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The n-Point Condition and Rough CAT(0)

Abstract

We show that for $n \geq 5$, a length space (X, d) satisfies a rough n-point condition if and only if it is rough CAT(0). As a consequence, we show that the class of rough CAT(0) spaces is closed under reasonably general limit processes such as pointed and unpointed Gromov-Hausdorff limits and ultralimits.

Keywords

CAT(0) space, rough CAT(0) space, Gromov hyperbolic space, Gromov-Hausdorff limit, ultralimit

MSC: 30L05, 51M05

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1. Introduction

Gromov hyperbolic spaces and CAT(0) spaces have been intensively studied; see [4], [6], [10], [3] and the references therein. Their respective theories display some common features, notably the canonical boundary topologies. Rough CAT(0) spaces, a class of length spaces that properly contains both CAT(0) spaces and those Gromov hyperbolic spaces that are length spaces, were introduced by the first author and Kurt Falk in a pair of papers to unify as much as possible of the theories of CAT(0) and Gromov hyperbolic spaces: the basic "finite distance" theory of rough CAT(0) spaces was developed in [1], and the boundary theory was developed in [2]. As in the earlier papers, we usually write rCAT(0) in place of rough CAT(0) below. Rough CAT(0) is closely related to the class of bolic spaces of Kasparov and Skandalis [8], [9] that was introduced in the context of their work on the Baum-Connes and Novikov Conjectures, and is also related to Gromov's class of CAT(-1,ε) spaces [7], [5].

One gap in the theory developed so far is the absence of results indicating that the class of rCAT(0) spaces is closed under reasonably general limit processes such as pointed and unpointed Gromov-Hausdorff limits and ultralimits. The purpose of this paper is to fill that gap.

The fact that the CAT(0) class is closed under such limit processes is a consequence of the following well-known result (for which, see [3, II.1.11]):

Theorem A.
A complete geodesic metric space X is CAT(0) if and only if it satisfies the 4-point condition.

In [1, Theorem 3.18], it was shown that a rough variant of the 4-point condition is quantitatively equivalent to a weak version of rCAT(0), and it follows that the class of weak rCAT(0) spaces is closed under reasonably general limit processes. However it seems difficult to decide whether or not all weak rCAT(0) spaces are necessarily rCAT(0). To establish similar limit closure properties for rCAT(0), we prove the following rough analogue to Theorem A; rough n-point conditions are defined in Section 2.

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Theorem 1.1.

Let (X, d) be a length space. If $n \geq 5$ and (X, d) satisfies a C-rough n-point condition for some $C \geq 0$, then (X, d) is C'-rCAT(0), where $C' = C + 2\sqrt{3}$. Conversely, if (X, d) is C_0-rCAT(0) for some $C_0 > 0$, then for all $n \geq 3$, (X, d) satisfies a C-rough n-point condition, where $C = (n-2)C_0$.

After some preliminaries in Section 2, we prove a pair of preparatory lemmas in Section 3. We then prove the main theorem and discuss its limit closure consequences in Section 4.

2. Preliminaries

Whenever we write $\mathbb{R}^2$ in this paper, we always mean the plane with the Euclidean metric attached. Throughout this section, X is a metric space with metric d attached; any extra assumptions on d will be explicitly stated.
A *h-short segment*, $h \geq 0$, in X is a path $\gamma : [0, L] \to X$, $L \geq 0$, satisfying

$$\operatorname{len}(\gamma) \geq d(\gamma(0), \gamma(L)) \geq \operatorname{len}(\gamma) - h.$$

We denote h-short segments connecting points $x, y \in X$ by $[x, y]_h$. It is convenient to use $[x, y]_h$ also for the image of this path, so instead of writing $z = \gamma(t)$ for some $0 \leq t \leq L$, we often write $z \in [x, y]_h$. Given such a path γ and point $z = \gamma(t)$, we denote by $[x, z]_h$ and $[z, y]_h$ respectively the subpaths $\gamma|_{[0,t]}$ and $\gamma|_{[t,L]}$, respectively; note that both of these are h-short segments. A 0-short segment is called a *geodesic segment*, and we write $[x, y]$ in place of $[x, y]_0$.
A metric space (X, d) is a *geodesic space* if for every $x, y \in X$, there exists at least one geodesic segment $[x, y]$. More generally, (X, d) is a *length space* if for every $x, y \in X$ and every $h > 0$, there exists a h-short path $[x, y]_h$.
A *h-short triangle* $T := T_h(x_1, x_2, x_3)$ with vertices $x_1, x_2, x_3 \in X$ is a collection of h-short segments $[x_1, x_2]_h$, $[x_2, x_3]_h$, and $[x_3, x_1]_h$ (the *sides* of T). Given such a h-short triangle T, a *comparison triangle* will mean a Euclidean triangle $\bar{T} := T(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$, such that $|\bar{x}_i - \bar{x}_j| = d(x_i, x_j)$, $i, j \in \{1, 2, 3\}$. Furthermore, we say that $\bar{u} \in [\bar{x}_i, \bar{x}_j]$ is a *comparison point* for $u \in [x_i, x_j]_h$, if

$$|\bar{x}_i - \bar{u}| \leq \operatorname{len}([x_i, u]_h) \quad \text{and} \quad |\bar{u} - \bar{x}_j| \leq \operatorname{len}([u, x_j]_h)\,.$$

A *geodesic triangle* $T = T(x, y, z)$ is just a 0-short triangle. Note that in this case if $\bar{T} := T(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ is a comparison triangle, and $\bar{u} \in [\bar{x}_i, \bar{x}_j]$ is a *comparison point* for $u \in [x_i, x_j]$, then $\bar{u} \in [\bar{x}_i, \bar{x}_j]$ is uniquely determined by the equation $|\bar{x}_i - \bar{u}| = d(u, x_i)$.
A geodesic space (X, d) is a *CAT(0) space* if given any geodesic triangle $T = T(x, y, z)$ with comparison triangle $\bar{T} = \bar{T}(\bar{x}, \bar{y}, \bar{z})$, and any two points $u \in [x, y]$ and $v \in [x, z]$, we have $d(u, v) \leq |\bar{u} - \bar{v}|$, where $\bar{u}$ and $\bar{v}$ are comparison points for u and v.

Definition 2.1.

Given $C > 0$, and a function $H : X \times X \times X \to (0, \infty)$, a length space (X, d) is said to be a C-*rCAT*$(0; H)$ space if the following *C-rough CAT(0) condition* is satisfied:

$$d(u, v) \leq |\bar{u} - \bar{v}| + C\,,$$

whenever

- $x, y, z \in X$;
- $T := T_h(x, y, z)$ is a h-short triangle, where $h = H(x, y, z)$;
- $\bar{T} := T(\bar{x}, \bar{y}, \bar{z})$ is a comparison triangle in $\mathbb{R}^2$ associated with T;
- u, v lie on different sides of T;
- $\bar{u}, \bar{v} \in \bar{T}$ are comparison points for u, v, respectively;

We call $(T_h(x,y,z),u,v)$ the *metric space data* and $(T(\bar{x},\bar{y},\bar{z}),\bar{u},\bar{v})$ the *comparison data*.

Definition 2.2.

Given $C>0$, a length space X is C-*rCAT*$(0;*)$ if there exists $H: X\times X\times X\to(0,\infty)$ such that X is C-rCAT$(0;H)$. (X,d) is C-*rCAT*(0) if it is C-rCAT$(0;H)$ with

$$H(x,y,z)=\frac{1}{1\vee d(x,y)\vee d(x,z)\vee d(y,z)}\,.$$

Let us make some remarks about the above definitions. First, every CAT(0) space is C-rCAT(0) and C'-rCAT$(0;*)$, with $C=2+\sqrt{3}$ and $C'>0$ arbitrary; this follows from Theorem 4.5 and Corollary 4.6 of [1]. Trivially C-rCAT(0) implies C-rCAT$(0;*)$. Conversely, C-rCAT$(0;*)$ implies C'-rCAT(0) for $C':=3C+2+\sqrt{3}$; see [1, Corollary 4.4].
The explicit H in the rCAT(0) condition has proved to be useful, but one situation where rCAT$(0;*)$ is needed is when the parameter C is close to 0. In particular, we show in Theorem 4.16 that if (X_n) is a sequence of C_n-rCAT$(0;*)$ spaces with $C_n\to 0$, then under rather general conditions the resulting limit space is necessarily CAT(0). *A fortiori*, we could change the C_n-rCAT$(0;*)$ hypothesis in this result to C_n-rCAT(0), but such a variant is of no real interest since a length space satisfying a C-rCAT(0) condition for $C<1/2$ has diameter at most C (as a hint, in a space of diameter larger than this, consider a triangle $T(x,x,x)$ containing a side $[x,x]$ that moves away from x and back again). In particular, only a one-point space can be C-rCAT(0) for all $C>0$. By contrast, the class of spaces that are C-rCAT$(0;*)$ for all $C>0$ is quite large: it includes, for instance, all CAT(0) spaces (as mentioned above), as well as examples such as the *deleted Euclidean plane* $D:=\mathbb{R}^2\setminus\{0\}$. That D is an example follows from a more general principle: if (X,d) is C-rCAT$(0;*)$ for all $C>0$, and a subspace (S,d) of X is also a length space then, *a fortiori*, S is C-rCAT$(0;*)$ for all $C>0$ because the set of h-short paths between $x,y\in S$ is a subset of the set of h-short paths between x and y in X.
We now introduce the concept of C-rough subembeddings (into $\mathbb{R}^2$), which we use to define rough n-point conditions.

Definition 2.3.

Let (X,d) be a metric space, $C\ge 0$ and $n\ge 3$ be an integer. Suppose $x_i\in X$ and $\overline{x}_i\in\mathbb{R}^2$ for $0\le i\le n$, with $x_0=x_n$ and $\overline{x}_0=\overline{x}_n$. We say that $(\overline{x}_1,\overline{x}_2,\dots,\overline{x}_n)$ is a C-*rough subembedding* of $(x_1,x_2,\dots,x_n)$ into $\mathbb{R}^2$ if

$$\begin{aligned} d(x_i,x_{i-1}) &= |\overline{x}_i-\overline{x}_{i-1}|\,, && 1\le i\le n\,, \\ d(x_1,x_i) &\le |\overline{x}_1-\overline{x}_i|\,, && 2\le i\le n\,, \quad \text{and} \\ d(x_i,x_j) &\le |\overline{x}_i-\overline{x}_j|+C\,, && 2\le i,j\le n\,. \end{aligned}$$

Definition 2.4.

Let $n\ge 3$ be an integer. A metric space (X,d) satisfies the C-*rough n-point condition*, where $C\ge 0$, if every n-tuple in X has a C-rough subembedding into $\mathbb{R}^2$. We say that X satisfies a rough n-point condition if it satisfies a C-rough n-point condition for some C. The *n-point condition* is the 0-rough n-point condition.

We note that our notion of a rough 5-point condition is somewhat analogous to the *mesoscopic curvature* notion of Delzant and Gromov [5] which they call CAT$_\varepsilon(\kappa)$, although that paper is concerned with $\kappa<0$, whereas our notion corresponds to $\kappa=0$.
Before proceeding further, let us discuss these conditions. If we vary just one of the parameters C and n in the C-rough n-point condition, it is easy to see that decreasing C or increasing n gives a stronger condition; note that to deduce the C-rough $(n-1)$-point condition from the C-rough n-point condition, we simply take $x_n=x_{n-1}$. The 3-point condition is satisfied by all metric spaces.
For geodesic spaces, the 4-point condition is equivalent to CAT(0); see [3, II.1.11]. For length spaces, a C-rough 4-point condition is quantitatively equivalent to a weaker version of rCAT(0) in which the C-rCAT(0) condition is assumed for metric space data $(T_h(x,y,z),u,v)$ only when v is one of the vertices x,y,z; see [1, Theorem 3.18]. However it seems difficult to decide whether or not weak rCAT(0) spaces are necessarily rCAT(0). We do not address that issue in this paper, but we will show that, among length spaces, rCAT(0) is quantitatively equivalent to a rough n-point condition

for $n > 4$. Thus the class of weak rCAT(0) spaces coincides with the class of length spaces satisfying a rough 4-point condition, and the class of rCAT(0) spaces coincides with the class of length spaces satisfying an n-point condition for any value (or all values) of $n > 4$, but we cannot say whether or not a rough 4-point condition implies a rough n-point condition for $n > 4$.
The proof of Theorem 4.2 will require the following simple results.

Lemma 2.5.

Suppose $A, B, B', C \in \mathbb{R}^2$ such that B and B' lie on opposite sides of the line through A and C. Suppose that the sum of the angles at C of the triangles $T(A, B, C)$ and $T(A, B', C)$ is at least π. Then

$$|B - C| + |C - B'| \le |B - A| + |A - B'| .$$

Lemma 2.5 is a part of the well-known Alexandrov's Lemma (for which see [3, I.2.16]) that is sufficient for our purposes. In fact, we need only the following immediate corollary; below, a *filled polygon* in the Euclidean plane means a polygon plus its bounded complementary component.

Corollary 2.6.

Suppose P, P' are filled polygons in $\mathbb{R}^2$ such that:

(a) P and P' share a side $[A, C]$, and lie on opposite sides of $[A, C]$;

(b) the sum of the interior angles at C in P and P' is at least π.

Let Q be the filled polygon $P \cup P'$, and let B, B' be the vertices other than A that are adjacent as vertices to C in P and P', respectively. Then the geodesic path from B to B' in Q consists of the union of the line segments $[B, C]$ and $[C, B']$.

3. Two lemmas

The proof of Theorem 1.1 requires the following two lemmas. The first is a restatement of [1, Lemma 3.12].

Lemma 3.1.

Let x, y be a pair of points in the Euclidean plane $\mathbb{R}^2$, with $l := |x - y| > 0$. Fixing $h > 0$, and writing $L := l + h$, let $\gamma : [0, L] \to \mathbb{R}^2$ be a h-short segment from x to y, parameterized by arclength. Then there exists a map $\lambda : [0, L] \to [x, y]$ such that $\lambda(0) = x$, $\lambda(L) = y$, and

$$\begin{aligned}
|\lambda(t) - x| &\le |\gamma(t) - x| , && 0 \le t \le L , \\
|\lambda(t) - y| &\le |\gamma(t) - y| , && 0 \le t \le L , \\
\delta(t) := \mathrm{dist}(\gamma(t), \lambda(t)) &\le M := \frac{1}{2}\sqrt{2lh + h^2} , && 0 \le t \le L .
\end{aligned}$$

In particular if $h = \varepsilon/(1 \vee l)$ for some $0 < \varepsilon \le 1$, then $\delta(t) \le \sqrt{3\varepsilon}/2$ for all $0 \le t \le L$.

Lemma 3.2.

Assume $x_i, x'_i \in \mathbb{R}^2$ for $i = 0, 1, 2$, with $u_i \in [x_0, x_i]$ and $u'_i \in [x'_0, x'_i]$ for $i = 1, 2$ and let

$$h = \frac{\varepsilon}{1 \vee |x'_0 - x'_1| \vee |x'_0 - x'_2|} ,$$

for some $0 < \varepsilon \le 1$. Suppose further that

$$\begin{aligned}
|x_1 - x_2| &= |x'_1 - x'_2| , \\
|x'_0 - x'_i| \le |x_0 - x_i| &\le |x'_0 - x'_i| + h, \qquad i = 1, 2 .
\end{aligned}$$

and

$$\frac{|u_i - x_0|}{|x_0 - x_i|} = \frac{|u_i' - x_0'|}{|x_0' - x_i'|}, \qquad i = 1, 2.$$

Then $|u_1 - u_2| \le |u_1' - u_2'| + \sqrt{3\varepsilon}$.

Proof. Set

$$s = \frac{|u_1 - x_0|}{|x_1 - x_0|} = \frac{|u_1' - x_1'|}{|x_1' - x_0'|}$$

and

$$t = \frac{|u_2 - x_0|}{|x_2 - x_0|} = \frac{|u_2' - x_0'|}{|x_2' - x_0'|}.$$

We assume without loss of generality that $s \le t$. An elementary calculation using the parallelogram law shows that given x, y, z in the Euclidean plane with $w \in [y, z]$ and $|w - y| = r|z - y|$ we have

$$|x - w|^2 = (1 - r)|x - y|^2 + r|x - z|^2 - r(1 - r)|y - z|^2. \tag{3.3}$$

Using (3.3) twice, we get

$$|u_1 - u_2|^2 = st|x_1 - x_2|^2 + t^2\left(1 - \frac{s}{t}\right)|x_0 - x_2|^2 - st\left(1 - \frac{s}{t}\right)|x_0 - x_1|^2 \tag{3.4}$$

and similarly

$$|u_1' - u_2'|^2 = st|x_1' - x_2'|^2 + t^2\left(1 - \frac{s}{t}\right)|x_0' - x_2'|^2 - st\left(1 - \frac{s}{t}\right)|x_0' - x_1'|^2. \tag{3.5}$$

Setting $|u_1 - u_2| = |u_1' - u_2'| + d$ and subtracting (3.5) from (3.4), we get

$$\begin{aligned}
2d|u_1' - u_2'| + d^2 &= t^2\left(1 - \frac{s}{t}\right)\left(|x_0 - x_2|^2 - |x_0' - x_2'|^2\right) - \\
&\quad - st\left(1 - \frac{s}{t}\right)\left(|x_0 - x_1|^2 - |x_0' - x_1'|^2\right) \\
&\le t^2\left(1 - \frac{s}{t}\right)\left(|x_0 - x_2|^2 - |x_0' - x_2'|^2\right) \\
&\le t^2\left(1 - \frac{s}{t}\right)\left(2h|x_0' - x_2'| + h^2\right) \le 3\varepsilon.
\end{aligned}$$

In particular $d \le \sqrt{3\varepsilon}$, as required. □

4. Proof and consequences

Here we prove Theorem 1.1 and discuss some consequences. First we need a definition.

Definition 4.1.
Suppose (S, d_S) is a metric space, and that for $i = 1, 2$, we have a metric space (X_i, d_i), a closed subspace $S_i \subset X_i$, and a surjective isometry $f_i : S \to S_i$. We then define the gluing of X_1 and X_2 along S_1, S_2 (denoted by $X = X_1 \sqcup_S X_2$) as the quotient of the disjoint union of X_1 and X_2 under the identification of $f_1(s)$ with $f_2(s)$ for each $s \in S$. The glued metric d on X is defined by the equations $d|_{X_i \times X_i} = d_i$, $i = 1, 2$ and

$$d(x_1, x_2) = \inf_{s \in S}\left(d_1(x_1, f_1(s)) + d_2(f_2(s), x_2)\right), \qquad x_1 \in X_1,\ x_2 \in X_2.$$

We note the following easily verified facts about $(X, d) := X_1 \sqcup_S X_2$ defined by gluing as above:

- d restricted to X_i, $i = 1, 2$, coincides with d_i;
- every geodesic segment in X_i, $i = 1, 2$, is also a geodesic segment in X.

We now prove the following slight improvement of Theorem 1.1.

Theorem 4.2.

Let (X, d) be a length space. If $n \geq 5$ and (X, d) satisfies a C-rough n-point condition for some $C \geq 0$, then (X, d) is C'-rCAT(0) and C''-rCAT(0; $*$), where $C' = C + 2\sqrt{3}$ and $C'' > C$ is arbitrary. Conversely, if (X, d) is C_0-rCAT(0; $*$) for some $C_0 > 0$, then for all $n \geq 3$, (X, d) satisfies a C-rough n-point condition, where $C = (n-2)C_0$.

Proof. Assume that (X, d) is a length space. We first prove the forward implication, so we assume that $n \geq 5$ and that (X, d) satisfies a C-rough n-point condition for some $C \geq 0$. It follows trivially that (X, d) satisfies a C-rough 5-point condition. Let $T := T_h(x, y, z)$ be a h-short geodesic triangle in X, where

$$h = H(x, y, z) := \frac{\varepsilon}{1 \vee d(x, y) \vee d(x, z) \vee d(y, z)}, \tag{4.3}$$

and $0 < \varepsilon \leq 1$ is fixed but arbitrary. Assume also that $u \in [x, y]_h$ and $v \in [x, z]_h$. Let (x', u', y', z', v') be a C-rough subembedding of $(x_1, x_2, x_3, x_4, x_5) = (x, u, y, z, v)$ into $\mathbb{R}^2$, so in particular we have

$$d(x, y) \leq |x' - y'|, \qquad d(x, z) \leq |x' - z'|, \qquad d(y, z) = |y' - z'|,$$

and

$$d(u, v) \;\leq\; |u' - v'| + C. \tag{4.4}$$

From the definition of a C-rough subembedding and the fact that T is h-short, it follows that the piecewise linear paths $\gamma_1 = [x', u'] \cup [u', y']$ and $\gamma_2 = [x', v'] \cup [v', z']$ are both h-short. Thus, by Lemma 3.1 we can choose $u'' \in [x', y']$ and $v'' \in [x', z']$ such that

$$|u' - u''| \leq \frac{\sqrt{3}\varepsilon}{2} \quad \text{and} \quad |v' - v''| \leq \frac{\sqrt{3}\varepsilon}{2} \tag{4.5}$$

and such that

$$|u'' - x'| \leq |u' - x'| \quad \text{and} \quad |u'' - y'| \leq |u' - y'| \tag{4.6}$$

and

$$|v'' - x'| \leq |v' - x'| \quad \text{and} \quad |v'' - z'| \leq |v' - z'|. \tag{4.7}$$

We illustrate the construction in Figure 1. The "squiggly lines" on the left are short paths, and significant distances between points not connected by paths are indicated by dashed lines. Note that $\{x', u', y'\}$ and $\{x', v', z'\}$ might not be collinear sets even though $u \in [x, y]_h$ and $v \in [x, z]_h$.
Now let $\bar{T} = T(\bar{x}, \bar{y}, \bar{z})$ be a comparison triangle for T and choose $\bar{u} \in [\bar{x}, \bar{y}], \bar{v} \in [\bar{x}, \bar{z}]$ satisfying:

$$\frac{|\bar{u} - \bar{x}|}{|\bar{x} - \bar{y}|} = \frac{|u'' - x'|}{|x' - y'|} \quad \text{and} \quad \frac{|\bar{v} - \bar{x}|}{|\bar{x} - \bar{z}|} = \frac{|v'' - x'|}{|x' - z'|}. \tag{4.8}$$

Since $|\bar{x} - \bar{y}| = d(x, y) \leq |x' - y'|$, it follows from (4.6) and (4.8) that

$$|\bar{u} - \bar{x}| \leq |u'' - x'| \leq |u' - x'|$$

and

$$|\bar{u} - \bar{y}| \leq |u'' - y'| \leq |u' - y'|,$$

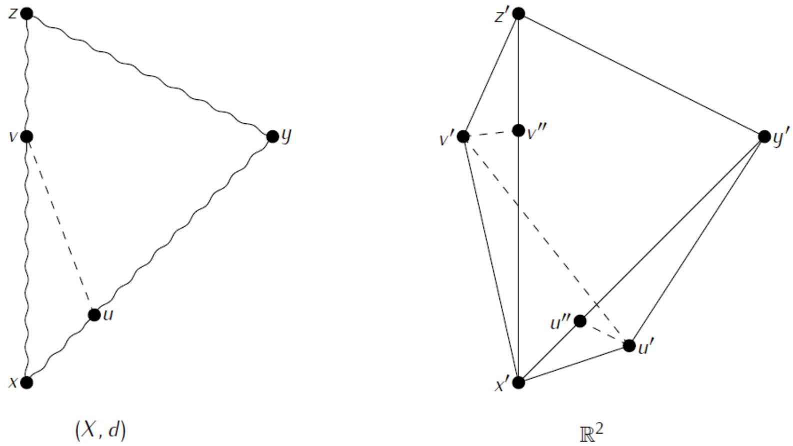

Fig 1. Rough subembedding.

so $\bar{u}$ is a comparison point for u. Similarly $\bar{v}$ is a comparison point for v. Finally, using (4.4) and (4.5), we see that

$$d(u,v) \le |u' - v'| + C \le |u'' - v''| + C + \sqrt{3\varepsilon}\,,$$

and so by Lemma 3.2, we get

$$d(u,v) \le |\bar{u} - \bar{v}| + C + 2\sqrt{3\varepsilon}\,.$$

Thus (X,d) is C'-rCAT(0; $*$), with $C' = C + 2\sqrt{3\varepsilon}$. Taking $\varepsilon = 1$, we see that X is C'-rCAT(0), where $C' = C + 2\sqrt{3}$. Letting $\varepsilon > 0$ be sufficiently small, we see that X is C''-rCAT(0; $*$).

We next proceed with the reverse implication, so let us assume that (X,d) is C'-rCAT(0; $*$). We will prove that (X,d) satisfies the C_n-rough n-point condition, where $C_n := (n-2)C'$ and $n \ge 3$.

The proof will involve induction, but using a stronger inductive hypothesis which involves not just a set of n points, but an n-gon with these points as vertices. Additionally, the inductive process requires us to establish simultaneously a CAT(0) version of the result. Note that it suffices to prove the result for sets of distinct points, since the desired conditions for n points with at least one repeated point follows immediately from the condition for $n-1$ points.

Given $u_1, u_2, \dots, u_n \in X$, $n \ge 3$, we say that P is a h-short n-gon (with vertices $u_1, u_2, \dots, u_n = u_0$) if P is the union of h-short paths $[u_{i-1}, u_i]_h$ for $i = 1, 2, \dots, n$. An n-gon is *geodesic* if it is 0-short. We say that h is *suitably small* if $h < H(u_i, u_j, u_k)$ for all $1 \le i, j, k \le n$.

Suppose

- Q is a geodesic n-gon with distinct vertices $(v_i)_{i=1}^n$ and associated metric d';
- P a h-short n-gon with distinct vertices $(u_i)_{i=1}^n$ and associated metric d;
- $F : Q \to P$ is a map with $F(v_i) = u_i$, $1 \le i \le n$.

Since a geodesic segment is isometrically equivalent to a segment on $\mathbb{R}$, we can view the restriction of F to a single side of Q as being a path, and hence define the path length $\operatorname{len}(F; x, y)$ to be the length of the associated path segment from $F(x)$ to $F(y)$. We call $F : Q \to P$ a *constant speed n-gon map* if P, Q, F are as above, and if for each $1 \le i \le n$ there is a constant K_i such that $\operatorname{len}(F; x, y) = K_i d'(x,y)$ whenever $x, y \in [v_{i-1}, v_i]$. It is easy to see that, given any P, Q as above, a constant speed n-gon map always exists.

Given the following data:

- a h-short n-gon P with distinct vertices $u_1, u_2, \dots, u_n \in X$, where (X,d) is a metric space and h is suitably small;

- a constant speed n-gon map $F : Q \to P$, where Q is a geodesic n-gon with distinct vertices $v_1, v_2, \dots, v_n \in Y$, and (Y, d') is a CAT(0) space,

we define a hypothesis $A_n(P, h; F, Q, d', C_n)$:

$$\begin{aligned}
u_i &= F(v_i)\,, && 1 \le i \le n\,, \\
d(u_{i-1}, u_i) &= d'(v_{i-1}, v_i)\,, && 1 \le i \le n\,, \\
d(u_1, u_i) &\le d'(v_1, v_i)\,, && 2 \le i \le n\,, \\
\operatorname{len}([F(x), u_i]_h) &\ge d'(x, v_i)\,, && x \in Q,\ \ v_i \text{ a vertex adjacent to } x\,, \qquad (4.9) \\
d(F(x), F(y)) &\le d'(x, y) + C_n\,, && x, y \in Q\,. \qquad (4.10)
\end{aligned}$$

The inductive hypothesis for n is that for all P, h as above, there exist data (F, Q, d') such that $A_n(P, h; F, Q, d', C_n)$ holds, and such that (Q, d') is a convex Euclidean n-gon in $\mathbb{R}^2$ with d' being the Euclidean metric. This implies the desired C_n-rough n-point embedding: the vertices of Q give the rough subembedding of the vertices of P. We have defined the hypothesis $A_n(P, h; F, Q, d', C_n)$ in the more general context of a CAT(0) space Y because we will need this along the way.
The CAT(0) version of our inductive hypothesis for n is that for all geodesic n-gons P as above, there exist data (F, Q, d') such that $A_n(P, 0; F, Q, d', 0)$ holds, and such that (Q, d') is a convex Euclidean n-gon in $\mathbb{R}^2$ with d' being the Euclidean metric. Note also that with $h = 0$ and $C_n = 0$ we get equality in (4.9), and (4.10) simplifies to

$$d(F(x), F(y)) \le |x - y|\,. \tag{4.11}$$

It is a routine task to use the C'-rCAT(0) condition to verify the inductive hypothesis for $n = 3$ (and CAT(0) to verify the CAT(0) variant of the inductive hypothesis for $n = 3$), so assume that it holds for $n = k \ge 3$. Let P be a given h-short $(k+1)$-gon, where h is sufficiently small. We draw a h-short path from u_1 to u_k that splits P into a h-short k-gon P_1 with vertices $u_1, \dots, u_k$, and a h-short triangle P_2 with vertices u_1, u_k, u_{k+1}. Let $F_i : Q_i \to P_i$, $i = 1, 2$ be the maps guaranteed by our inductive hypothesis for $n = k$ and the easy case $n = 3$, where Q_1 is a convex k-gon with vertices $v_1, v_2, \dots, v_k \in \mathbb{R}^2$ and Q_2 is a triangle with vertices v_1, v_k, v_{k+1}. By use of isometries of $\mathbb{R}^2$, we may assume that the sides from v_1 to v_k in Q_1 and in Q_2 are the same, and that Q_1 and Q_2 are on opposite sides of this line segment (so the interiors of Q_1 and Q_2 are disjoint).
We now let (Q', d') be the metric space formed by gluing Q_1 and Q_2 together along $S = [v_1, v_k]$, so $Q' = Q_1 \sqcup_S Q_2$. Let Q be the $(k+1)$-gon with vertices $v_1, v_2, \dots, v_{k+1}$ and define $F : Q \to P$ by

$$F(x) = \begin{cases} F_1(x)\,, & x \in Q_1 \cap Q\,, \\ F_2(x)\,, & x \in Q_2 \cap Q\,. \end{cases}$$

Note that the fact that each F_i is a constant speed map ensures that F is well-defined.
We wish to prove $A_{k+1}(P, h; F, Q, d', C_{k+1})$. In view of the construction, it suffices to verify (4.10), and for this we may assume that $x \in Q_1$ and $y \in Q_2$. Let γ be the geodesic in Q connecting x to y. It follows that $\gamma = [x, v] \cup [v, y]$, where $v \in [v_1, v_k]$.
Using (4.10) for P_1 and P_2 and the definition of the gluing metric d' on Q, we thus get

$$\begin{aligned}
d(F(x), F(y)) &\le d(F(x), F(v)) + d(F(v), F(y)) \\
&\le |x - v| + C_k + |v - y| + C_3 \\
&= d'(x, y) + C_{k+1}\,.
\end{aligned}$$

We can deduce $A_{k+1}(P, 0; F, Q, d', 0)$ from the CAT(0) version of our inductive hypothesis by essentially the same argument.
If Q happens to be convex, we are done with the proof so assume that Q is not convex. Then the interior angle at either v_1 or v_k exceeds π. Assume without loss of generality that the interior angle at v_1 is larger than π. By Corollary 2.6,

the union of the two geodesic segments $[v_{k+1}, v_1]$ and $[v_1, v_2]$ is also a geodesic segment and so, by eliminating v_1 as a vertex, we may consider Q to be a geodesic k-gon with vertices $v_2, v_3, \dots, v_{k+1}$. We also note that Q is CAT(0) since Q_1 and Q_2 are CAT(0) and the gluing set $[v_1, v_k]$ is convex; see [3, II.11.1]. Applying the CAT(0) version of our induction assumption to Q, we get a map $G : R \to Q$, where R is a convex k-gon in $\mathbb{R}^2$ with vertices $w_2, w_3, \dots, w_{k+1}$ satisfying:

$$\begin{aligned} v_i &= G(w_i)\,, && 2 \le i \le k+1\,, \\ |v_{i-1} - v_i| &= |w_{i-1} - w_i|\,, && 3 \le i \le k+1\,, \\ |v_2 - v_{k+1}| &\le |w_2 - w_{k+1}|\,, && \\ |G(y) - v_i| &= |y - w_i|\,, && y \in R,\quad w_i \text{ a vertex adjacent to } y\,, \\ |G(y) - G(z)| &\le |y - z|\,, && y, z \in Q\,. \end{aligned}$$

We now view R as a convex $(k+1)$-gon by identifying $G^{-1}(v_1)$ as an extra vertex (with interior angle π). Then $F \circ G$ is the desired mapping (for both the rCAT(0) and CAT(0) variants of our inductive hypothesis). Thus we have established the inductive hypothesis for $n = k+1$ and we are done with the proof. □

For completeness we state a CAT(0) variant of Theorem 1.1.

Theorem 4.12.

A complete geodesic space (X, d) satisfies the n-point condition for fixed $n \ge 4$ if and only if it is CAT(0).

Proof. Since Theorem A already tells us that the 4-point condition is equivalent to CAT(0), it suffices to prove that CAT(0) implies the n-point condition for each $n > 4$. But this follows from the CAT(0) version of our inductive hypothesis which was established in the proof for all $n \in \mathbb{N}$. □

Remark 4.13.

By examining the above proof, we see that if X is C-rCAT(0; ∗), then X is C'-rCAT(0; H') with $C' = 3C + 2\sqrt{3\varepsilon}$, $0 < \varepsilon \le 1$, and

$$H'(x, y, z) := \frac{\varepsilon}{1 \vee d(x,y) \vee d(x,z) \vee d(y,z)}\,. \tag{4.14}$$

Taking $\varepsilon = 1$, this slightly strengthens [1, Corollary 4.4] which states that C-rCAT(0; ∗) implies C'-rCAT(0) for $C' := 3C + 2 + \sqrt{3}$. Also interesting is the case $\varepsilon = 1 \wedge (C^2/3)$: this shows that the C-rCAT(0; H) condition with arbitrary H implies the $(5C)$-rCAT(0; H') condition with the explicit H' given by (4.14).

As mentioned in the Introduction, CAT(0) is preserved by various limit operations, including pointed Gromov-Hausdorff limits and ultralimits [3, II.3.10]. The trick is to use the 4-point condition and the concept of a 4-point limit. A very similar argument, with the 4-point condition replaced by our rough 5-point condition, will give us similar results for rCAT(0) spaces. We begin with a definition of n-point limits.

Definition 4.15.

A metric space (X, d) is an n-point limit of a sequence of metric spaces (X_m, d_m), $m \in \mathbb{N}$, if for every $\{x_i\}_{i=1}^n \subset X$, and $\varepsilon > 0$, there exist infinitely many integers m and points $x_i(m) \in X_m$, $1 \le i \le n$, such that $|d(x_i, x_j) - d_m(x_i(m), x_j(m))| < \varepsilon$ for $1 \le i, j \le n$.

We are now ready to state a 5-point limit result. Note that, since any n-point limit (X, d) of $((X_m, d_m))_{m=1}^\infty$ is also an n'-point limit of this sequence of spaces for all $n' \le n$, the following result also holds if 5 is replaced by any larger integer. The proof of this result, which is very similar to the corresponding result for CAT(0) and 4-point limits given in [3, II.3.9], is included for completeness.

Theorem 4.16.

Suppose the length space (X,d) is a 5-point limit of (X_m,d_m), $m\in\mathbb{N}$, where (X_m,d_m) is C_m-rCAT$(0;*)$ for some constant C_m. If $C_m\le C$ for all $m\in\mathbb{N}$, then (X,d) is a $\widetilde{C}$-rCAT(0) space, where $\widetilde{C}=3C+2\sqrt{3}$. If $C_m\to 0$, and (X,d) is complete, then (X,d) is a CAT(0) space.

Proof. Suppose first that $C_m\le C$ for all $m\in\mathbb{N}$. Let $(x_i)_{i=1}^5$ be an arbitrary 5-tuple of points in (X,d), and suppose that it is the 5-point limit of the 5-tuples $(x_i(m))_{i=1}^5$ in X_m. By passing to a subsequence if necessary, we may assume that $d(x_i(m),x_j(m))\to d(x_i,x_j)$ for all $1\le i,j\le 5$.
By Theorem 4.2, every (X_m,d_m) satisfies a C'-rough 5-point condition, where $C':=3C$, so there exists a C'-rough subembedding $(\overline{x}_1(m),\overline{x}_2(m),\dots,\overline{x}_5(m))$ of $(x_1(m),x_2(m),\dots,x_5(m))$ into $\mathbb{R}^2$, for all $m\in\mathbb{N}$. Since translation is an isometry in $\mathbb{R}^2$, we may assume that the points $\overline{x}_1(m)$ coincide for all $m\in\mathbb{N}$. Thus all 5-tuples are contained in a disk of finite radius and by passing to a subsequence if necessary we may assume that $\overline{x}_i(m)$ converges to some point $\overline{x}_i$ as $m\to\infty$, for all $1\le i\le m$. It follows readily that $(\overline{x}_i)_{i=1}^5$ is a C'-rough subembedding of $(x_i)_{i=1}^5$ in $\mathbb{R}^2$. Thus (X,d) satisfies the C'-rough 5-point condition. By again using Theorem 4.2, we deduce that (X,d) is a $\widetilde{C}$-rCAT(0) space where $\widetilde{C}=C'+2\sqrt{3}$.
If in fact $C_m\to 0$, then (X_m,d_m) satisfies a $(3C_m)$-rough 5-point condition, and it follows as above that (X,d) satisfies the 0-rough 5-point condition, and hence the 4-point condition. This together with completeness and approximate midpoints (as follows from the fact that (X,d) is a length space) implies that (X,d) is a CAT(0) space: see [3, II.1.11]. □

With Theorem 4.16 in hand, it is now routine to deduce the following corollary.

Corollary 4.17.

Suppose (X,d) is a length space and suppose (X_m,d_m), $m\in\mathbb{N}$, form a sequence of C-rCAT(0) spaces. Writing $\widetilde{C}=3C+2\sqrt{3}$, the following results hold.

(a) If (X,d) is a (pointed or unpointed) Gromov-Hausdorff limit of (X_m,d_m) then (X,d) is a $\widetilde{C}$-rCAT(0) space.

(b) If (X,d) is an ultralimit of (X_m,d_m), then (X,d) is a $\widetilde{C}$-rCAT(0) space.

(c) If X is rCAT(0), then the asymptotic cone $\mathrm{Cone}_\omega X:=\lim_\omega(X,d/m)$ is a CAT(0) space for every non-principal ultrafilter ω.

Note that in the proof of Corollary 4.17(c), we need the fact that $\mathrm{Cone}_\omega X$ is complete, but this is true because ultralimits are always complete [3, I.5.53].
In each part of Corollary 4.17, the existence of an approximate midpoint for arbitrary $x,y\in X$ (meaning a point z such that $d(x,z)\vee d(y,z)\le\varepsilon+d(x,y)/2$ for fixed but arbitrary $\varepsilon>0$) follows easily from the hypotheses, and so (X,d) is easily seen to be a length space if it is complete. Thus Corollary 4.17 generalizes the $\kappa=0$ case of [3, II.3.10](1), (2), where the spaces are assumed to be CAT(0) rather than rCAT(0) and the limit space (X,d) is assumed to be complete rather than a length space.

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