}$, which predicts that $\rho_{xy}^c = 3/5, \rho_{xx}^c = 1/5$ which can be compared with the experimental results in [8].

The assumption that $\Gamma_0(2)$ is a symmetry also allows one to derive a selection rule for allowed quantum Hall transitions $\nu : p_1/q_1 \rightarrow p_2/q_2$, [11]. By assumption $\nu : p_1/q_1 \rightarrow p_2/q_2$ can be obtained from $\nu : 0 \rightarrow 1$ by some $\gamma = \begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \in \Gamma_0(2)$. Now $\gamma(0) = \frac{b}{d} = \frac{p_1}{q_1}$, $\gamma(1) = \frac{a+b}{2c+d} = p_2/q_2$, so $\gamma = \begin{pmatrix} p_2 - p_1 & p_1 \\ q_2 - q_1 & q_1 \end{pmatrix}$. The condition $det\gamma = 1$ now requires $p_2 q_1 - p_1 q_2 = 1$, which is a selection rule which is well observed in the experimental data, see for example [12]. The critical conductivity for the transition is

predicted by $\Gamma_0(2)$ symmetry to be

$$\sigma_c = \gamma\left(\frac{1+i}{2}\right) = \frac{p_2 q_2 + p_1 q_1 + i}{(q_1^2 + q_2^2)} \tag{6}$$

in units of e^2/h. This action of $\Gamma_0(2)$ is in fact an extension to the complex conductivity plane of the "law of corresponding states" of reference [13].

4 Crossover in the Quantum Hall Effect

By making certain assumptions about the analytic form of the crossover, compatible with $\Gamma_0(2)$ symmetry, one can make very strong predictions which can then be used to test the assumptions. We shall define β-functions for σ using the dimensionless variable v defined previously, but in order to be as general as possible we shall define

$$\beta = \frac{d\sigma}{ds} \tag{7}$$

where s is some real analytic monotonic function of v. In general, β might depend on both σ and its complex conjugate, $\bar{\sigma}$, but if we make the assumption that β is analytic (holomorphic) and depends only on σ we get some very tight restrictions on the form of β (a possible form of the beta-function with is not holomorphic was proposed in [14]).[1] Since

$$\beta(\gamma(\sigma)) = \frac{d\gamma(\sigma)}{dS} = \frac{1}{(2c\sigma + d)^2}\beta(\sigma) \tag{8}$$

β is what is known in the mathematical literature as a modular form of weight -2 for the group $\Gamma_0(2)$ (See e.g. Ran-kin [16]). We shall now follow an analysis similar to that of Ritz [17]. Such modular forms can be written as a ratio of two polynomials

$$\beta(\sigma) = \frac{c}{f'}\prod_{i=1}^{N}(f - \alpha_i)^{m_i} \tag{9}$$

where $m_i \in \mathbf{Z}$ and c and α_i are constants. The function $f(\sigma)$ is defined to be $f(\sigma) = -\frac{\theta_3^4\theta_4^4}{\theta_2^8}$ where $\theta_2 = \sum_n e^{\pi i(n+1/2)^2\sigma}, \theta_3 = \sum_n e^{\pi i n^2\sigma}, \theta_4 = \sum_n(-1)^n e^{\pi i n^2\sigma}$ are Jacobi θ-functions, and $f' = \frac{df}{d\sigma}$. f has the property of being invariant under $\Gamma_0(2)$, $f(\gamma(\sigma)) = f(\sigma)$, as shown in [16].

We can obtain information about the possible values of α_i by appealing to experiment. By assumption the β-function should vanish at $\sigma = 1$ and $\sigma = 2$, since these are are Hall plateaux. Equivalently, since $\sigma \to \sigma - 1$ is a $\Gamma_0(2)$ transformation, the β-function should vanish at $\sigma = 0$ and $\sigma = 1$. Experimentally [8] there is also a critical point at $\sigma_c = \frac{1+i}{2}$ (see figure 2). Now analytically $f(0) = f(1) = 0$ and $f(\frac{1+i}{2}) = 1/4$, therefore $\alpha_i = 0$

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[1]After this talk was presented at Faro another non-analytic β-function was proposed in [15].

or 1/4. We shall make the minimal assumption that there are no other critical points. Thus

$$\beta(\sigma) = \frac{c}{f'} f^n (f - 1/4)^m \qquad (10)$$

for some integers n and m and a constant, c. Further constraints can be placed on β by making some further, rather reasonable, assumptions.

- β is finite as $\sigma \to i\infty \Rightarrow n + m \leq 1$ (since $f \to -\infty$ and $f' \to -2\pi i f$ as $\sigma \to i\infty$).

- $\beta \to 0$ as $\sigma \to 0 \Rightarrow n \geq 1$, and so $m \leq 0$ (since $f \to -16 e^{-\frac{i\pi}{\sigma}}$ and $f' \to i\pi \frac{f}{\sigma^2}$ as $\sigma \to 0$, with positive imaginary part).

- $\frac{d\sigma_{xx}}{ds} \leq 0$ and $\frac{d\sigma_{xy}}{ds} = 0$ as $\sigma \to i\infty \Rightarrow c$ is real.

Without loss of generality, we can choose $c = 1$. This leads to the form

$$\beta(\sigma) = \frac{(-1)^{\tilde{m}+n}}{f'} \frac{f^n}{(f - 1/4)^{\tilde{m}}} \qquad (11)$$

where $\tilde{m} = -m \geq 0$ and $1 \leq n \leq 1 + \tilde{m}$ (further details are given in [18]). The simplest case is $\tilde{m} = 0, n = 1$ and there are arguments [18] that this is the best choice compatible with experiment, because it gives the fastest approach to Hall plateaux. One then has

$$\frac{d\sigma}{ds} = -f/f' \qquad (12)$$

which was considered in a completely different context, that of QCD, by Latorre and Lütken in [19]. The resulting renormalisation group flow can be determined by integrating

$$\frac{ds}{d\sigma} = -f'/f \Rightarrow s(\sigma) = -\ln f/f_0 + s_0 \qquad (13)$$

where s_0 is a constant and $f_0 = f(\sigma(s_0))$. Since s is real, f/f_0 must be real and thus $arg(f)$ is constant along the trajectories. The flow is shown in the figure below, which is simply a contour plot of $arg(f)$, and reproduces the topology of Khmel'nitskii's flow diagram, but with the vertical axis normalised so that there are unstable fixed points at $\sigma_c = \frac{1+i}{2}$ and its images under $\Gamma_0(2)$.
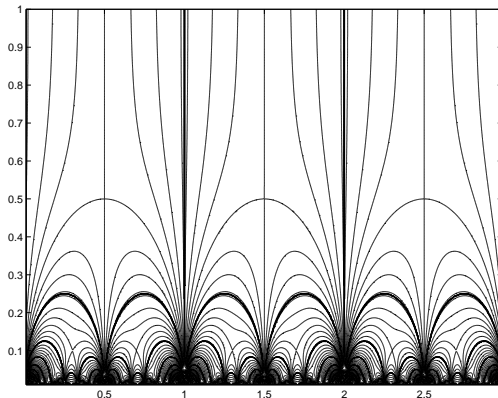

Figure 2. The flow diagram for the complex conductivity

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We can obtain an explicit form for $\sigma(s)$ using well known relations between Jacobi θ-functions and complete elliptic integrals of the second kind

$$K(k) = \int_0^{\pi 12} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},\tag{14}$$

see for example Whittaker and Watson [20],

$$\theta_2^2 = \frac{2kK(k)}{\pi} \quad \theta_3^2 = \frac{2K(k)}{\pi} \quad \theta_4^2 = \frac{2k'K(k)}{\pi}$$

where $e^{i\pi\sigma} = e^{-\pi K'(k)/K(k)}$ and $K'(k) = K(k')$ with $(k')^2 = 1 - k^2$ the complimentary modulus. Using these relations, and equation (13), one can show that, for the transition $\nu : 0 \to 1$,

$$\sigma(s) = \frac{K'(w)\{K'(w) + iK(w)\}}{(K(w))^2 + (K'(w))^2}\tag{15}$$

where $w^2 = \frac{1 \pm \sqrt{1 - e^{-s}}}{2}$, with $s > 0$ ($s < 0$ corresponds to the vertical lines through $\sigma_c = \frac{n+i}{2}$ in figure 2, which are irrelevant directions —details are given in [18]).

To make contact with experiment, we must say something about how s depends on $\Delta\nu$ or, equivalently, ΔB. Experimentally the slope of σ_{xy} as a function of $\Delta\nu$ is finite and non-zero at $\Delta\nu = 0$, (i.e. at $\sigma_c = \frac{1+i}{2}$). Analytically one finds $\beta(\sigma) \propto \frac{1}{\sqrt{s}}$ near $s = 0$. If one takes $s \propto \Delta\nu^2$, $s = (\frac{A\Delta\nu}{T^\mu})^2$ near $\Delta\nu = 0$ where A is a constant, then $\frac{d\sigma}{d(\Delta\nu)}$ is perfectly finite at $\Delta\nu = 0$. Equation (15) can now be used to determine σ for the crossover between any Quantum Hall plateau $\nu : p_1/q \to p_2/q_2$, by acting on it with an element, γ, of $\Gamma_0(2)$. A subtlety is that one should also transform ν by the *inverse* γ^{-1}. The final result is

$$\sigma = \frac{p_1 q_1 (K(w))^2 + p_2 q_2 (K'(w))^2 + iK(w)K'(w)}{q_1^2(K(w))^2 + q_2^2(K'(w))^2}$$

$$w^2 = 1 - \text{sign}\,(\Delta\nu)\sqrt{\frac{1 - e^{-(\frac{A\Delta\nu}{\zeta T^\mu})^2}}{2}}\tag{16}$$

where $\zeta(\Delta\nu) = \alpha\{(q_1 - q_2)\Delta\nu + \alpha\}$ with $\alpha = (p_2 - p_1) - (q_2 - q_1)\nu_c$. A plot of $\sigma(\Delta\nu)$ for the transition $\Delta\nu : 1 \to 2$ is given in figure 3, which shows remarkable argument with the experimental data in figure 1 of reference [8]. The plot was obtained by using the experimental value $\mu = 0.45$ and the choice $A = 55$ which appears to give a good fit to the data in [8].
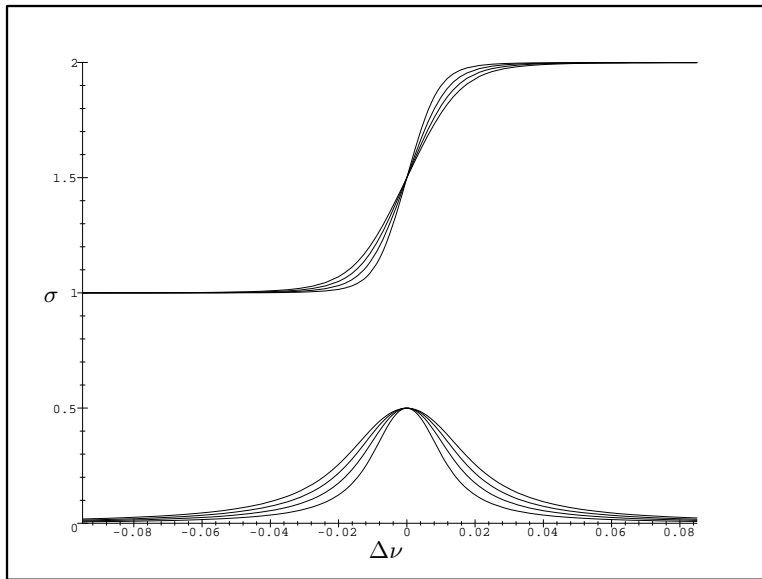
Figure 3: Crossover of conductivity for $\nu : 1 \to 2$ at the four temperatures $T = 42$, 70, 101 and $137mK$ with $A = 55$ and $\mu = 0.45$. To be compared with the experimental data in figure 1 of [8].

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