Fundamental Limitations in Control over a Communication Channel [1]

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Abstract

Fundamental limitations in feedback control is a well established area of research. In recent years it has been extended to the study of limitations imposed by the consideration of a communication channel in the control loop. Previous results characterised these limitations in terms of a minimal data transmission rate necessary for stabilisation. In this paper a signal-to-noise ratio (SNR) approach is used to obtain a tight condition for the linear time invariant output feedback stabilisation of a continuous-time, unstable, non minimum phase (NMP) plant with time-delay over an additive Gaussian coloured noise communication channel. By working on a linear setting the infimal SNR for stabilisability is defined as the infimal achievable H_2 norm between the channel noise input and the channel signal input. The result gives a guideline in estimating the severity of the fundamental SNR limitation imposed by the plant unstable poles, NMP zeros, time-delay as well as the channel NMP zeros, bandwidth, and channel noise colouring.

Key words: Fundamental constraints; H-2 norm; Signal-to-noise ratio; Networks; Input-output stability.

1 Introduction

The study of fundamental limitations in control design is an established area of research with important early results from Bode (1945) and Horowitz (1963). For a linear time invariant (LTI) plant it is well understood that its unstable

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poles, non minimum phase (NMP) zeros and time-delay will cause unavoidable limitations in performance (see for example Seron et al. (1997) and references therein). In more recent years, the study of fundamental limitations has been extended to problems of control over communication channels and has attracted growing interest (see for example Antsaklis and Baillieul (2004) and the recent survey by Nair et al. (2007)).

A communication channel in the feedback loop can impose a number of constraints to control design. For example, it is shown in Nair and Evans (2004) that for a noiseless (error-free) channel the minimal data transmission rate necessary and sufficient for stabilisation must satisfy a lower bound expressed as a function of the open loop unstable poles of the plant. This result is obtained using fairly complex information theoretic arguments, but is valid for a large class of feedback controllers assumed only to be casual.

A framework for the study of control over a communication channel with a constraint on its signal-to-noise ratio (SNR) has been developed in Braslavsky et al. (2007). In that paper, the authors derive the infimal SNR required to stabilise an unstable plant by feedback over an additive white Gaussian noise (AWGN) communication channel. It is shown that the limitations to control imposed by a SNR constraint can be studied in a completely linear scenario, linking with a large body of well-established tools and techniques. For the problem of feedback stabilisation, such limitations are quantified by the optimal H_2 norm of the closed-loop transfer function between the channel noise input and the channel signal input, and are expressed as a function of the plant unstable poles, NMP zeros, and time-delay. It is also shown in Braslavsky et al. (2007) that for state feedback, or for a minimum-phase, delay-free plant, the infimal SNR condition matches the exact same requirement on channel transmission data rate derived in the general formulation by Nair and Evans (2004).

Time-delays are ubiquitous in real systems and constitute a fundamental obstacle to feedback stabilisation. In Braslavsky et al. (2007) time-delays are treated only in discrete-time, wherein they are simpler because they appear as an increase in relative degree of a finite dimensional transfer function. It is shown in Braslavsky et al. (2007) that the presence of time-delay increases the SNR required to stabilise an unstable system. In Braslavsky et al. (2005), time-delays are treated in continuous time, with a similar conclusion, albeit with the considerably greater technical complexity required to treat infinite dimensional systems. It follows from Braslavsky et al. (2005) that for a system with one unstable pole the increase in SNR is exponential in the magnitude of the time-delay and that of the unstable pole. A consistent deterioration in control performance imposed by time-delays has been characterised in Nair et al. (2007) in a (discrete-time) data-rate constrained control framework.

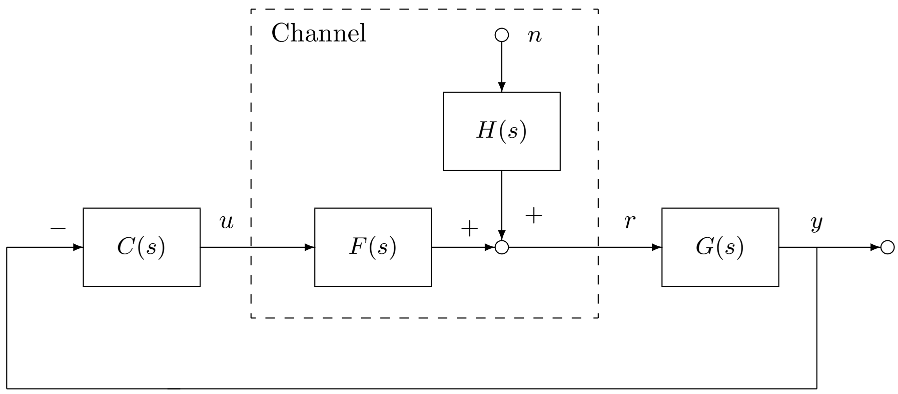

Fig. 1. Stabilisation via output feedback over an ACGN channel with bandwidth limitation.

The present paper extends the continuous-time results by Braslavsky et al. (2007) to systems with time-delay and feedback over bandwidth-limited additive coloured Gaussian noise (ACGN) channels. ACGN channels with a bandwidth limitation are a simple, yet more realistic model for a feedback communication link than an AWGN channel. The imposition of a bandwidth limitation, for example, is common in practice to avoid interference between different channels in a communication network.

We consider the control structure with feedback over an ACGN communication channel shown in figure 1. Of the two possible locations for the ACGN channel (measurement path and actuation path), we consider the actuation path location, as in Figure 1. Such a setting is common in practice and arises, for example, when actuators are far from the controller and have to communicate through a communication network. Nonetheless, in a single-input single-output (SISO) LTI setting both forms are equivalent, and it is a simple matter to restate the results for the case of measurement performed over a communication channel.

Our main contribution (Theorem 2) is a necessary and sufficient condition on the channel SNR for closed-loop stabilisability using LTI feedback. The SNR required for stabilisability is characterised as the infimal H_2 norm of the transfer function between n and u in the feedback loop of Figure 1, and is given as an explicit function of the plant unstable poles, NMP zeros and time-delay, and the channel transfer functions $F(s)$ and $H(s)$. We express this infimal H_2 norm in a formula that shows the exponential dependence on the time-delay and the unstable poles of the plant, in agreement with results in Braslavsky et al. (2005) and Mirkin and Raskin (2003). The formulas in these papers, however, are in terms of a state-space description of the plant and involve a Grammian-type definite integral, which can be readily computed numerically, but does not show any explicit dependency on the system unstable poles, NMP zeros, and time-delays. On the other hand, the formula proposed here provides a direct quantification of the impact of these plant parameters on optimal H_2

performance, and thereby on the problem of closed-loop stabilisation by feedback over a SNR constrained communication channel. A preliminary version of this result has been communicated in Rojas et al. (2006).

Terminology: let $\mathbb{C}^-$, $\bar{\mathbb{C}}^-$, $\mathbb{C}^+$ and $\bar{\mathbb{C}}^+$ denote respectively the open-left, closed-left, open-right and closed-right halves of the complex plane $\mathbb{C}$. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers and $\mathbb{R}_o^+$ the set of positive real numbers including zero. A continuous-time signal is denoted by $x(t)$, $t \in \mathbb{R}_o^+$, and its Laplace transform by $X(s)$, $s \in \mathbb{C}$. Where the meaning is clear from the context, we omit the argument of $x(t)$ or $X(s)$. The expectation operator is denoted by $\mathcal{E}$. A rational transfer function $P(s)$ of a continuous-time system is termed minimum phase (MP) if all its zeros lie in $\bar{\mathbb{C}}^-$, and is non minimum phase (NMP) if it has zeros in $\mathbb{C}^+$. The H_∞ norm of a system $P(s)$, denoted by $\|P\|_{H_\infty}$, is given by $\|P\|_{H_\infty} = \sup_{\omega\in\mathbb{R}} |P(j\omega)|$. Define L_2 as the space of functions $f : j\mathbb{R} \to \mathbb{C}$ such that $\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(j\omega)|^2 d\omega < \infty$; H_2 as the space of functions $f : \mathbb{C}^+ \to \mathbb{C}$ such that $\sup_{\sigma>0} \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(\sigma + j\omega)|^2 d\omega < \infty$; and also define $H_2^\perp$ as the space of functions $f : \mathbb{C}^- \to \mathbb{C}$ such that $\sup_{\sigma<0} \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(\sigma + j\omega)|^2 d\omega < \infty$.

2 SNR Constrained Stabilisation

Consider the control feedback loop in Figure 1 for a continuous-time plant with time-delay defined as

$$G(s) = G_1(s)e^{-s\tau}, \tag{1}$$

where $G_1(s)$ is a rational transfer function with relative degree $n_g \geq 1$, which contains m distinct unstable poles ($p_i \in \mathbb{C}^+, i = 1, \cdots, m$), q distinct NMP zeros ($z_j \in \mathbb{C}^+, j = 1, \cdots, q$) and $\tau \in \mathbb{R}_o^+$. The assumption of distinct zeros and distinct poles simplifies the proof of the main result but is not essential to it, and may be relaxed.

We assume in the present paper the channel model to be the bandwidth-limited ACGN channel as in Figure 1. The signals involved in the channel model are $u(t)$ the channel input, $r(t)$ the channel output, and $n(t)$ a zero-mean stationary white Gaussian noise process with power spectral density Φ. The channel transfer function $F(s)$ modelling the bandwidth limitation and channel dynamics is assumed to be stable, with f distinct NMP zeros ($w_j \in \mathbb{C}^+, j = 1, \cdots, f$) and relative degree $n_f \geq 0$. The channel transfer function $H(s)$, colouring the additive white Gaussian noise, is assumed to be stable, minimum phase and with relative degree $n_h \geq 0$. The channel input is required to satisfy the power constraint

$$\mathcal{P} > \|u\|_{Pow}^2, \tag{2}$$

for some predetermined input power level $\mathcal{P} > 0$, where $\|u\|^2_{Pow} \triangleq \mathcal{E}\{u^2(t)\}$. A power constraint such as (2) may arise from a range of factors such as electronic hardware limitations or regulatory constraints introduced to minimise interference to other communication system users. The bandwidth-limited ACGN channel is thus characterised by two stable transfer functions, $F(s)$ and $H(s)$, and two parameters: the admissible input power level $\mathcal{P}$, and the noise spectral density Φ.

We restrict our attention to the case where the overall feedback system is stabilised, such that for any distribution of initial conditions, the distribution of all signals converges exponentially fast to a stationary distribution. Without loss of generality, we therefore consider directly the properties of the stationary distribution of the relevant signals. Denote the power spectral density of $u(t)$ by $S_u(\omega)$. The power in the channel input is related to its spectral density by

$$\|u\|^2_{Pow} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_u(\omega)\, d\omega. \tag{3}$$

The closed-loop transfer function $T_{un}(s)$ from channel noise $n(t)$ to channel input $u(t)$ is equal to $-(T(s)/F(s))H(s)$, where $T(s) = C(s)G(s)F(s)/(1 + C(s)G(s)F(s))$ is the complementary sensitivity function of the output feedback loop, thus $T_{un}(s)$ is

$$T_{un}(s) = -(T(s)/F(s))H(s) = -\frac{C(s)G(s)}{1 + C(s)G(s)F(s)}H(s). \tag{4}$$

If the feedback system is stable, then the power of the channel input signal is given by

$$\|u\|^2_{Pow} = \|T_{un}\|^2_{H_2}\,\Phi.$$

We see that the input power constraint (2) may be restated as a constraint imposed on the transfer function (4) by the admissible channel SNR, specifically

$$\frac{\mathcal{P}}{\Phi} > \|T_{un}\|^2_{H_2}\,. \tag{5}$$

Notice that the proposed SNR involves Φ, the power spectral density of the channel noise and not its power, which is ill-defined in continuous-time (this can be seen by directly replacing u with n in (3)).

Let $\mathcal{K}$ denote the class of all proper controllers $C(s)$ that internally stabilise the feedback system of Figure 1.

Problem 1 (Continuous-Time SNR Constrained Output Feedback Stabilisation). Find a proper rational function $C(s) \in \mathcal{K}$ such that the transfer function (4) satisfies the constraint (5) imposed by the admissible channel SNR.

To solve Problem 1 we need to first introduce the following Blaschke products

$$B_p(s) = \prod_{i=1}^{m} \frac{s - p_i}{s + \bar{p}_i}, \quad B_z(s) = \prod_{j=1}^{q} \frac{s - z_j}{s + \bar{z}_j} \prod_{j=1}^{f} \frac{s - w_j}{s + \bar{w}_j}, \tag{6}$$

containing respectively the $\mathbb{C}^+$ poles of $G(s)$ and the $\mathbb{C}^+$ zeros of $G(s)$ and $F(s)$. The residue of $B_p^{-1}(s)$ at $s = p_i$ is given by

$$\mathrm{Res}_{s=p_i} B_p^{-1}(s) := 2Re\,\{p_i\} \prod_{\substack{j=1 \\ j \neq i}} \frac{p_i + \bar{p}_j}{p_i - p_j}. \tag{7}$$

The following theorem presents the main result of the paper: a closed-form expression of the infimal SNR required for stabilisability.

Theorem 2 *Consider the feedback system of Figure 1 with $T_{un}(s)$ defined as in* (4)*. Assume also that there are no unstable pole-zero cancellations in $G(s)$, or between $G(s)$ and $F(s)$. Then, for the feedback system to be stabilisable, the channel SNR $\mathcal{P}/\Phi$ must satisfy*

$$\frac{\mathcal{P}}{\Phi} > \inf_{C(s) \in \mathcal{K}} \|T_{un}\|_{H_2}^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{r_i \bar{r}_j}{p_i + \bar{p}_j} e^{(p_i + \bar{p}_j)\tau}, \tag{8}$$

where

$$r_i = \mathrm{Res}_{s=p_i} B_p^{-1}(s) B_z^{-1}(p_i)\, F^{-1}(p_i)\, H(p_i)\,. \tag{9}$$

PROOF. Consider a coprime factorisation for $F(s)G(s)$ as

$$F(s)G(s) = \frac{e^{-s\tau} N(s)}{M(s)}, \tag{10}$$

where $N(s), M(s) \in RH_\infty$. Further, without loss of generality, consider

$$\begin{aligned} N(s) &= B_z(s) N_o(s) F(s), \\ M(s) &= B_p(s) M_o(s), \end{aligned} \tag{11}$$

where $N_o(s), M_o(s) \in RH_\infty$, $N_o(s)$ and $M_o(s)$ are stable and MP transfer functions, $B_p(s)$, $B_z(s)$ are as defined in (6).

Following Braslavsky et al. (2005, Lemma 3.1), a Youla parameterisation of all controllers that stabilise $G(s)$ is given by

$$C(s) = \frac{X(s) + M(s)Q(s)}{Y(s) - e^{-s\tau} N(s) Q(s)}, \tag{12}$$

where $X(s)$ is in RH_∞, $Q(s), Y(s)$ are in H_∞ and $X(s)$ and $Y(s)$ satisfy the Bezout identity

$$e^{-s\tau} N(s) X(s) + M(s) Y(s) = 1. \tag{13}$$

A demonstration of the Bezout identity (13) can be found for example in Meinsma and Zwart (2000, Lemma 3.2). Replacing these factorisations for $F(s)G(s)$ and $C(s)$ into (4) gives

$$T_{un}(s) = -\Big(e^{-s\tau}B_z(s)N_o(s)F(s)X(s) \\ +e^{-s\tau}B_p(s)B_z(s)M_o(s)N_o(s)F(s)Q(s)\Big)\,F^{-1}(s)H(s).$$

Since $B_p(s)$ and $B_z(s)$ are all pass they have norm one, we have

$$\inf_{Q(s)\in H_\infty} \|T_{un}\|^2_{H_2} = \inf_{Q(s)\in H_\infty} \left\|e^{-s\tau}B_p^{-1}N_oXH + e^{-s\tau}M_oN_oQH\right\|^2_{L_2}. \tag{14}$$

Since $e^{-s\tau}$ has magnitude one at all frequencies, the norm expression on the RHS of equation (14) is not affected by it

$$\inf_{Q(s)\in H_\infty} \|T_{un}\|^2_{H_2} = \inf_{Q(s)\in H_\infty} \left\|B_p^{-1}N_oXH + M_oN_oQH\right\|^2_{L_2}. \tag{15}$$

Also notice that the second term inside the norm expression in (15) belongs to H_2, whilst the first term is a mixed term that can be decomposed as

$$B_p^{-1}(s)N_o(s)X(s)H(s) = \Gamma^\perp(s) + \Gamma(s), \tag{16}$$

where $\Gamma(s)$ is in H_2, whilst $\Gamma^\perp(s)$ is in $H_2^\perp$ and therefore by Lemma 3 in Doyle et al. (1992)

$$\inf_{Q(s)\in H_\infty} \|T_{un}\|^2_{H_2} = \left\|\Gamma^\perp\right\|^2_{H_2^\perp} + \inf_{Q(s)\in H_\infty} \|\Gamma + M_oN_oQH\|^2_{H_2}. \tag{17}$$

By means of a partial fraction expansion and the Bezout identity in (13), it is possible to quantify $\Gamma^\perp(s)$

$$\Gamma^\perp(s) = \sum_{i=1}^{m} \frac{r_i e^{p_i\tau}}{s-p_i},$$

where

$$r_i = \mathrm{Res}_{s=p_i} B_p^{-1}(s)B_z^{-1}(p_i)\,F^{-1}(p_i)\,H(p_i).$$

Note that from (13) we have $N_o(p_i)X(p_i) = F^{-1}(p_i)B_z^{-1}(p_i)e^{p_i\tau}$ at any p_i, $\forall i = 1,\cdots,m$ unstable poles of $G(s)$. The result for the first norm term on the RHS of equation (17), by use of the Residue theorem (see for example Churchill and Brown (1990, pp. 169–172)), is

$$\left\|\Gamma^\perp\right\|^2_{H_2^\perp} = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{r_i\bar{r}_j}{p_i+\bar{p}_j}e^{(p_i+\bar{p}_j)\tau}.$$

Replacing in (17) will give

$$\inf_{Q(s)\in H_\infty} \|T_{un}\|^2_{H_2} = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{r_i\bar{r}_j}{p_i+\bar{p}_j}e^{(p_i+\bar{p}_j)\tau} + \inf_{Q(s)\in H_\infty} \|\Gamma + M_oN_oQH\|^2_{H_2}. \tag{18}$$

By choosing $Q(s)$ we can make the expression $\Gamma(s) + M_o(s)N_o(s)Q(s)H(s)$ arbitrarily small in H_2 obtaining the infimal norm that can be achieved in (18) as

$$\inf_{Q(s)\in H_\infty} \|T_{un}\|_{H_2}^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{r_i \bar{r}_j}{p_i + \bar{p}_j} e^{(p_i+\bar{p}_j)\tau},$$

which completes the proof. $\square$

Theorem 2 gives a necessary and sufficient condition for the infimal channel SNR for which the system can be stabilised by LTI feedback control. This condition is expressed in a form that explicitly shows the dependence on the plant unstable poles, NMP zeros and time-delay, thereby allowing a direct characterisation of their impact in the proposed feedback control over a communication channel problem. We see from this formula that open loop NMP zeros and time-delays worsen the constraints imposed by unstable poles to LTI feedback stabilisation. This conclusion holds in contrast to the result by Nair and Evans (2004), which shows that only the unstable poles impose constraints to feedback stabilisation—NMP zeros and time-delays have no effect—if non-linear, time-varying feedback is allowed. Moreover, it has been pointed out in Braslavsky et al. (2007, § IV), that such relaxation of constraints by using time-varying feedback may come at the expense of stability robustness.

An explicit formula for the optimal H_2 norm similar to that in (8) has been recently reported in Bakhtiar and Hara (2007), however, only for a plant with a single unstable pole. The formula in (8) holds for a plant with multiple (distinct) unstable poles.

Remark 3 *Observe from the result of Theorem 2 that $F(s)$ and $H(s)$ are relevant to the analysis through the factor $F^{-1}(s)H(s)$. A SNR constrained output feedback stabilisation problem defined by the triplet of models $\{G(s), F(s), H(s)\}$ (a bandwidth-limited, coloured noise case) has the same solution of a SNR constrained output feedback stabilisation problem defined by the triplet of models $\{G(s), 1, F^{-1}(s)H(s)\}$ (a coloured noise case with no bandwidth-limitation).* $\star\star\star$

The following examples illustrate the application of Theorem 2 in the analysis of the limiting effects of channel and plant parameters on the stability by LTI feedback over a communication channel, as measured by the required channel SNR.

Example 4 *Consider a plant model $G(s)$ with an unstable pole at $p = 5$, no NMP zeros and $\tau = 0$. The LTI filters used to model the finite bandwidth*

$\star\star\star$ A.J. Rojas wishes to thank one of the anonymous reviewer of Rojas (2006) for the remark.

and coloured noise of the communication link are both chosen to be low-pass Butterworth filters of order 4 *(with the filters bandwidth defined by the* $-3[dB]$ *cut-off frequency).*

Figure 2 shows the effect of bandwidth limitation on one axis and coloured noise on the other axis. The vertical scale is the SNR value in decibels required to guarantee stabilisability. Two facts can be appreciated from Figure 2. First, a

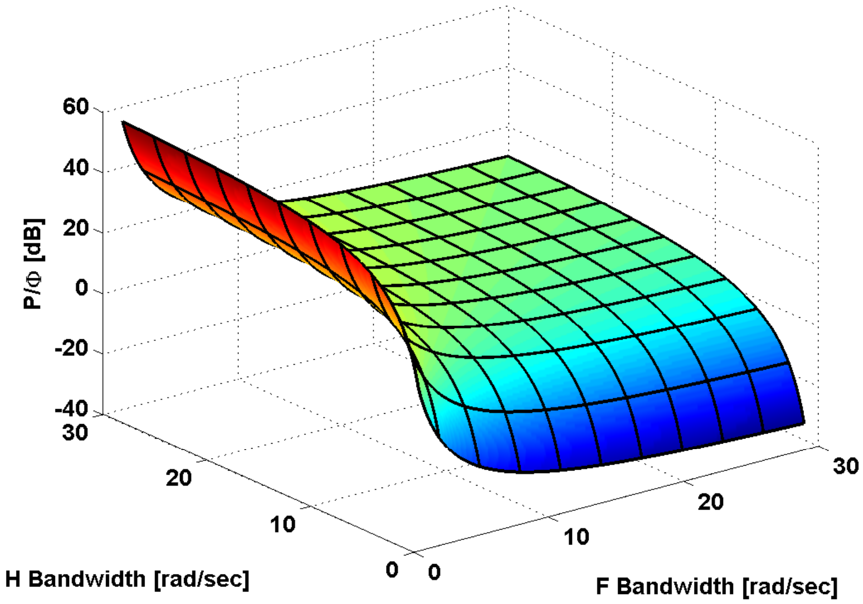

Fig. 2. SNR bound for stabilisability, unstable pole at 5. Bandwidth-limited coloured noise case.

reduction in the available bandwidth of the communication channel forces an increase in the value of SNR required to guarantee stabilisability. Second, if the noise is coloured by a low pass filter, the reduction of its cut-off frequency has the opposite effect of reducing the value of SNR required for stabilisability. The overall result approaches the case of SNR for an infinite bandwidth AWGN communication channel, that is $10\log_{10}(2p) = 10[dB]$ *in this case.*

More generally, for the case of one unstable real pole p and two possible selections for MP filters $F(s)$ *and* $H(s)$*, say* $(F_1(s), H_1(s))$ *and* $(F_2(s), H_2(s))$*, with* $|F_1(j\omega)^{-1}H_1(j\omega)| \geq |F_2(j\omega)^{-1}H_2(j\omega)| \quad \forall\omega$*, it is possible to verify through the Poisson integral formula, see for example Churchill and Brown (1990); Seron* *et al. (1997), that*

$$\log|F_1^{-1}(p)H_1(p)| = \frac{1}{\pi}\int_{-\infty}^{\infty}\log|F_1^{-1}(j\omega)H_1(j\omega)|\frac{p}{p^2+\omega^2}d\omega \geq$$
$$\frac{1}{\pi}\int_{-\infty}^{\infty}\log|F_2^{-1}(j\omega)H_2(j\omega)|\frac{p}{p^2+\omega^2}d\omega = \log|F_2^{-1}(p)H_2(p)|. \tag{19}$$

The result in (19) *is equivalent to* $|F_1^{-1}(p)H_1(p)| \geq |F_2^{-1}(p)H_2(p)|$ *and since for the case studied in the present example the SNR required for stabilisability is given by* $2p|F^{-1}(p)H(p)|^2$*, we can conclude that the first pair of LTI filters will always demand a higher SNR for stabilisability than the second pair.*

The following example considers an infinite bandwidth AWGN channel (i.e. $F(s) = 1$ and $H(s) = 1$) and a MP unstable plant with time-delay, recovering results by Braslavsky et al. (2005).

Example 5 *Consider the case of two unstable real poles* p_1 *and* p_2*, and an infinite bandwidth AWGN communication channel. For this selection equation* (8) *becomes:*

$$\frac{\mathcal{P}}{\Phi} > \inf_{C(s)\in\mathcal{K}}\|T_{un}\|_{H_2}^2 = \frac{r_1^2}{2p_1}e^{2p_1\tau} + \frac{2r_1r_2}{p_1+p_2}e^{(p_1+p_2)\tau} + \frac{r_2^2}{2p_2}e^{2p_2\tau} =$$
$$2p_1\left(\frac{p_1+p_2}{p_1-p_2}\right)^2 e^{2p_1\tau} - \frac{8p_1p_2}{p_1+p_2}\left(\frac{p_1+p_2}{p_1-p_2}\right)^2 e^{(p_1+p_2)\tau} + 2p_2\left(\frac{p_1+p_2}{p_1-p_2}\right)^2 e^{2p_2\tau}. \tag{20}$$

The expression obtained in (20) *matches the result in Example* 2.2*, Equation* (21)*, in Braslavsky et al. (2005).*

The implementation of a stabilising controller for an unstable plant with time delay requires special care, as we illustrate in the following example.

Example 6 *Consider an AWGN channel with infinite bandwidth and the plant*

$$G(s) = \frac{e^{-s\tau}}{s-p}, \tag{21}$$

where $p \in \mathbb{R}^+$ *and* $\tau \in \mathbb{R}_o^+$*. A coprime factorisation for* $G(s)$ *as in* (10) *is given by*

$$N(s) = \frac{1}{s+p}, \quad M(s) = \frac{s-p}{s+p}. \tag{22}$$

From the Bezout Identity (13)*, we can observe the interpolation condition that* $X(s)$ *has to satisfy at* $s = p$ *and from it we can propose a suitable choice for*

$X(s)$ itself.

$$X(p) = 2pe^{p\tau} \Rightarrow X(s) = \frac{4p^2 e^{p\tau}}{s+p}. \tag{23}$$

From the knowledge of $X(s)$ we can obtain $Y(s)$

$$Y(s) = (1 - e^{-s\tau}N(s)X(s))M^{-1}(s) = \frac{s+p}{s-p} - \frac{4p^2 e^{p\tau} e^{-s\tau}}{(s+p)(s-p)}, \tag{24}$$

and by following the proof of Theorem 2 for the present example is possible to verify that the infimum of the H_2 term on the RHS of (18) is achieved by $\hat{Q}(s) = 2pe^{p\tau}$. Thus, from (12), we can obtain the stabilising controller $C(s)$ that achieves the infimal SNR as

$$\hat{C}(s) = \frac{X + M\hat{Q}}{Y - e^{-s\tau}N\hat{Q}} = \frac{2pe^{p\tau}(s-p)}{(s+p) - 2pe^{p\tau}e^{-s\tau}}. \tag{25}$$

At this point we can observe from (25), via a L'Hôpital's argument, that

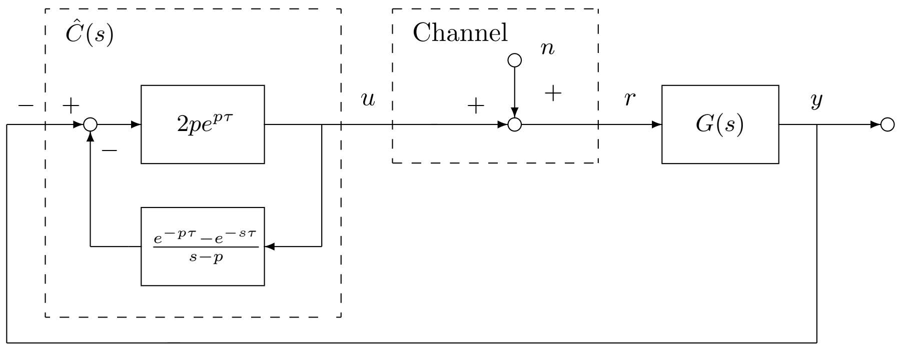

Fig. 3. Stabilisation via output feedback with a modified Smith predictor interpretation of the controller in (25) over an AWGN channel with infinite bandwidth.

$\lim_{s\to p}\hat{C}(s)$ *is well defined. Nonetheless the same L'Hôpital's argument implies that an unstable cancellation at $s = p$ is taking place, not between the controller and the plant, but within the controller itself. As a consequence, the stabilising controller in (25) is not implementable in its current form (if $\tau \neq 0$). A standard approach to circumvent this difficulty is to use a Padé's approximant for the plant time-delay, see for example Goodwin et al. (2001, p.429), so that we obtain a rational controller in which this internal cancellation can be explicitly accounted for. A different approach is to consider that the controller can be interpreted as a modified Smith predictor with the primary controller given by $2pe^{p\tau}$ and the (distributed time-delay) predictor*

$$\frac{e^{-p\tau} - e^{-s\tau}}{s-p} = e^{-p\tau}\int_0^\tau e^{(p-s)\sigma}d\sigma,$$

as in Figure 3. The above predictor expression can be then approximated using, for example, the results in Mirkin (2004).

3 Conclusion

We have addressed the problem of stabilisability of a NMP continuous-time LTI unstable plant with time-delay over an ACGN communication channel. The main result is expressed as a lower bound on the channel SNR, below which stability is not achievable. The lowest SNR bound is quantified by an expression of the optimal transfer function H_2-norm between the channel noise input and the channel signal input. Most of the ideas presented here have corresponding dual concepts that apply to discrete-time systems (see for example Rojas et al. (2006) for the discrete-time version of Theorem 2). Further investigation will consider multiplicative model error, multiple channels/single user scenario and sampled-data systems (a discussion for the infinite bandwidth AWGN channel case can already be found in Braslavsky et al. (2007, Sec.4)).

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