

# On the second parameter of an $(m, p)$ -isometry

Philipp Hoffmann\*, Michael Mackey and Mícheál Ó Searcoid

November 3, 2011

*Appeared in<sup>1</sup>: Integr. Equ. Oper. Theory 71 (2011), 389-405*

## Abstract

A bounded linear operator  $T$  on a Banach space  $X$  is called an  $(m, p)$ -isometry if it satisfies the equation  $\sum_{k=0}^m (-1)^k \binom{m}{k} \|T^k x\|^p = 0$ , for all  $x \in X$ . In this paper we study the structure which underlies the second parameter of  $(m, p)$ -isometric operators. We concentrate on determining when an  $(m, p)$ -isometry is a  $(\mu, q)$ -isometry for some pair  $(\mu, q)$ . We also extend the definition of  $(m, p)$ -isometry, to include  $p = \infty$  and study basic properties of these  $(m, \infty)$ -isometries.

**Keywords:** Banach space, operator,  $m$ -isometry,  $(m, p)$ -isometry

## 1 Introduction

Let  $H$  be a Hilbert space over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ ,  $B(H)$  the algebra of bounded linear operators  $T$  on  $H$ . Let further the symbol  $\mathbb{N}$  denote the natural numbers including 0 and denote  $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ . For  $m \in \mathbb{N}^+$  an operator  $T \in B(H)$  is called an  $m$ -isometry if

$$\sum_{k=0}^m (-1)^k \binom{m}{k} T^{*k} T^k = 0, \quad (1.1)$$

where  $T^*$  denotes the Hilbert adjoint of  $T$ . (We exclude the trivial case of  $m = 0$ .) This condition is obviously a generalization of the notion of an isometry.

---

\*Philipp Hoffmann is currently a PhD-student at the School of Mathematical Sciences, University College Dublin (National University of Ireland). He is funded by the school as a Research Demonstrator

<sup>1</sup>The final publication is available at [www.springerlink.com](http://www.springerlink.com)  
<http://www.springerlink.com/content/jx47988153826m44/>

In particular, 1-isometries are isometries and an isometry is an  $m$ -isometry for every  $m \in \mathbb{N}^+$ . Operators of this type have been introduced (for  $m = 2$ ) by Agler in [1] and, independently, by Richter in [11]. They have been studied extensively for general  $m$  by Agler and Stankus in three papers [2, 3, 4]. Basic properties of  $m$ -isometries include the facts that every  $m$ -isometry is an  $(m + 1)$ -isometry, that  $m$ -isometries are bounded below and that their spectrum  $\sigma(T) \subseteq \mathbb{K}$  lies in the closed unit disc. The dynamics of  $m$ -isometric operators have been studied in [6] and [10].

If  $H$  is a complex Hilbert space, condition (1.1) can be rewritten as

$$\sum_{k=0}^m (-1)^k \binom{m}{k} \|T^k x\|^2 = 0, \quad \forall x \in H, \quad (1.2)$$

and this formulation can be interpreted in an arbitrary Banach space. Operators which satisfy (1.2) have been studied by Sid Ahmed on complex Banach spaces in [12] and on general function spaces and  $\ell_p$  spaces by Botelho in [8]. In a Banach space, however, there is no intrinsic motivation for the square of the norm which appears in the definition of an  $m$ -isometry. Bayart [5] introduced a more general term: an operator  $T \in B(X)$  (where  $X$  is a Banach space over  $\mathbb{K}$ ) is called an  $(m, p)$ -isometry, if there exist an  $m \in \mathbb{N}^+$  and a  $p \in [1, \infty)$ , such that

$$\sum_{k=0}^m (-1)^k \binom{m}{k} \|T^k x\|^p = 0, \quad \forall x \in X. \quad (1.3)$$

Bayart showed that all basic properties of  $m$ -isometries on Hilbert spaces (which we should now refer to as  $(m, 2)$ -isometries) carry over to  $(m, p)$ -isometries on Banach spaces and, further, that  $(m, p)$ -isometries are never  $N$ -supercyclic if  $X$  is of infinite dimension and complex. In this paper, we do not impose the restriction  $p \geq 1$  and thus will call an operator  $T \in B(X)$  an  $(m, p)$ -isometry, if there exists an  $m \in \mathbb{N}^+$  and a  $p \in (0, \infty)$ , such that (1.3) is satisfied. Most results in the literature remain valid, with their existing proofs, for  $p$  in this extended range. (Example 1.2 is an exception.) Nevertheless, we have found it appropriate to include short alternative proofs of several known results.

Most examples of non-trivial  $(m, p)$ -isometries in the literature are given in the setting of Hilbert space and satisfy (1.2). Botelho gives some examples involving  $\ell_2$  direct sums of Banach spaces in [8]. We will refer to two motivating examples from [12].

**Example 1.1.**  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  on  $(\mathbb{C}^2, \|\cdot\|_2)$  is a  $(3, 2)$ -isometry.

It is easy to check that this example is neither a  $(3, 1)$ -isometry, nor a  $(3, 3)$ -isometry. One may ask whether this operator is a  $(3, p)$ -isometry for any  $p \neq 2$ . We will see that it is not, but that  $T$  is a  $(5, 4)$ -isometry, a  $(7, 6)$ -isometry, a  $(9, 8)$ -isometry and so on.

In the first part of this paper, we will concentrate on determining the pairs  $(\mu, q)$ , with  $\mu \in \mathbb{N}$ ,  $\mu \geq 2$  and  $q \in (0, \infty)$ , for which an  $(m, p)$ -isometry is also a

$(\mu, q)$ -isometry.

The second example ([12, Example 2.4]) is a weighted right shift operator on the Hilbert space  $\ell_2$ . The example can be adapted to general  $\ell_p$  spaces,  $1 \leq p < \infty$ .

**Example 1.2.** Let  $p \in [1, \infty)$ ,  $\lambda \geq 1$  and  $T_p \in B(\ell_p)$  be a weighted right-shift operator with weight sequence  $(\lambda_n)_{n \in \mathbb{N}}$ . That is, for  $x = (x_k)_{k \in \mathbb{N}} \in \ell_p$

$$(T_p x)_n = \begin{cases} 0, & \text{if } n = 0, \\ \lambda_n x_{n-1}, & \text{if } n \geq 1 \end{cases}$$

where  $(\lambda_n)_{n \in \mathbb{N}}$  is given by

$$\lambda_n = \left( \frac{1 + (n+1)(\lambda^2 - 1)}{1 + n(\lambda^2 - 1)} \right)^{1/p}.$$

(Note that for  $\lambda \geq 1$ , this is well-defined.) Then  $T_p$  is a  $(2, p)$ -isometry. This follows from the fact that  $(\lambda_{n+2}\lambda_{n+1})^p - 2(\lambda_{n+1})^p = -1$ .

Remark that for  $\lambda > 1$  we have  $\|T_p x\| > \|x\|$ , for all non-zero  $x$  and, in particular,  $T_p$  is not an isometry for  $\lambda > 1$ .

One may now like to consider the analogue of this example in  $\ell_\infty$ . As  $p$  tends to infinity, the weight sequence becomes a sequence of ones and  $T_\infty$  will just be the right shift operator, which is an isometry. However, it is natural to seek a non-isometric operator which is a  $(2, \infty)$ -isometry - a term which so far has not been defined. In the later parts of this paper, we will give a natural definition of  $(m, \infty)$ -isometries and we will show that their basic properties coincide with those of  $(m, p)$ -isometries.

## 2 Preliminaries

In this section, we cite some basic results concerning  $(m, p)$ -isometric operators proven by Bayart in [5]. There, it is assumed that  $p \geq 1$ , but on inspection it is clear that that this restriction is unnecessary and one can allow  $p \in (0, \infty)$ . However, we will provide alternative proofs for (2.3) and Propositions 2.1 and 2.2 in the next section. (Then (2.4) follows and (2.2) obviously does not depend on the range of  $p$ .)

For  $T \in B(X)$ ,  $p \in (0, \infty)$  and  $l \in \mathbb{N}$  define the functions  $\beta_l^{(p)}(T, \cdot) : X \rightarrow \mathbb{R}$ , by

$$\beta_l^{(p)}(T, x) := \frac{1}{l!} \sum_{j=0}^l (-1)^{l-j} \binom{l}{j} \|T^j x\|^p.$$

(This is analogous to [5].)  $T$  is an  $(l, p)$ -isometry, iff  $\beta_l^{(p)}(T, \cdot) \equiv 0$ .

For  $k, n \in \mathbb{N}$  denote the (descending) Pochhammer symbol by  $n^{(k)}$ , i.e.

$$n^{(k)} := \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n > 0, k > n, \\ \binom{n}{k} k!, & \text{if } n > 0, k \leq n. \end{cases}$$

Then for  $n > 0, k > 0$  and  $k \leq n$  we have

$$n^{(k)} = n(n-1) \cdots (n-k+1).$$

By [5, (2)] we have, for all  $T \in B(X)$  and for all  $n \in \mathbb{N}$ ,

$$\|T^n x\|^p = \sum_{k=0}^n n^{(k)} \beta_k^{(p)}(T, x), \quad \forall x \in X. \quad (2.1)$$

Further, by [5, page 3], the functions  $\beta_l^{(p)}(T, \cdot)$  satisfy

$$l!(\beta_l^{(p)}(T, Tx) - \beta_l^{(p)}(T, x)) = (l+1)!\beta_{l+1}^{(p)}(T, x), \quad \forall x \in X. \quad (2.2)$$

This shows that an  $(m, p)$ -isometry is an  $(m+1, p)$ -isometry. Hence, if  $T$  is an  $(m, p)$ -isometry, we have [5, Proposition 2.1], on discarding terms from (2.1), that

$$\|T^n x\|^p = \sum_{k=0}^{m-1} n^{(k)} \beta_k^{(p)}(T, x), \quad \forall x \in X \text{ and all } n \in \mathbb{N}. \quad (2.3)$$

Finally, we get equation [5, (4)] for an  $(m, p)$ -isometry  $T$ :

$$\lim_{n \rightarrow \infty} \frac{\|T^n x\|^p}{n^{m-1}} = \beta_{m-1}^{(p)}(T, x), \quad \forall x \in X. \quad (2.4)$$

This implies the following useful proposition (see also [6, Corollary 2.4]):

**Proposition 2.1.** *Let  $T \in B(X)$  be an  $(m, p)$ -isometry such that for each  $x \in X$  there exists a real number  $C(x) > 0$ , with*

$$\|T^n x\| \leq C(x), \quad \forall n \in \mathbb{N}.$$

*Then  $T$  is an isometry.*

*Proof.* Since  $T$  is an  $(m, p)$ -isometry, we have

$$0 \leq \beta_{m-1}^{(p)}(T, x) = \lim_{n \rightarrow \infty} \frac{\|T^n x\|^p}{n^{m-1}} \leq \lim_{n \rightarrow \infty} \frac{(C(x))^p}{n^{m-1}} = 0, \quad \forall x \in X.$$

Thus,  $T$  is an  $(m-1, p)$ -isometry. Applying the same argument sufficiently often gives that  $T$  is an isometry.  $\square$

Observe that we have not made use of the uniform boundedness principle above. We deliberately avoid appealing to the linearity or continuity of  $T$  where possible.

Using (2.4), Bayart showed the following in [5, Proposition 2.4.(b)]:

**Proposition 2.2.** *If  $T \in B(X)$  is an invertible  $(m, p)$ -isometry and  $m$  is even then  $T$  is an  $(m - 1, p)$ -isometry.*

This implies in particular, since  $(m, p)$ -isometries are bounded below by definition (see also [12, Lemma 2.1]), that there are no non-trivial examples for  $(2, p)$ -isometries in finite dimensions.

Since referring to an operator as an  $(m, p)$ -isometry does not exclude the possibility of this operator being an  $(m - 1, p)$ -isometry, some statements could become convoluted and this motivates the following terminology.

**Definition 2.3.** Let  $T \in B(X)$  be an  $(m, p)$ -isometry and not an  $(m - 1, p)$ -isometry. Then we call  $T$  a *strict  $(m, p)$ -isometry* or say that  $T$  is  $(m, p)$ -*strict*.

### 3 $(m, p)$ -isometries - an alternative approach

There is little about the basic theory of  $(m, p)$ -isometries which depends on linearity or continuity of the operator in question. Therefore, we find it useful to present an alternative and perhaps more natural approach which simplifies many of the proofs.

**Notation 3.1.** Throughout,  $\mathcal{P}^n$  denotes the space of real polynomial functions of degree less than or equal to  $n$ . Let  $\mathfrak{F}$  denote the set of real functions whose domain is a subset of  $\mathbb{R}$  that is invariant under the mapping  $s : x \rightarrow x + 1$ . We define  $D$  on  $\mathfrak{F}$  by setting  $Df := f - (f \circ s)$  for each  $f \in \mathfrak{F}$ . Note that  $Df \in \mathfrak{F}$ , so that  $D : \mathfrak{F} \rightarrow \mathfrak{F}$  and we can form successive iterates  $D^n f$  on  $\mathfrak{F}$ . Then  $D^m f = \sum_{k=0}^m (-1)^k \binom{m}{k} (f \circ s^k)$  for all  $f \in \mathfrak{F}$  and  $m \in \mathbb{N}$ .

Let now  $\mathfrak{A}$  be the set of all real sequences and note that  $\mathfrak{A} \subseteq \mathfrak{F}$  and  $D(\mathfrak{A}) \subseteq \mathfrak{A}$  and  $D^m a = \left( \sum_{k=0}^m (-1)^k \binom{m}{k} a_{n+k} \right)_{n \in \mathbb{N}}$ , for all  $a \in \mathfrak{A}$  and  $m \in \mathbb{N}$ .

Denote by  $\mathfrak{A}^+$  the positive cone of  $\mathfrak{A}$ . For  $a = (a_n)_{n \in \mathbb{N}} \in \mathfrak{A}^+$  and  $p \in (0, \infty)$ , the sequence  $(a_n^p)_{n \in \mathbb{N}} \in \mathfrak{A}^+$  will be denoted by  $a^p$ . Then, for each  $p \in (0, \infty)$  and  $m \in \mathbb{N}^+$ , we define

$$(i) \quad \mathfrak{A}_{m,p} := \{a \in \mathfrak{A}^+ \mid D^m a^p = 0\},$$

$$(ii) \quad \widehat{\mathfrak{A}}_{m,p} := \{a \in \mathfrak{A}^+ \mid D^m a^p = 0 \text{ and } D^{m-1} a^p \neq 0\};$$

and, for each  $a \in \mathfrak{A}^+$ , we define

$$(iii) \quad \rho(a) := \{(m, p) \mid m \in \mathbb{N}^+, p \in (0, \infty), a \in \mathfrak{A}_{m,p}\},$$

$$(iv) \quad \hat{\rho}(a) := \{(m, p) \mid m \in \mathbb{N}^+, p \in (0, \infty), a \in \widehat{\mathfrak{A}}_{m,p}\}.$$

The following result is surely well-known but, lacking a reference, we provide a proof for completeness.

**Proposition 3.2.** (i)  $D(\mathcal{P}^0) = \{0\}$  and  $D(\mathcal{P}^{m+1}) = \mathcal{P}^m$  for all  $m \in \mathbb{N}$ ;

(ii) for  $a \in \mathfrak{A}$  and  $m \in \mathbb{N}$ ,  $D^{m+1}a = 0$  if, and only if, there exists  $P \in \mathcal{P}^m$  such that  $a = P|_{\mathbb{N}}$ . Moreover, at most one real polynomial function  $P$  satisfies  $a = P|_{\mathbb{N}}$ .

*Proof.* Suppose  $m \in \mathbb{N}$ . It is easy to check that  $D(\mathcal{P}^0) = \{0\}$  and that  $D(\mathcal{P}^{m+1}) \subseteq \mathcal{P}^m$ . For the reverse inclusion, suppose  $P \in \mathcal{P}^m \setminus \{0\}$ . We want to solve the equation  $P = Q - (Q \circ s)$  for some  $Q \in \mathcal{P}^{d+1}$ , where  $d$  is the degree of  $P$ . Every real polynomial function has a unique representation as a linear combination of power functions; equating the coefficients of those power functions on the two sides of the equation  $P = Q - (Q \circ s)$ , we get  $d + 1$  linear equations in the  $d + 1$  unknown coefficients (excluding the constant term) of the proposed polynomial function  $Q$ . Observe that each of the equations has a different number of the unknowns in it, making the equations linearly independent. So there are solutions for  $Q$ , giving  $\mathcal{P}^m \subseteq D(\mathcal{P}^{m+1})$ , as required, and we have proved (i).

By iteration, (i) gives  $D^{m+1}(\mathcal{P}^m) = \{0\}$  for all  $m \in \mathbb{N}$  and the backward implication of (ii) follows. The forward implication is certainly true for  $m = 0$ ; we suppose it true for  $m = k \in \mathbb{N}$ . Consider  $a \in \mathfrak{A}$  and suppose  $D^{k+2}a = 0$ . Then  $D^{k+1}(Da) = 0$  and, by hypothesis, there exists  $P \in \mathcal{P}^k$  with  $Da = P|_{\mathbb{N}}$ . By (i), there exists  $Q \in \mathcal{P}^{k+1}$  with  $P = DQ$ . Then  $Da = (DQ)|_{\mathbb{N}} = D(Q|_{\mathbb{N}})$ , giving  $D(a - Q|_{\mathbb{N}}) = 0$ , and then there is a constant function  $C : \mathbb{R} \rightarrow \mathbb{R}$  such that  $a - Q|_{\mathbb{N}} = C|_{\mathbb{N}}$ . So  $a = (Q + C)|_{\mathbb{N}}$ . Since  $Q + C \in \mathcal{P}^{k+1}$ , the result follows by induction. Uniqueness of  $P$  is ensured by the fact that a polynomial function is fully determined by its values on an infinite set.  $\square$

**Corollary 3.3.** A sequence  $a$  is an element of  $\mathfrak{A}_{m,p}$  if, and only if, there exists  $P \in \mathcal{P}^{m-1}$  such that  $P|_{\mathbb{N}} = a^p$  (that is,  $P(n) = a_n^p$  for all  $n \in \mathbb{N}$ ); in this case, the polynomial function  $P$  is uniquely determined by the equation  $P|_{\mathbb{N}} = a^p$ .

*Remark 3.4.* It is immediate from this characterisation that  $a \in \mathfrak{A}_{m,p}$  implies  $a \in \mathfrak{A}_{m+1,p}$ . Thus  $(\mathfrak{A}_{n,p})_{n \in \mathbb{N}}$  is an increasing sequence of sets and  $\mathfrak{A}_{m,p} = \widehat{\mathfrak{A}}_{m,p} \dot{\cup} \widehat{\mathfrak{A}}_{m-1,p} \dot{\cup} \cdots \dot{\cup} \widehat{\mathfrak{A}}_{1,p} \dot{\cup} \{0\}$ .

**Corollary 3.5.** Suppose  $a \in \mathfrak{A}_{m,p}$  and let  $P \in \mathcal{P}^{m-1}$  be the unique polynomial determined by  $P|_{\mathbb{N}} = a^p$  in 3.3. Then  $a \in \widehat{\mathfrak{A}}_{d+1,p}$  where  $d$  is the degree of  $P$ . Conversely, if  $a \in \widehat{\mathfrak{A}}_{m,p}$ , the degree of  $P$  is  $m - 1$ . Moreover,  $P$  has positive leading coefficient.

*Proof.* The first parts are a clear corollary of our notation. That the leading coefficient of  $P$  is positive follows from the fact that  $a^p$  is a positive sequence interpolated on  $\mathbb{N}$  by  $P$ .  $\square$

*Remark 3.6.* Notice that a linear operator  $T \in B(X)$  is an  $(m, p)$ -isometry if, and only if, for every  $x \in X$ ,  $(\|T^n x\|)_{n \in \mathbb{N}} \in \mathfrak{A}_{m,p}$ . Moreover,  $T$  is a strict  $(m, p)$ -isometry if, in addition, for some  $x_0 \in X$ ,  $(\|T^n x_0\|)_{n \in \mathbb{N}} \in \widehat{\mathfrak{A}}_{m,p}$ .

These facts allow us to retrieve most of the basic properties of  $(m, p)$ -isometries in an elementary way. For example, Remark 3.4 shows that an  $(m, p)$ -isometry is an  $(m + 1, p)$  isometry. Proposition 2.1 follows from the fact that a polynomial which is bounded on  $\mathbb{N}$  is constant. We can give a simple unified proof for the following reproducing formulae for an  $(m, p)$ -isometry. Part (i) essentially appears as [6, Theorem 2.1] while part (ii), as already mentioned, appears in [5, Proposition 2.1].

**Proposition 3.7.** *Let  $T \in B(X)$  be an  $(m, p)$ -isometry. Then for all  $n \in \mathbb{N}^+$  with  $n \geq m$  and for all  $x \in X$ :*

$$(i) \quad \|T^n x\|^p = \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{n}{k} \binom{n-1-k}{m-1-k} \|T^k x\|^p,$$

$$(ii) \quad \|T^n x\|^p = \sum_{k=0}^{m-1} n^{(k)} \beta_k^{(p)}(T, x),$$

$$(iii) \quad \|T^n x\|^p = \frac{\sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} \frac{1}{n-k} \|T^k x\|^p}{\sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} \frac{1}{n-k}}.$$

*Proof.* As  $T$  is an  $(m, p)$ -isometry the sequence  $a_n = \|T^n x\|^p$  is interpolated by some polynomial  $P \in \mathcal{P}^{m-1}$ . Evidently,  $P$  must also be the (unique) Lagrange polynomial of degree less than or equal to  $m - 1$  which interpolates  $\{(k, a_k) \mid k = 0, 1, \dots, m - 1\}$ . Using the normal form of the Lagrange polynomial to calculate  $P(n)$  yields (i), using the Newton form yields (ii), while using the barycentric form we get (iii).  $\square$

As one more example, we give an alternative derivation of Proposition 2.2 which states that if  $T \in B(X)$  is an invertible  $(m, p)$ -isometry and  $m$  is even, then  $T$  is an  $(m - 1, p)$ -isometry.

*Proof (of Proposition 2.2).* Fix  $x \in X$  and choose  $P \in \mathcal{P}^{m-1}$  with  $P|_{\mathbb{N}} = (\|T^n x\|^p)_{n \in \mathbb{N}}$ . Since  $T$  is invertible, we actually have  $P|_{\mathbb{Z}} = (\|T^n x\|^p)_{n \in \mathbb{Z}}$ . (Indeed, if  $l$  is a negative integer then the sequence  $(\|T^n(T^l x)\|^p)_{n \in \mathbb{N}}$  is interpolated on  $\mathbb{N}$  by a polynomial  $Q \in \mathcal{P}^{m-1}$  but this sequence has  $(\|T^n x\|^p)_{n \in \mathbb{N}}$  as a subsequence and so, by uniqueness of polynomials,  $Q$  must be a translation of  $P$ , that is  $Q(t) = P(t + l)$  for all  $t \in \mathbb{R}$  and so  $\|T^l x\|^p = Q(0) = P(l)$  as claimed.) In particular,  $P$  is positive at every (negative) integer and thus cannot have odd degree. Since  $P \in \mathcal{P}^{m-1}$  and  $m - 1$  is odd, we actually have  $P \in \mathcal{P}^{m-2}$ . As  $x$  was arbitrary in  $X$ , it follows that  $T$  is an  $(m - 1, p)$ -isometry.  $\square$

## 4 $(m, p)$ - and $(\mu, q)$ -isometries

We are interested in determining when an  $(m, p)$ -isometry is simultaneously a  $(\mu, q)$ -isometry, where  $m, \mu \in \mathbb{N}^+$  and  $p, q \in (0, \infty)$ . We have already seen that an  $(m, p)$ -isometry is an  $(m + k, p)$ -isometry, for all  $k \in \mathbb{N}$  and so it is natural to rephrase the problem in terms of strict  $(m, p)$ -isometries. We begin with the following.

**Proposition 4.1.** *Suppose  $m, \mu \in \mathbb{N}^+$  and  $p, q \in (0, \infty)$  and  $\widehat{\mathfrak{A}}_{m,p} \cap \widehat{\mathfrak{A}}_{\mu,q} \neq \emptyset$ . Then  $(m - 1)q = (\mu - 1)p$ .*

*Proof.* Suppose  $a \in \widehat{\mathfrak{A}}_{m,p} \cap \widehat{\mathfrak{A}}_{\mu,q}$ . From Corollary 3.3 we know that there exist polynomials  $P$  and  $Q$  with  $P|_{\mathbb{N}} = a^p$  and  $Q|_{\mathbb{N}} = a^q$ , and (by 3.5) that these polynomials have degree  $m - 1$  and  $\mu - 1$  respectively. Clearly  $m = 1$  forces  $\mu = 1$  and vice versa. So assume now  $m, \mu > 1$ . Then  $\lim_{n \rightarrow \infty} a_n^p / n^{m-1}$  exists and is positive, being the leading coefficient of  $P$  and, similarly,  $\lim_{n \rightarrow \infty} a_n^q / n^{\mu-1}$  exists and is positive (being the leading coefficient of  $Q$ ). Therefore the limits  $\lim_{n \rightarrow \infty} a_n^{pq} / n^{q(m-1)}$  and  $\lim_{n \rightarrow \infty} a_n^{pq} / n^{p(\mu-1)}$  both exist and are positive, forcing the powers of  $n$  in their denominators to be equal.  $\square$

**Lemma 4.2.** *Suppose  $a \in \widehat{\mathfrak{A}}_{m,p}$  and  $a \in \widehat{\mathfrak{A}}_{\mu,q}$ , with  $m, \mu > 1$ . Then  $a \in \widehat{\mathfrak{A}}_{d+1,r}$ , where  $d = \gcd(m - 1, \mu - 1)$  and  $r = \frac{d}{(m-1)}p$ .*

*Proof.* Let  $P$  and  $Q$  be the polynomials of degree  $m - 1$  and  $\mu - 1$  respectively (3.3, 3.5) that satisfy  $P|_{\mathbb{N}} = a^p$  and  $Q|_{\mathbb{N}} = a^q$ . Then  $P^{\mu-1}|_{\mathbb{N}} = a^{p(\mu-1)}$  and  $Q^{m-1}|_{\mathbb{N}} = a^{q(m-1)}$ , so that, since  $p(\mu - 1) = q(m - 1)$  by 4.1, we have  $P^{\mu-1}|_{\mathbb{N}} = Q^{m-1}|_{\mathbb{N}}$  and then  $P^{\mu-1} = Q^{m-1}$ , because a polynomial is fully determined by its values on  $\mathbb{N}$ . So  $P$  and  $Q$  have the same zeroes. If a zero has multiplicity  $\xi$  in  $P$  and  $\zeta$  in  $Q$ , then  $\zeta(m - 1) = \xi(\mu - 1)$ , so that the integer  $(m - 1)/d$  divides  $\xi$ . This is true for all zeroes of  $P$ , so  $P = R^{(m-1)/d}$  where  $R$  is a polynomial of degree  $d$ . Then  $a^{p d / (m-1)} = R|_{\mathbb{N}}$  and 3.3 and 3.5 give  $a \in \widehat{\mathfrak{A}}_{d+1,r}$ .  $\square$

**Proposition 4.3.** *Suppose  $a \in \mathfrak{A}^+$ . Then exactly one of the following occurs:*

- (i)  $\hat{\rho}(a) = \rho(a) = \emptyset$ ;
- (ii)  $\hat{\rho}(a) = \{1\} \times (0, \infty)$  and  $\rho(a) = \mathbb{N}^+ \times (0, \infty)$ ;
- (iii) there exist unique  $m_0 \in \mathbb{N} \setminus \{0, 1\}$  and  $p_0 \in (0, \infty)$  such that
 
$$\hat{\rho}(a) = \{(k(m_0 - 1) + 1, kp_0) \mid k \in \mathbb{N}^+\}$$
 and
 
$$\rho(a) = \{(m, kp_0) \mid k \in \mathbb{N}^+, m > k(m_0 - 1)\}.$$

*Proof.* It is evident that  $\rho(a)$  is empty if and only if  $\hat{\rho}(a)$  is empty and that (ii) occurs if, and only if,  $a$  is a constant sequence, so we suppose that  $a$  is not constant and that  $\hat{\rho}(a) \neq \emptyset$  and show that (iii) occurs.

Let  $m_0$  be the least integer such that  $(m_0, p_0) \in \hat{\rho}(a)$  for some  $p_0 \in (0, \infty)$ . Then, since  $a$  is not constant,  $m_0 > 1$  and  $p_0$  is unique by 4.1. Now  $a \in \widehat{\mathfrak{A}}_{m_0, p_0}$  and, by 3.5, the real polynomial  $P$  with  $P|_{\mathbb{N}} = a^{p_0}$  satisfies  $\deg P = m_0 - 1$ . Then for each  $k \in \mathbb{N}^+$ ,  $a^{kp_0} = P^k|_{\mathbb{N}}$ , with  $\deg P^k = k(m_0 - 1)$  and we invoke

3.3 and 3.5 again to get  $\{(k(m_0 - 1) + 1, kp_0) \mid k \in \mathbb{N}^+\} \subseteq \hat{\rho}(a)$ . If conversely  $(\mu, q) \in \hat{\rho}(a)$ , by 4.2  $(d+1, r) \in \hat{\rho}(a)$  for  $d = \gcd(m_0 - 1, \mu - 1)$ . The minimality of  $m_0$  forces  $d = m_0 - 1$  and then there exists a  $k \in \mathbb{N}^+$  with  $\mu - 1 = k(m_0 - 1)$ . The form of  $q$  follows from 4.1. So  $\hat{\rho}(a) = \{(k(m_0 - 1) + 1, kp_0) \mid k \in \mathbb{N}^+\}$ , as required. Observing 3.4, we deduce easily that  $\rho(a) = \{(m, kp_0) \mid k \in \mathbb{N}^+, m > k(m_0 - 1)\}$ .  $\square$

Our aim is now to translate these results into the language of  $(m, p)$ -isometric operators by considering the sequences  $(\|T^n x\|^p)_{n \in \mathbb{N}}$  for a given operator  $T$  and  $x \in X$ . For  $m \in \mathbb{N}^+$  and  $p \in (0, \infty)$ , it is useful to define the subsets  $X_{m,p}^T$  of  $X$  by  $X_{m,p}^T := \{x \in X \mid (\|T^n x\|)_{n \in \mathbb{N}} \in \widehat{\mathfrak{A}}_{m,p}\}$ . Let's note some basic properties of these sets before we state a fundamental decomposition theorem for  $(m, p)$ -isometric operators.

**Lemma 4.4.** *Suppose  $m, \mu \in \mathbb{N}^+$  and  $p, q \in (0, \infty)$ . Then, with reference to the operator  $T \in B(X)$ ,*

$$(i) \quad X_{1,p}^T = X_{1,q}^T;$$

$$(ii) \quad X_{m,p}^T \subseteq X_{k(m-1)+1, kp}^T \text{ for all } k \in \mathbb{N}^+;$$

$$(iii) \quad \text{if } X_{m,p}^T \cap X_{\mu,q}^T \text{ is not empty, then } (m-1)q = (\mu-1)p \text{ and, provided } m, \mu > 1, X_{m,p}^T \cap X_{\mu,q}^T = X_{d+1,r}^T, \text{ where } d = \gcd(m-1, \mu-1) \text{ and } r = \frac{d}{m-1}p = \frac{d}{\mu-1}q;$$

$$(iv) \quad \text{if } m > 1 \text{ and } X_{m,p}^T \cap X_{m,q}^T \neq \emptyset, \text{ then } p = q.$$

*Proof.* (i)  $X_{1,p}^T = \{x \in X \setminus \{0\} \mid \|T^n x\|^p = \|x\|^p, \forall n \in \mathbb{N}\}$ .

(ii) If  $m = 1$ , this is clear, so suppose  $m > 1$ , let  $x \in X_{m,p}^T$  and set  $a := (\|T^n x\|)_{n \in \mathbb{N}}$ . Then  $(m, p) \in \hat{\rho}(a)$  and, using the representation of  $\hat{\rho}(a)$  given in 4.3, it follows that  $(k(m-1) + 1, kp) \in \hat{\rho}(a)$  also, so that  $x \in X_{k(m-1)+1, kp}^T$ . So  $X_{m,p}^T \subseteq X_{k(m-1)+1, kp}^T$ .

(iii) Suppose  $X_{m,p}^T \cap X_{\mu,q}^T \neq \emptyset$ ; then  $(m-1)q = (\mu-1)p$  by 4.1 and, if  $m, \mu > 1$ , then  $X_{m,p}^T \cap X_{\mu,q}^T \subseteq X_{d+1,r}^T$  by 4.2 and the reverse inclusion is got from (ii).

(iv) This follows from the equation  $(m-1)q = (m-1)p$  in (iii).  $\square$

**Theorem 4.5.**  *$T \in B(X)$  is a strict  $(m, p)$ -isometry if, and only if, there is an increasing finite sequence  $(\nu_i)_{1 \leq i \leq n}$  in  $\mathbb{N}^+$  with  $\nu_n = m$  such that the sets  $X_{\nu_i, p}^T$  are non-empty and satisfy*

$$X_{\nu_1, p}^T \dot{\cup} \dots \dot{\cup} X_{\nu_n, p}^T \dot{\cup} \{0\} = X.$$

*This partition is independent of the parameters, that is to say, if  $T$  is also a strict  $(\mu, q)$ -isometry and its associated increasing sequence is  $(\lambda_i)_{1 \leq i \leq n'}$ , then*

$n = n'$  and  $X_{\nu_i, p}^T = X_{\lambda_i, q}^T$  for  $1 \leq i \leq n$ ; moreover,  $\lambda_i = 1 + \frac{(\nu_i - 1)q}{p}$  for  $1 \leq i \leq n$ .

*Proof.* Remarks 3.4 and 3.6 argue the necessity of the existence of such a partition and its sufficiency is clear from the definitions. Suppose now that  $T$  is also  $(\mu, q)$ -strict. Notice that if  $x \in X_{\nu_r, p}^T$  then  $x \in X_{\lambda_s, q}^T$  for some  $\lambda_s \in \{1, \dots, \mu\}$  and Proposition 4.1 guarantees that  $(\nu_r - 1)q = (\lambda_s - 1)p$ . In particular, we get the same  $\lambda_s$ , namely  $1 + \frac{(\nu_r - 1)q}{p}$ , for all  $x \in X_{\nu_r, p}^T$  and so  $X_{\nu_r, p}^T \subseteq X_{\lambda_s, q}^T$ . By symmetry,  $X_{\lambda_s, q}^T \subseteq X_{\nu_t, p}^T$  for some  $t$  and the disjoint nature of partitions implies  $t = r$  and therefore  $X_{\nu_r, p}^T = X_{\lambda_s, q}^T$ . So the two partitions are identical; their specified orders also match because the increasing order of  $(\nu_i)_{1 \leq i \leq n}$  is mirrored in the order of  $\left(1 + \frac{(\nu_i - 1)q}{p}\right)_{1 \leq i \leq n}$ . Therefore  $X_{\nu_i, p}^T = X_{\lambda_i, q}^T$  for  $1 \leq i \leq n$ .  $\square$

We use 4.4 and 4.5 freely in the following corollaries.

**Corollary 4.6.** *Suppose  $T \in B(X)$  is a strict  $(m, p)$ -isometry. Then  $T$  is also a strict  $(k(m - 1) + 1, kp)$ -isometry for all  $k \in \mathbb{N}^+$ .*

*Proof.* Suppose  $k \in \mathbb{N}^+$  and let  $(\nu_i)_{1 \leq i \leq n}$  be the sequence associated with  $T$  as an  $(m, p)$ -isometry. We have  $X_{\nu_i, p}^T \subseteq X_{k(\nu_i - 1) + 1, kp}^T$  for  $1 \leq i \leq n$  by 4.4. Since  $\{X_{\nu_i, p}^T \mid 0 \leq i \leq n\}$  is a (disjoint) partition of  $X \setminus \{0\}$ , the inclusions are in fact not proper and  $\{X_{k(\nu_i - 1) + 1, kp}^T \mid 1 \leq i \leq n\}$  is the same partition, making  $T$  a strict  $(k(m - 1) + 1, kp)$ -isometry.  $\square$

**Corollary 4.7.** *Suppose  $T \in B(X)$  is a strict  $(m, p)$ -isometry and a strict  $(\mu, q)$ -isometry. Then  $(m - 1)q = (\mu - 1)p$  and, provided  $m, \mu > 1$ ,  $T$  is a strict  $(d + 1, r)$ -isometry, where  $d = \gcd(m - 1, \mu - 1)$  and  $r = \frac{d}{m - 1}p$ .*

*Proof.*  $m = 1$  forces  $\mu = 1$  and vice versa. Assume now  $m, \mu > 1$ . Let  $(\nu_i)_{1 \leq i \leq n}$  and  $(\lambda_i)_{1 \leq i \leq n}$  be the sequences associated with  $T$  as a strict  $(m, p)$ -isometry and a strict  $(\mu, q)$ -isometry, respectively. Then  $(\lambda_i - 1)p = (\nu_i - 1)q$  and in particular  $(\mu - 1)p = (m - 1)q$ . Further, by 4.4,  $X_{1, p}^T = X_{1, q}^T = X_{1, r}^T$  and  $X_{\nu_i, p}^T = X_{\lambda_i, q}^T = X_{\delta_i + 1, s_i}^T$ , where  $\delta_i = \gcd(\nu_i - 1, \lambda_i - 1)$  and  $s_i = \frac{\delta_i}{(\nu_i - 1)}p$ , for  $i = 2, \dots, n$ . Now, since  $\lambda_n = \mu$  and  $\nu_n = m$ , the equations  $(\lambda_i - 1)p = (\nu_i - 1)q$  give  $(\lambda_i - 1)(m - 1) = (\nu_i - 1)(\mu - 1)$ . Therefore,  $\frac{(\lambda_i - 1)}{\delta_i} \cdot \frac{(m - 1)}{d} = \frac{(\nu_i - 1)}{\delta_i} \cdot \frac{(\mu - 1)}{d}$  and unique prime factorization (except if  $i = 1 = \nu_i$ ) gives  $\frac{(m - 1)}{d} = \frac{(\nu_i - 1)}{\delta_i}$ . Thus,  $s_i = r$  and, moreover,  $(\delta_i + 1)_{1 \leq i \leq n}$  is increasing; in particular  $d + 1 = \delta_n + 1$  is the maximum value in  $(\delta_i + 1)_{1 \leq i \leq n}$ . So  $X_{1, r}^T \dot{\cup} \dots \dot{\cup} X_{\delta_i + 1, s_i}^T \dot{\cup} \dots \dot{\cup} X_{d + 1, r}^T = X \setminus \{0\}$  and  $T$  is a strict  $(d + 1, r)$ -isometry.  $\square$

Note the necessity of  $(m - 1)q = (\mu - 1)p$  in 4.7 implies immediately that if  $m > 1$ , a strict  $(m, p)$ -isometry cannot be a strict  $(m, q)$ -isometry for  $q \neq p$ . Further, simultaneous  $(m, p)$ -strictness and  $(\mu, q)$ -strictness with  $\mu > m$  ( $\mu < m$ ) requires  $q > p$  ( $q < p$ ). In particular, if  $m > 1$ , an  $(m, p)$ -isometry which is also an  $(m, q)$ -isometry for  $q > p$  must be an  $(m - 1, p)$ -isometry.

Notice further, that if  $m - 1$  and  $\mu - 1$  in 4.7 are relatively prime then the partition of  $X$  is  $X = X_{m,p}^T \dot{\cup} X_{1,p}^T \dot{\cup} \{0\} = X_{\mu,q}^T \dot{\cup} X_{1,q}^T \dot{\cup} \{0\}$ . Indeed, assuming  $m$  and  $\mu$  are at least 2,  $X_{\nu,p}^T = X_{\lambda,q}^T$  implies  $\frac{\lambda-1}{\nu-1} = \frac{\mu-1}{m-1}$  ( $= \frac{q}{p}$ ) and then  $2 \leq \nu < m$ ,  $2 \leq \lambda < \mu$  would imply a common factor of  $\mu - 1$  and  $m - 1$ . We elaborate on this theme in the next result, producing several other ways of determining whether or not we have found the smallest  $m$  such that  $T$  is an  $(m, p)$ -isometry (for some  $p$ ).

**Corollary 4.8.** *Suppose  $T \in B(X)$  is a strict  $(m, p)$ -isometry with associated sequence  $(\nu_i)_{1 \leq i \leq n}$ . Suppose that the terms of  $(\nu_i - 1)_{1 \leq i \leq n}$  greater than 0 have no common prime factor and that there are at least two such terms. Then for each  $\mu \in \mathbb{N}^+$  such that  $T$  is a  $(\mu, q)$ -isometry,  $\mu - 1$  is an integer multiple of  $m - 1$  and, in particular,  $m \leq \mu$ .*

*Proof.* Suppose  $T$  is a  $(\mu, q)$ -isometry. We may assume that it is strict and let  $(\lambda_i)_{1 \leq i \leq n}$  be the associated sequence. The hypothesis forces  $m, \mu > 1$ , so  $p/q = (m - 1)/(\mu - 1)$  is rational; write  $p/q$  as  $u/v$  where  $u, v \in \mathbb{N}^+$  and  $\gcd(u, v) = 1$ . For  $\nu_i > 1$ , we have  $(\lambda_i - 1)p = (\nu_i - 1)q$  and therefore  $(\lambda_i - 1)u = (\nu_i - 1)v$ , ensuring that  $u$  divides  $\nu_i - 1$ . Since  $\nu_i > 1$  is arbitrary, our hypothesis forces  $u = 1$  and then  $\mu - 1 = v(m - 1)$ .  $\square$

There is nothing strange about  $\mu - 1$  being a multiple of  $m - 1$  in 4.8. The result identifies  $m$  as the least integer for which  $T$  is a  $(m, p)$ -isometry for any  $p$ . If, for example, the associated sequence has two consecutive integers greater than 1 as terms, in particular if  $X_{m-1,p}^T \neq \emptyset$ , or if there are sufficiently many terms in the sequence ( $n > m/2 + 1$ ), or if  $X_{2,p}^T \neq \emptyset$ , then  $m$  is that smallest number. The reason why it is important to identify this minimum value is shown in the next result.

**Theorem 4.9.** *Suppose  $T \in B(X)$  is a strict  $(m, p)$ -isometry for some  $m > 1$  and  $p \in (0, \infty)$ . Then there exist  $m_0 > 1$  and  $p_0 \in (0, \infty)$  such that  $T$  is a strict  $(\mu, q)$ -isometry if, and only if,  $(\mu, q) = (k(m_0 - 1) + 1, kp_0)$  for some  $k \in \mathbb{N}^+$ .*

*Proof.* Let  $m_0$  be the least integer for which there exists  $p_0$  such that  $T$  is an  $(m_0, p_0)$ -isometry. Sufficiency is in 4.6. Towards necessity, assume  $T$  is  $(\mu, q)$ -strict. Then, by 4.7,  $T$  is  $(d + 1, r)$ -strict for  $d = \gcd(m_0 - 1, \mu - 1)$ . Hence,  $d + 1 \leq m_0$ , and the minimality of  $m_0$  forces  $d = m_0 - 1$ . So, there exists a  $k \in \mathbb{N}^+$  with  $\mu - 1 = k(m_0 - 1)$  and  $q = kp_0$  then follows from 4.7.  $\square$

Let us consider again Examples 1.1 and 1.2 in light of these results.

**Example 4.10.** (i)  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  on  $(\mathbb{C}^2, \|\cdot\|_2)$  is a strict  $(3, 2)$ -isometry.

*Strictness can be asserted because otherwise  $T$  would be a  $(2, 2)$ -isometry, and then by Proposition 2.2 a  $(1, 2)$ -isometry, hence an isometry, which it is not. For the same reason, the index 3 is minimal in the sense of Theorem 4.9. By Theorem 4.9,  $T$  is a strict  $(2k + 1, 2k)$ -isometry, for any  $k \in \mathbb{N}^+$ , and is not a  $(\mu, q)$ -isometry for any other pair  $(\mu, q)$ .*

- (ii)  $T_p \in B(\ell_p)$  in Example 1.2 is a strict  $(2, p)$ -isometry, not a  $(2, q)$ -isometry, for any  $q \neq p$ , but a strict  $(m, (m-1)p)$ -isometry, for any  $m \in \mathbb{N}$ ,  $m \geq 2$  (and for no other pair  $(\mu, q)$ ).

## 5 $(m, \infty)$ -isometric operators

In this section, motivated by the consideration of  $\ell_\infty$  in Example 1.2, we give a natural extension of the definition of  $(m, p)$ -isometric operators where we now allow the second parameter  $p$  to become infinite.

Consider an  $(m, p)$ -isometry  $T \in B(X)$ . We have the following obvious equivalence for all  $x \in X$ :

$$\begin{aligned} & \sum_{k=0}^m (-1)^k \binom{m}{k} \|T^k x\|^p = 0 \\ \Leftrightarrow & \left( \sum_{\substack{k \leq m \\ k \text{ even}}} \binom{m}{k} \|T^k x\|^p \right)^{1/p} = \left( \sum_{\substack{k \leq m \\ k \text{ odd}}} \binom{m}{k} \|T^k x\|^p \right)^{1/p}. \end{aligned}$$

Taking the limit as  $p$  tends to infinity, prompts the following.

**Definition 5.1.** An operator  $T \in B(X)$  is called an  $(m, \infty)$ -isometry, if

$$\max_{\substack{k=0, \dots, m \\ k \text{ even}}} \|T^k x\| = \max_{\substack{k=0, \dots, m \\ k \text{ odd}}} \|T^k x\|, \quad \forall x \in X. \quad (5.1)$$

Clearly a  $(1, \infty)$ -isometry is an isometry and vice versa. For  $m > 1$  however, condition (5.1) does not impose boundedness on a linear map  $T$ . We will see that  $(m, \infty)$ -isometries share many of the basic properties of  $(m, p)$ -isometries even though the two classes have an essentially trivial intersection.

It is obvious on replacing  $x$  in (5.1) by  $T^\ell x$  for  $\ell \in \mathbb{N}$  that  $T$  is an  $(m, \infty)$ -isometry if, and only if, for all  $x \in X$ ,

$$\max_{\substack{k=\ell, \dots, m+\ell \\ k \text{ even}}} a_k = \max_{\substack{k=\ell, \dots, m+\ell \\ k \text{ odd}}} a_k, \quad \forall \ell \in \mathbb{N}, \quad (5.2)$$

where the sequence  $a$  is given by  $(a_k)_{k \in \mathbb{N}} := (\|T^k x\|)_{k \in \mathbb{N}}$ .

We begin our treatment by stating a major property of this kind of operator.

**Theorem 5.2.** *Let  $T \in B(X, \|\cdot\|)$  be an  $(m, \infty)$ -isometry. Then there exists a norm on  $X$ , equivalent to  $\|\cdot\|$ , under which  $T$  is an isometry.*

This condition is not sufficient for an operator to be an  $(m, \infty)$ -isometry (see Remark 5.5). Before we can prove this theorem, we need some preliminary results. The following lemma provides an alternative description of  $(m, \infty)$ -isometries. Throughout,  $\pi(k) = k \bmod 2$  denotes the parity of  $k \in \mathbb{N}$ .

**Lemma 5.3.** *Let  $a \in \mathfrak{A}$  and  $m \in \mathbb{N}^+$ . Then the following are equivalent.*

(i)  $a$  satisfies (5.2), i.e.  $\max_{\substack{k=\ell, \dots, m+\ell \\ k \text{ even}}} a_k = \max_{\substack{k=\ell, \dots, m+\ell \\ k \text{ odd}}} a_k, \forall \ell \in \mathbb{N}$

(ii)  $a$  attains a maximum and  $\max_{k \in \mathbb{N}} a_k = \max_{\substack{k=\ell, \dots, m-1+\ell \\ \pi(k)=\pi(m-1+\ell)}} a_k, \forall \ell \in \mathbb{N}$ .

*Proof.* If  $a$  satisfies (ii), then, for all  $\ell \in \mathbb{N}$ ,

$$\max_{k \in \mathbb{N}} a_k = \max_{\substack{k=\ell, \dots, m-1+\ell \\ \pi(k)=\pi(m-1+\ell)}} a_k \leq \max_{\substack{k=\ell, \dots, m+\ell \\ \pi(k)=\pi(m-1+\ell)}} a_k \leq \max_{k \in \mathbb{N}} a_k$$

and also

$$\max_{k \in \mathbb{N}} a_k = \max_{\substack{k=\ell+1, \dots, m+\ell \\ \pi(k)=\pi(m+\ell)}} a_k \leq \max_{\substack{k=\ell, \dots, m+\ell \\ \pi(k)=\pi(m+\ell)}} a_k \leq \max_{k \in \mathbb{N}} a_k$$

and the ensuing equalities give  $\max_{\substack{k=\ell, \dots, m+\ell \\ \pi(k)=\pi(m-1+\ell)}} a_k = \max_{\substack{k=\ell, \dots, m+\ell \\ \pi(k)=\pi(m+\ell)}} a_k$ ; in other

words,  $a$  satisfies (5.2).

For the reverse implication, suppose  $a$  satisfies (5.2) and fix  $n > m$ . By virtue of (5.2),  $\max_{k=n-m, \dots, n} a_k$  is attained for at least two indices (one even and one odd).

Hence, there exists an  $r < n$ , such that  $a_r \geq a_n$ . It follows that the sequence indeed has a maximum and that it is attained over the first  $m$  indices, that is,  $\max_{k \in \mathbb{N}} a_k = \max_{k=0, \dots, m-1} a_k = \max_{k=0, \dots, m} a_k$ . This last maximum is achieved at either an even or odd index, but in accordance with (5.2), it is achieved both over even and over odd indices. For later reference, remark that this means the maximum of the sequence  $a$  is unchanged if the  $a_0$  term is discarded. In any case, we can write,

$$\max_{k \in \mathbb{N}} a_k = \max_{k=0, \dots, m} a_k = \max_{\substack{k=0, \dots, m \\ \pi(k)=\pi(m-1)}} a_k = \max_{\substack{k=0, \dots, m-1 \\ \pi(k)=\pi(m-1)}} a_k.$$

This establishes that the equation in (ii) holds for  $\ell = 0$ .

Now remark that property (5.2) is inherited by the subsequence  $(a_{k+1})_{k \in \mathbb{N}} =: a'$  obtained on discarding the  $a_0$  term (or indeed any finite number of the leading terms) from  $a$  so that we can apply the above argument to  $a'$  (as already remarked above, the maximum of  $a'$  is the same as that of  $a$ ) to gain that the equation in (ii) holds for  $\ell = 1$ . This process can be repeated ad infinitum in order to assert (ii).  $\square$

Let us translate this into the language of  $(m, \infty)$ -isometric operators:

**Corollary 5.4.** *Let  $T \in B(X)$ . Then  $T$  is an  $(m, \infty)$ -isometry if, and only if, for all  $\ell \in \mathbb{N}$  and all  $x \in X$ ,*

$$\max_{k \in \mathbb{N}} \|T^k x\| = \max_{\substack{k=\ell, \dots, m-1+\ell \\ \pi(k)=\pi(m-1+\ell)}} \|T^k x\|$$

*Remark 5.5.* The backward implication in Lemma 5.3 requires the parity statement and may be false without it. Consider, for example, the sequence  $(b_k)_{k \in \mathbb{N}}$  with  $b_k = \pi(k)$ . Clearly, for  $m = 2$ ,  $\max_{k \in \mathbb{N}} b_k = \max_{k=\ell, \dots, \ell+m-1} b_k$  for all  $\ell \in \mathbb{N}$ , but  $(b_k)_{k \in \mathbb{N}}$  fails (5.2) for any  $m \in \mathbb{N}^+$ .

As one might then expect, the parity statement is required for the backward implication in 5.4 as well. For example, consider  $m = 2$  and any non-isometric operator  $S \in B(X)$  with  $S^2 = I$ , say  $S = \begin{pmatrix} 4 & 5 \\ -3 & -4 \end{pmatrix}$  on  $(\mathbb{C}^2, \|\cdot\|_\infty)$ . Such  $S$  satisfies  $\max(\|S^k x\|)_{k \in \mathbb{N}} = \max\{\|x\|, \|Sx\|\} = \max\{\|S^\ell x\|, \|S^{\ell+1}x\|\}$ , for all  $\ell \in \mathbb{N}$  and all  $x \in X$ . However, it follows directly from the definition that an  $(m, \infty)$ -isometry  $T$  which satisfies  $T^2 = I$  is an isometry. Hence,  $S$  is not a  $(2, \infty)$ -isometry, nor indeed an  $(m, \infty)$ -isometry for any  $m \in \mathbb{N}^+$ .

An operator is called *power bounded* if there exists  $C > 0$  such that  $\|T^n\| \leq C$  for all  $n \in \mathbb{N}$ . The following statement is a trivial consequence of 5.4.

**Corollary 5.6.** *Let  $T \in B(X)$  be an  $(m, \infty)$ -isometry. Then, for all  $n \in \mathbb{N}$ ,*

$$\|T^n x\| \leq \max_{k=0, \dots, m-1} \|T^k x\|, \quad \forall x \in X.$$

*In particular,  $T$  is power bounded by  $C := \max_{k=0, \dots, m-1} \|T^k\|$ .*

We can now easily prove the main result of this section.

*Proof of Theorem 5.2.* Define  $|\cdot| : X \rightarrow [0, \infty)$ , by  $|x| = \max_{k=0, \dots, m-1} \|T^k x\|$ . Since the maximum preserves the triangle inequality and  $T$  is linear,  $|\cdot|$  is a norm on  $X$ . Now, by definition  $\|x\| \leq |x|$ , for all  $x \in X$ . Furthermore, by Corollary 5.6, for  $C := \max_{k=0, \dots, m-1} \|T^k\|$ , we have

$$\|x\| \leq |x| \leq C \cdot \|x\|, \quad \forall x \in X.$$

Thus, the two norms are equivalent. (In particular,  $(X, |\cdot|)$  is a Banach space.) Corollary 5.4 implies that  $T$  is an isometry with respect to  $|\cdot|$ .  $\square$

*Remark 5.7.* The use here of linearity and boundedness of  $T$  is essential as the result obviously fails for non-continuous self-maps of  $X$  satisfying 5.1.

Theorem 5.2 and Corollary 5.6 enable us to deduce easily basic properties of  $(m, \infty)$ -isometric operators which coincide with those of  $(m, p)$ -isometries. We will do so in the next section. First we want to look at some examples of, and sufficiency conditions for,  $(m, \infty)$ -isometric operators.

**Proposition 5.8.** *Let  $T \in B(X)$  and  $m \in \mathbb{N}$ ,  $m \geq 2$ . Suppose that*

$$\|T^m x\| = \|T^{m-1} x\| \quad \text{and} \quad \|T^m x\| \geq \|T^k x\|, \quad k = 0, \dots, m-2, \quad \forall x \in X.$$

*Then  $T$  is an  $(m, \infty)$ -isometry. If  $m = 2$  then this condition is also necessary.*

*Proof.* The first part follows from the definition (5.1) of an  $(m, \infty)$ -isometry. Let  $T$  be an  $(2, \infty)$ -isometry. By definition  $\|Tx\| = \max\{\|T^2x\|, \|x\|\}$ , for all  $x \in X$ . Hence,  $\|Tx\| \geq \|T^2x\|$  and  $\|Tx\| \geq \|x\|$ , for all  $x \in X$ . Replacing  $x$  with  $Tx$  yields  $\|T^2x\| = \max\{\|T^3x\|, \|Tx\|\}$  and therefore  $\|T^2x\| \geq \|Tx\|$  for all  $x \in X$ . So we have equality, which proves the statement.  $\square$

For  $m > 2$  the above condition is not a necessary one for an  $(m, \infty)$ -isometry, as is demonstrated by Example 5.12.

**Proposition 5.9.** *Let  $T \in B(X)$  and  $n \in \mathbb{N}$  be odd. If  $T^n$  is an isometry then  $T$  is a  $(m, \infty)$ -isometry for all  $m \geq 2n - 1$ .*

*Proof.* Suppose  $T^n$  is an isometry. Then  $\|T^kx\| = \|T^{n+k}x\|$  for all  $x \in X$ , for all  $k \in \mathbb{N}$ . If  $k$  is even,  $n + k$  is odd and vice versa. Hence,

$$\{\|T^kx\| : k \in \{0, \dots, 2n - 1\} \text{ even}\} = \{\|T^kx\| : k \in \{0, \dots, 2n - 1\} \text{ odd}\}$$

and  $T$  is a  $(2n - 1, \infty)$ -isometry. Similarly (or by 6.3), one sees that  $T$  is an  $(m, \infty)$ -isometry, for all  $m \geq 2n - 1$ .  $\square$

*Remark 5.10.* Proposition 5.9 is not in general true for even  $n$ . Consider again a non-isometric operator  $S$ , with  $S^2 = I$ . In particular,  $S^2$  is an isometry, but  $S$  cannot be a  $(3, \infty)$ -isometry as this would imply it is an isometry (Remark 5.5).

**Example 5.11.** *Let  $m \in \mathbb{N}$ ,  $m \geq 1$ ,  $p \in [1, \infty]$  and  $T \in B(\ell_p)$  be a weighted right-shift operator with a weight sequence  $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{C}$  such that*

$$|\lambda_n| \geq 1, \text{ for } n = 1, \dots, m - 1, \text{ and } |\lambda_n| = 1, \text{ for } n \geq m.$$

*For readability, we will write  $\mu_n = \lambda_n$ , if  $n \geq m$  (i.e., for those  $\lambda_n$  which are definitely of modulus 1).*

*That is, for all  $x = (x_0, x_1, x_2, x_3, \dots) \in \ell_p$ ,*

$$(Tx)_n = \begin{cases} 0, & n = 0, \\ \lambda_n x_{n-1}, & n < m, \\ \mu_n x_{n-1}, & n \geq m. \end{cases}$$

*Hence,*

$$\begin{aligned} Tx &= (0, \lambda_1 x_0, \lambda_2 x_1, \lambda_3 x_2, \dots, \lambda_{m-1} x_{m-2}, \mu_m x_{m-1}, \mu_{m+1} x_m, \dots), \\ T^2x &= (0, 0, \lambda_2 \lambda_1 x_0, \lambda_3 \lambda_2 x_1, \lambda_4 \lambda_3 x_2, \dots, \\ &\quad \lambda_{m-1} \lambda_{m-2} x_{m-3}, \mu_m \lambda_{m-1} x_{m-2}, \mu_{m+1} \mu_m x_{m-1}, \dots), \\ &\quad \vdots \\ T^{m-1}x &= (0, \dots, 0, \lambda_{m-1} \cdots \lambda_1 x_0, \mu_m \lambda_{m-1} \cdots \lambda_2 x_1, \dots, \\ &\quad \mu_{2m-3} \cdots \mu_m \lambda_{m-1} x_{m-2}, \mu_{2m-2} \cdots \mu_m x_{m-1}, \dots), \\ T^m x &= (0, \dots, 0, \mu_m \lambda_{m-1} \cdots \lambda_1 x_0, \mu_{m+1} \mu_m \lambda_{m-1} \cdots \lambda_2 x_1, \dots, \\ &\quad \mu_{2m-2} \cdots \mu_m \lambda_{m-1} x_{m-2}, \mu_{2m-1} \cdots \mu_m x_{m-1}, \dots), \end{aligned}$$

where the first  $k$  coefficients of  $T^k x$  are zero.

Since  $|\lambda_n| \geq 1$ , for  $n < m$ , and  $|\mu_n| = 1$ , for  $n \geq m$ , we see that  $T$  satisfies  $\|T^m x\| = \|T^{m-1} x\|$  and  $\|T^m x\| \geq \|T^k x\|$ , for  $k = 0, \dots, m-2$ ,  $\forall x \in X$ . It is therefore an  $(m, \infty)$ -isometry by Proposition 5.8. Furthermore, it is obvious that  $T$  is not an isometry for  $m \geq 2$ , if there exists an  $n_0 \geq 1$ , such that  $|\lambda_{n_0}| \neq 1$ .

**Example 5.12.** Let  $T = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$  be defined on  $(\mathbb{K}^2, \|\cdot\|)$ , where  $\|\cdot\|$  denotes an arbitrary norm. Since  $T^3 x = -x$  for all  $x \in \mathbb{K}^2$ , Proposition 5.9 implies that  $T$  is a  $(5, \infty)$ -isometry. It is easy to see that in general  $T$  is not a  $(4, \infty)$ -isometry (e.g. consider  $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $\|\cdot\| = \|\cdot\|_1$ ). Thus, borrowing terminology from the previous section, we see  $T$  is a strict  $(5, \infty)$ -isometry.

## 6 $(m, \infty)$ - and $(m, p)$ -isometries

Since  $(m, \infty)$ -isometries arise as an analogue of  $(m, p)$ -isometric operators, two natural questions present themselves. First, which properties of  $(m, p)$ -isometries are also enjoyed by  $(m, \infty)$ -operators, and second, what is the setwise intersection of these two classes. The answer to the latter question is immediate on comparing Corollary 5.6 with Proposition 2.1.

**Proposition 6.1.** *An  $(m, p)$ -isometry  $T \in B(X)$  is not a  $(\mu, \infty)$ -isometry for any  $\mu \geq 1$ ,  $\mu \in \mathbb{N}$ , unless it is an isometry.*

Nevertheless, we shall see that virtually all basic properties of  $(m, p)$ -isometric operators (as stated in [5] and [12]) do hold for  $(m, \infty)$ -isometries.

**Proposition 6.2.** (i) *Let  $T \in B(X)$  be a  $(1, \infty)$ -isometry. Then  $T$  is an isometry.*

(ii) *Let  $T \in B(X)$  be an isometry. Then  $T$  is an  $(m, \infty)$ -isometry for all  $m \in \mathbb{N}$ ,  $m \geq 1$ .*

(iii) *An  $(m, \infty)$ -isometry is bounded below (hence injective) and has therefore closed range.*

(iv) *If  $T \in B(X)$  is a  $(2, \infty)$ -isometry, then  $\|Tx\| \geq \|x\|$ , for all  $x \in X$ .*

*Proof.* The only statement which might not be immediately clear from the definitions is that an  $(m, \infty)$ -isometry is bounded below, but this follows directly from Theorem 5.2.<sup>2</sup>  $\square$

**Proposition 6.3.** *Suppose  $T \in B(X)$  is an  $(m, \infty)$ -isometry. Then  $T$  is an  $(m+1, \infty)$ -isometry.*

<sup>2</sup>One can also easily derive this directly from the definition, by assuming the opposite and choosing a suitable sequence  $(x_n)_{n \in \mathbb{N}}$  with  $\|x_n\| = 1$  and  $\|Tx_n\| \rightarrow 0$ .

*Proof.* By 5.4,  $\max_{k \in \mathbb{N}} \|T^k x\|$  exists and, for all  $\ell \in \mathbb{N}$  and  $x \in X$ , we have

$$\max_{k \in \mathbb{N}} \|T^k x\| = \max_{\substack{k=\ell+1, \dots, m+\ell \\ \pi(k)=\pi(m+\ell)}} \|T^k x\| \leq \max_{\substack{k=\ell, \dots, m+\ell \\ \pi(k)=\pi(m+\ell)}} \|T^k x\| \leq \max_{k \in \mathbb{N}} \|T^k x\|$$

and the ensuing equality gives the result by invoking 5.4 again.  $\square$

Let now  $T \in B(X)$  be an invertible operator. For  $\ell \in \mathbb{Z}$ , replacing  $x$  by  $T^\ell x$  in (5.1) gives us that  $T$  is an  $(m, \infty)$ -isometry if, and only if, for all (equivalently, for some)  $\ell \in \mathbb{Z}$ ,

$$\max_{\substack{k=\ell, \dots, \ell+m \\ k \text{ even}}} \|T^k x\| = \max_{\substack{k=\ell, \dots, \ell+m \\ k \text{ odd}}} \|T^k x\| \quad \forall x \in X. \quad (6.1)$$

Following a mild modification of the proof, Lemma 5.3 remains valid for a  $\mathbb{Z}$ -indexed sequence  $a$  if  $\ell \in \mathbb{N}$  and  $k \in \mathbb{N}$  are replaced respectively by  $\ell \in \mathbb{Z}$  and  $k \in \mathbb{Z}$ .

**Proposition 6.4.** *Suppose  $T \in B(X)$  is an invertible  $(m, \infty)$ -isometry.*

- (i)  $T^{-1}$  is an  $(m, \infty)$ -isometry.
- (ii) If  $m$  is even,  $T$  is an  $(m-1, \infty)$ -isometry.

*Proof.* (i) This follows from (6.1) on setting  $\ell = -m$ .

(ii) Suppose  $m$  is even. By 5.4,  $\max_{k \in \mathbb{N}} \|T^k x\|$  exists and, for all  $\ell \in \mathbb{N}$  and  $x \in X$ ,

$$\max_{k \in \mathbb{N}} \|T^k x\| = \max_{\substack{k=\ell-1, \dots, m-2+\ell \\ \pi(k)=\pi(m-2+\ell)}} \|T^k x\| = \max_{\substack{k=\ell, \dots, m-2+\ell \\ \pi(k)=\pi(m-2+\ell)}} \|T^k x\|,$$

$\ell-1$  being valid in the middle expression because  $T$  is invertible and  $\ell-1$  being dropped in the last because  $\pi(\ell-1) \neq \pi(m-2+\ell)$  for even  $m$ . The result then follows by invoking 5.4 again.  $\square$

Example 5.12 shows that Proposition 6.4(ii) does not hold in general for odd  $m$ . Theorem 5.2 allows us to easily deduce further similarities between  $(m, p)$ - and  $(m, \infty)$ -isometric operators.

**Proposition 6.5.** *Let  $X$  be a complex Banach space and  $T \in B(X)$  an  $(m, \infty)$ -isometry. Then the approximate point spectrum  $\sigma_{ap}(T)$  of  $T$  lies in the unit circle  $\mathbb{T}$  and the spectral radius  $r(T)$  is equal to 1.*

*Proof.* The approximate point spectrum contains the boundary of the spectrum (see for example [9, Chapter VII. Proposition 6.7]) and spectral properties of an operator do not change under equivalent norms. Thus, the statement follows from Theorem 5.2.  $\square$

*Remark 6.6.* The restriction to the complex case in 6.5 has actually no significance. One can prove that the approximate spectrum contains the boundary of the spectrum generally for bounded operators on real or complex Banach spaces.

Recall now that for a bounded operator  $T \in B(X)$  and a subset  $E \subseteq X$ , the *orbit* of  $E$  under  $T$ , is defined by  $\text{Orb}(E, T) := \bigcup_{n \in \mathbb{N}} T^n(E)$ .  $T$  is called *hypercyclic* if there exists an  $x \in X$  such that  $\text{Orb}(\{x\}, T)$  is dense in  $X$  and  *$N$ -supercyclic* if there exists a subspace  $E \subseteq X$  with  $\dim E = N$ , such that  $\text{Orb}(E, T)$  is dense in  $X$ .

The following follows immediately from Theorem 5.2.

**Proposition 6.7.** *Let  $T \in B(X)$  be an  $(m, \infty)$ -isometry. Then  $T$  is not hypercyclic.*

However, analogous to the  $(m, p)$ -isometry case, we can do better. Bayart proves in [5, Theorem 3.4] that an isometry on a complex infinite-dimensional Banach space is not  $N$ -supercyclic, for any  $N \geq 1$ . Hence, Theorem 5.2 implies:

**Proposition 6.8.** *On a complex infinite-dimensional Banach space  $X$ , an  $(m, \infty)$ -isometry  $T \in B(X)$  is not  $N$ -supercyclic, for any  $N \geq 1$ .*

We conclude that almost all basic properties (Proposition 2.1 being a notable exception) of  $(m, p)$ -isometric operators for  $p \in (0, \infty)$  on Banach spaces are also valid for  $p = \infty$ .

### Acknowledgement

The authors wish to thank the referee for suggestions which led to a significant improvement of this paper.

## References

- [1] J. Agler, A disconjugacy theorem for Toeplitz operators, *Am. J. Math.* 112(1) (1990), 1-14.
- [2] J. Agler and M. Stankus,  $m$ -isometric transformations of Hilbert space, I, *Integr. equ. oper. theory* 21 (1995), 383-429.
- [3] J. Agler and M. Stankus,  $m$ -isometric transformations of Hilbert space, II, *Integr. equ. oper. theory* 23 (1995), 1-48.
- [4] J. Agler and M. Stankus,  $m$ -isometric transformations of Hilbert space, III, *Integr. equ. oper. theory* 24 (1996), 379-421.
- [5] F. Bayart,  $m$ -Isometries on Banach Spaces, to appear in *Math. Nachrichten*.
- [6] T. Bermúdez, I. Marrero, A. Martínón, On the Orbit of an  $m$ -Isometry, *Integr. equ. oper. theory* 64 (2009), 487-494
- [7] T. Bermúdez, A. Martínón, E. Negrín, Weighted Shift Operators Which are  $m$ -Isometries, *Integr. equ. oper. theory* 68 (2010), 301-312
- [8] F. Botelho, On the existence of  $n$ -isometries on  $\ell_p$  spaces, *Acta Sci. Math. (Szeged)* 76 : 1-2 (2010), 183-192

- [9] J. B. Conway, A Course in Functional Analysis, Springer-Verlag, NewYork-Berlin-Heidelberg-Tokyo, Second edition (1990)
- [10] M. Faghih Ahmadi, K. Hedayatian, Hypercyclicity and Supercyclicity of  $m$ -Isometric Operators, to appear in *Rocky Mountain J. Math.*
- [11] S. Richter, Invariant subspaces of the Dirichlet shift, *J. reine angew. Math.* 386 (1988), 205-220.
- [12] O.A. Sid Ahmed,  $m$ -isometric Operators on Banach Spaces, *Asian-Eur. J. Math.* 3 (2010), no. 1, 1-19.

School of Mathematical Sciences, University College Dublin, Ireland

Philipp Hoffmann  
*email: philipp.hoffmann@ucdconnect.ie*

Michael Mackey  
*email: michael.mackey@ucd.ie*

Mícheál Ó Searcóid  
*micheal.osearcoid@ucd.ie*