REVERSIBILITY IN THE GROUP OF HOMEOMORPHISMS OF THE CIRCLE

NICK GILL, ANTHONY G. O'FARRELL AND IAN SHORT

ABSTRACT. We identify those elements of the homeomorphism group of the circle that can be expressed as a composite of two involutions.

1. Introduction

We describe an element g of a group G as *reversible* in G if it is conjugate in G to its own inverse. We say that g is *strongly reversible* in G if there exists an involution τ in G such that $\tau g \tau = g^{-1}$. This language has developed from the theory of finite groups, where the terms *real* and *strongly real* replace *reversible* and *strongly reversible.* (The word *real* is used because an element g of a finite group is reversible if and only if each irreducible character of G takes a real value when applied to g.) Notice that g is strongly reversible if and only if it can be expressed as a composite of two involutions. The strongly reversible elements of the homeomorphism group of the real line were determined by Jarczyk and Young; see [4, 6, 7]. The purpose of this paper is to determine the strongly reversible maps in the group of homeomorphisms of the circle.

Let $\mathbb{S}$ denote the unit circle in $\mathbb{R}^2$ centred on the origin. Denote by $\mathrm{H}(\mathbb{S})$ the group of homeomorphisms of $\mathbb{S}$. There is a subgroup $\mathrm{H}^+(\mathbb{S})$ of $\mathrm{H}(\mathbb{S})$ consisting of orientation preserving homeomorphisms. The subgroup $\mathrm{H}^+(\mathbb{S})$ has a single distinct coset $\mathrm{H}^-(\mathbb{S})$ in $\mathrm{H}(\mathbb{S})$ which consists of orientation reversing homeomorphisms. We classify the strongly reversible maps in both groups $\mathrm{H}^+(\mathbb{S})$ and $\mathrm{H}(\mathbb{S})$. A classification of the reversible elements of $\mathrm{H}^+(\mathbb{S})$ and $\mathrm{H}(\mathbb{S})$ can be extracted from a conjugacy classification in these two groups. We describe the conjugacy classes of $\mathrm{H}^+(\mathbb{S})$ and $\mathrm{H}(\mathbb{S})$ in §2, and comment briefly on reversibility.

For points a and b in $\mathbb{S}$, we write (a,b) to indicate the open anticlockwise interval from a to b in $\mathbb{S}$. Let $[a,b]$ denote the closure of (a,b). For a proper open interval I in $\mathbb{S}$, we say that $u<v$ *in* I if $(u,v)\subset I$. To classify the strongly reversible maps in $\mathrm{H}^+(\mathbb{S})$ we need the notion of the *signature* of an orientation preserving homeomorphism which has a fixed point. If f is such a homeomorphism, then each point x in $\mathbb{S}$ is either a fixed point of f or else it lies in an open interval component I in the complement of the fixed point set of f. The signature Δ_f of f is the function from $\mathbb{S}$ to $\{-1,0,1\}$ given by the equation

$$\Delta_f(x) = \begin{cases} 1 & \text{if } x<f(x) \text{ in } I, \\ 0 & \text{if } f(x)=x, \\ -1 & \text{if } f(x)<x \text{ in } I. \end{cases}$$

The second and third authors were supported by Science Foundation Ireland, grant 05/RFP/MAT0003.

The strongly reversible maps in $\mathrm{H}^+(\mathbb{S})$ are classified according to the next theorem, proven in §3.

Theorem 1.1. *An element f of $\mathrm{H}^+(\mathbb{S})$ is strongly reversible if and only if either it is an involution or else it has a fixed point and there is an orientation preserving homeomorphism h of rotation number $\frac{1}{2}$ such that $\Delta_f = -\Delta_f \circ h$.*

Elements in $\mathrm{H}^+(\mathbb{S})$ that cannot be expressed as a composite of two involutions (elements that are not strongly reversible) can nevertheless be expressed as a composite of three involutions. The next theorem is proven in §4.

Theorem 1.2. *Each member of $\mathrm{H}^+(\mathbb{S})$ can be expressed as a composite of three orientation preserving involutions.*

We move on to describe the strongly reversible elements in the larger group $\mathrm{H}(\mathbb{S})$. There are orientation preserving homeomorphisms of $\mathbb{S}$ which are strongly reversible in $\mathrm{H}(\mathbb{S})$, but not strongly reversible in $\mathrm{H}^+(\mathbb{S})$. Before we state our theorem on strong reversibility in $\mathrm{H}(\mathbb{S})$, we introduce some notation which is explained in more detail in §2. For a homeomorphism f, the *degree* of f, denoted $\deg(f)$, is equal to 1 if f preserves orientation, and -1 if f reverses orientation. Let $\rho(f)$ denote the rotation number of an orientation preserving homeomorphism f. The rotation number is an element of $[0,1)$. If $\rho(f)=0$ then f has a fixed point. If $\rho(f)$ is rational then f has a periodic point, in which case the minimal period of f (the smallest positive integer n such that f^n has fixed points) is denoted by n_f. If $\rho(f)$ is irrational then we denote the minimal set of f, that is, the smallest non-trivial f invariant compact subset of $\mathbb{S}$, by K_f. This set is either a perfect and nowhere dense subset of $\mathbb{S}$ (a Cantor set) or else equal to $\mathbb{S}$. In the former case we define I_f to be the set of inaccessible points of K_f, and in the latter case we define I_f to be $\mathbb{S}$. There is a continuous surjective map $w_f : \mathbb{S} \to \mathbb{S}$ of degree 1 with the properties: (i) w_f maps I_f homeomorphically onto $w_f(I_f)$; (ii) w_f maps each closed interval component in the complement of I_f to a point; (iii) $w_f f = R_\theta w_f$, where R_θ is the anticlockwise rotation by $\theta = 2\pi\rho(f)$. The next theorem is proven in §5.

Theorem 1.3. *Let f be an orientation preserving member of $\mathrm{H}(\mathbb{S})$. Either*

(i) $\rho(f)=0$, *in which case f is strongly reversible if and only if there is a homeomorphism h such that $\Delta_f = -\deg(h)\cdot\Delta_f \circ h$, and either h preserves orientation and has rotation number $\frac{1}{2}$ or else reverses orientation;*

(ii) $\rho(f)$ *is non-zero and rational, in which case f is strongly reversible if and only if f^{n_f} is strongly reversible by an orientation reversing involution;*

(iii) $\rho(f)$ *is irrational, in which case f is strongly reversible if and only if $w_f(I_f)$ has a reflectional symmetry.*

Using Theorem 1.3 we also show that an orientation preserving circle homeomorphism is strongly reversible by an orientation *reversing* involution if and only if it is reversible by an orientation *reversing* homeomorphism.

It remains to state a result on strong reversibility of orientation reversing homeomorphisms. Each orientation reversing homeomorphism has exactly two fixed points. The next theorem is proven in §6.

Theorem 1.4. *An orientation reversing homeomorphism f is strongly reversible if and only if there is an orientation reversing homeomorphism h that interchanges the pair of fixed points of f and satisfies $hf^2h^{-1} = f^{-2}$.*

Fine and Schweigert proved in [2, Theorem 25] that each member of $\mathrm{H}(\mathbb{S})$ can be expressed as a composite of three involutions (this result follows quickly from Theorems 1.2 and 1.3).

We now describe the structure of this paper. Section 2 contains unoriginal material; it consists of a brief review of a conjugacy classification in $\mathrm{H}^+(\mathbb{S})$ and $\mathrm{H}(\mathbb{S})$. All subsequent sections contain new results. Sections 3 and 4 are about $\mathrm{H}^+(\mathbb{S})$ and sections 5 and 6 are about $\mathrm{H}(\mathbb{S})$.

2. Conjugacy classification

Two conjugacy invariants which can be used to determine the conjugacy classes in $\mathrm{H}(\mathbb{S})$ are the rotation number and signature, introduced in §1. We describe here only those properties of these two quantities that we will use. For more information on rotation numbers, see [3]; for more information on signatures, see [2].

For an orientation preserving homeomorphism f, choose any point x in $\mathbb{S}$, and let θ_n be the angle in $[0, 2\pi)$ measured anticlockwise between $f^{n-1}(x)$ and $f^n(x)$. The *rotation number* of f, denoted $\rho(f)$, is the unique number in $[0, 1)$ such that the expression

$$(\theta_1 + \cdots + \theta_n) - 2\pi n\rho(f)$$

is bounded for all n. The quantity $\rho(f)$ is independent of x. The rotation number is invariant under conjugation in $\mathrm{H}^+(\mathbb{S})$, and $\rho(f^n) = n\rho(f) \pmod 1$, for each integer n.

A straightforward consequence of the definition of $\rho(f)$ is that $\rho(f) = 0$ if and only if f has a fixed point. In this case, $\mathbb{S}$ can be partitioned into a closed set $\mathrm{fix}(f)$, consisting of fixed points of f, and a countable collection of open intervals on each of which f is free of fixed points. Now suppose that $\rho(f) = p/q$, where p and q are coprime positive integers. Then f has periodic points, that is, there is a positive integer n for which f^n has fixed points. The smallest such n, denoted n_f, is the *minimal period* of f and is equal to q.

The remaining possibility is that $\rho(f)$ is irrational. In this case we define K_f to be the unique minimal set in the poset consisting of f invariant compact subsets of $\mathbb{S}$ ordered by inclusion. We describe K_f as the *minimal set* of f. Either $K_f = \mathbb{S}$ or else K_f is a perfect subset of $\mathbb{S}$ with empty interior—a Cantor set. In the latter case there is a sequence of open intervals (a_i, b_i), for $i = 1, 2, \dots$, such that $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ when $i \neq j$, and K_f is the complement of $\bigcup_{i=1}^{\infty}(a_i, b_i)$. The set I_f of *inaccessible points* of K_f is the complement of $\bigcup_{i=1}^{\infty}[a_i, b_i]$. If $K_f = \mathbb{S}$ then we define $I_f = \mathbb{S}$. There is a continuous surjective map w_f of $\mathbb{S}$ of degree 1 such that $w_f f = R_\theta w_f$, where $\theta = 2\pi\rho(f)$. The map w_f is a homeomorphism when restricted to I_f, and it maps each interval $[a_i, b_i]$ to a single point. The map w_f is unique up to post composition by rotations.

Now suppose that f is an orientation preserving homeomorphism which has a fixed point. The signature of f was defined in §1. The signature Δ_f takes the value 0 on

$\mathrm{fix}(f)$, and elsewhere it takes either the value -1 or the value 1. Useful properties of the signature are encapsulated in the next elementary lemma.

Lemma 2.1. *If f is a member of $\mathrm{H}^+(\mathbb{S})$ with a fixed point, and h is a member of $\mathrm{H}(\mathbb{S})$, then*

(i) $\Delta_{hfh^{-1}} = \deg(h) \cdot \Delta_f \circ h^{-1}$,
(ii) $\Delta_{f^{-1}} = -\Delta_f$.

We are now in a position to state criteria which determine whether two circle homeomorphisms are conjugate. The results are stated in such a way that one can deduce from them when two orientation preserving circle homeomorphisms are conjugate in each of the groups $\mathrm{H}^+(\mathbb{S})$ and $\mathrm{H}(\mathbb{S})$. The result on irrational rotation numbers follows from [5, Theorem 2.3]. The result on orientation preserving homeomorphisms with fixed points is similar to [2, Theorem 10]. The other two theorems are well-known; they can both be proven directly. Recall that the degree of a circle homeomorphism is 1 if the map preserves orientation, and -1 if it reverses orientation.

Theorem 2.2. *Two orientation preserving circle homeomorphisms f and g, each of which has a fixed point, are conjugate by a homeomorphism of degree ϵ if and only if there is a homeomorphism h of degree ϵ such that $\Delta_g = \epsilon\Delta_f \circ h$.*

Note in particular that a map g in $\mathrm{H}^+(\mathbb{S})$ with fixed points is conjugate in $\mathrm{H}^+(\mathbb{S})$ to all of its powers.

Theorem 2.3. *Two orientation preserving circle homeomorphisms f and g, both of which have the same non-zero rational rotation number, are conjugate by a homeomorphism of degree ϵ if and only if f^{n_f} is conjugate to g^{n_g} by a homeomorphism of degree ϵ.*

Since f and g have the same non-zero rational rotation number, the integers n_f and n_g in Theorem 2.3 are equal.

Recall that an orthogonal map of the circle of degree 1 is a rotation, and an orthogonal map of the circle of degree -1 is a reflection in a line through the origin.

Theorem 2.4. *Two orientation preserving circle homeomorphisms f and g, both of which have the same irrational rotation number, are conjugate by a homeomorphism of degree ϵ if and only if there is an orthogonal map of degree ϵ that maps $w_f(I_f)$ to $w_g(I_g)$.*

It remains to consider conjugacy between orientation reversing maps.

Theorem 2.5. *Two orientation reversing circle homeomorphisms f and g are conjugate in $\mathrm{H}(\mathbb{S})$ if and only if f^2 and g^2 are conjugate in $\mathrm{H}(\mathbb{S})$ by a homeomorphism that maps the pair of fixed points of f to the pair of fixed points of g.*

It follows from Theorems 2.3 and 2.5 that all non-trivial involutions in $\mathcal{H}^+$ are conjugate, and all orientation reversing involutions in $\mathcal{H}$ are conjugate. (These statements can easily be seen directly.)

We briefly remark on the reversible elements in $\mathrm{H}^+(\mathbb{S})$ and $\mathrm{H}(\mathbb{S})$. Suppose that f and h are members of $\mathrm{H}^+(\mathbb{S})$ such that $hfh^{-1} = f^{-1}$. Then

$$\rho(f^{-1}) = \rho(hfh^{-1}) = \rho(f) \pmod 1.$$

But $\rho(f^{-1}) = -\rho(f) \pmod 1$, hence $\rho(f)$ is equal to either 0 or $\frac{1}{2}$. One can construct examples of reversible elements in $\mathrm{H}^+(\mathbb{S})$ with either of these two rotation numbers (there are examples at the end of §5). On the other hand, if h reverses orientation and still $hfh^{-1} = f^{-1}$, then

$$\rho(hfh^{-1}) = -\rho(f) \pmod 1,$$

so, in this case, the rotation number tells us nothing about reversibility. Notice that if we compare Theorems 2.2, 2.3, and 2.4 with Theorem 1.3, we see that that an orientation preserving map f is reversible by an orientation *reversing* map if and only if f is strongly reversible by an orientation *reversing* involution. There are, however, orientation preserving homeomorphisms that are not strongly reversible in $\mathcal{H}$, but are nevertheless reversible by orientation *preserving* maps; one example is given at the end of §5.

3. Proof of Theorem 1.1

The following two theorems deal with strong reversibility in $\mathrm{H}^+(\mathbb{S})$ for the two rotation numbers 0 and $\frac{1}{2}$ separately.

A result similar to Theorem 3.2, below, has been proven by Jarzcyk [4] and Young [7] for homeomorphisms of the real line. We use an elementary lemma in the proof of Theorem 3.2.

Lemma 3.1. *If f and g are orientation preserving homeomorphisms such that $\Delta_f = \Delta_g$ then there is an orientation preserving homeomorphism k that fixes each of the fixed points of f and g such that $kfk^{-1} = g$.*

Proof. We define a homeomorphism k as follows. On each fixed point x of f and g, define $k(x) = x$. On each open interval component (a, b) of $\mathbb{S} \setminus \mathrm{fix}(f)$, the signature function takes either the value 1 for both functions f and g, or else it takes the value -1 for both functions. In either case we can choose an orientation preserving homeomorphism k_0 of (a, b) such that $k_0 f k_0^{-1} = g$ for $x \in (a, b)$. We then define $k(x) = k_0(x)$ for $x \in (a, b)$. We have constructed the required function k. Of course, the existence of a conjugation between f and g follows from Theorem 2.2, but we also needed the property of k that it fixes each element of $\mathrm{fix}(f)$. □

Theorem 3.2. *An element f of $\mathrm{H}^+(\mathbb{S})$ with a fixed point is strongly reversible in $\mathrm{H}^+(\mathbb{S})$ if and only if there is a homeomorphism h in $\mathrm{H}^+(\mathbb{S})$ with rotation number $\frac{1}{2}$ such that $\Delta_f = -\Delta_f \circ h$.*

Proof. If $\sigma f \sigma = f^{-1}$ for an orientation preserving involution σ, then $\Delta_f = -\Delta_f \circ \sigma$, by Lemma 2.1 (i). Conversely, suppose that there is a homeomorphism h in $\mathrm{H}^+(\mathbb{S})$ with rotation number $\frac{1}{2}$ such that $\Delta_f = -\Delta_f \circ h$. By Lemma 2.1, $\Delta_{h^{-1}fh} = \Delta_{f^{-1}}$. Using Lemma 3.1 we can construct a map k in $\mathrm{H}^+(\mathbb{S})$ that fixes each fixed point of f, and satisfies $k^{-1}h^{-1}fhk = f^{-1}$.

Now choose a fixed point p of f. Then $\Delta_f(h(p)) = -\Delta_f(p) = 0$, so $h(p)$ is a fixed point of f. The points p and $h(p)$ are distinct because h has no fixed points. Define

$$\mu(x) = \begin{cases} hk(x) & \text{if } x \in [p, h(p)], \\ k^{-1}h^{-1}(x) & \text{if } x \in [h(p), p]. \end{cases}$$

One can check that μ is an involution in $\mathrm{H}^+(\mathbb{S})$ and $\mu f \mu = f^{-1}$. □

We move on to orientation preserving homeomorphisms with rotation number $\frac{1}{2}$.

Theorem 3.3. *An element of* $\mathrm{H}^+(\mathbb{S})$ *with rotation number* $\frac{1}{2}$ *is strongly reversible in* $\mathrm{H}^+(\mathbb{S})$ *if and only if it is an involution.*

Proof. All involutions are strongly reversible by the identity map. Conversely, let f be a homeomorphism with rotation number $\frac{1}{2}$, and let σ be an involution in $\mathrm{H}^+(\mathbb{S})$ such that $\sigma f \sigma = f^{-1}$. Choose an element x of $\mathrm{fix}(f^2)$. Then $f(x)$ is also an element of $\mathrm{fix}(f^2)$, and by interchanging x and $f(x)$ if necessary, we may assume that $\sigma(x) \in (x, f(x)]$. Suppose that $\sigma(x) \neq f(x)$. Since σ maps $(\sigma(x), x)$ onto $(x, \sigma(x))$, we have that $\sigma f(x) \in (x, \sigma(x))$. Likewise, f maps $(x, f(x))$ onto $(f(x), x)$, therefore $f\sigma(x) \in (f(x), x)$. However, $f\sigma(x) = \sigma f^{-1}(x) = \sigma f(x)$, and yet $(x, \sigma(x)) \cap (f(x), x) = \emptyset$. This is a contradiction, therefore $\sigma(x) = f(x)$. This means that σf is an orientation preserving involution which fixes x; hence it is the identity map. Therefore $f = \sigma$, as required. □

In contrast to Theorem 3.3 there are reversible homeomorphisms with rotation number $\frac{1}{2}$ that are not strongly reversible. An example is given at the end of §5.

We have all the ingredients for a proof of Theorem 1.1.

Proof of Theorem 1.1. If f is a strongly reversible member of $\mathrm{H}^+(\mathbb{S})$ then it is reversible, so it must have rotation number equal to either 0 or $\frac{1}{2}$. If it has rotation number 0 then there is an orientation preserving homeomorphism h with rotation number $\frac{1}{2}$ σ such that $\Delta_f = -\Delta_f \circ h$, by Theorem 3.2. If f has rotation number $\frac{1}{2}$ then it is an involution, by Theorem 3.3. The converse implication follows immediately from Theorem 3.2. □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. Choose an element f of $\mathrm{H}^+(\mathbb{S})$ that is not an involution. There exists a point x in $\mathbb{S}$ such that x, $f(x)$, and $f^2(x)$ are three distinct points. By replacing f with f^{-1} if necessary we can assume that x, $f(x)$, and $f^2(x)$ occur in that order anticlockwise around $\mathbb{S}$. Notice that $f^{-1}(x)$ lies in $(f(x), x)$. Choose a point y in $(x, f(x))$ that is sufficiently close to $f(x)$ that $f^{-1}(y) > f^2(x)$ in $(f(x), x)$. We construct an orientation preserving homeomorphism g from $[x, f(x)]$ to $[f(x), x]$ such that $g(y) = f(y)$, $g(t) < \min(f(t), f^{-1}(t))$ in (x, y), and $f(t) < g(t) < f^{-1}(t)$ in $(y, f(x))$. A graph of such a function is shown in Figure 1. Define an involution σ in $\mathrm{H}^+(\mathbb{S})$ by the equation

$$\sigma(t) = \begin{cases} g(t) & \text{if } t \in [x, f(x)], \\ g^{-1}(t) & \text{if } t \in [f(x), x]. \end{cases}$$

Let us determine the fixed points of σf. For a point t in $[x, f(x)]$ we have that $\sigma f(t) = t$ if and only if $f(t) = g(t)$. This means that either $t = x$ or $t = y$. For a point t in $(f(x), x)$

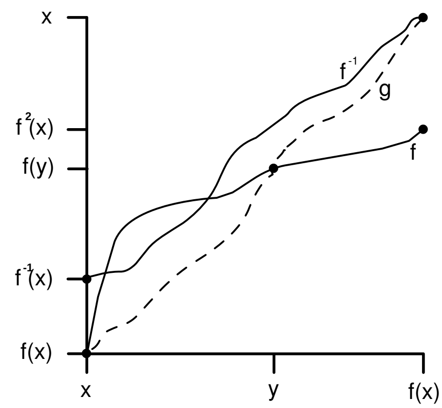

FIGURE 1.

we have that $\sigma f(t) = t$ if and only if $\sigma f g(w) = g(w)$, where $w = g^{-1}(t)$ is a point in $(x, f(x))$. Therefore $g(w) = f^{-1}(w)$. This equation has no solutions in $(x, f(x))$, hence σf has no fixed points in $(f(x), x)$. It is straightforward to obtain the direction of flow on the complement of $\{x, y\}$ and we find that

$$\Delta_{\sigma f}(t) = \begin{cases} 0 & \text{if } t = x, y, \\ 1 & \text{if } t \in (x, y), \\ -1 & \text{if } t \in (y, x). \end{cases}$$

By Theorem 3.2, σf is expressible as a composite of two involutions in $\mathrm{H}^+(\mathbb{S})$. Therefore f is expressible as a composite of three involutions in $\mathrm{H}^+(\mathbb{S})$. □

A simple corollary of Theorem 1.2 is that $\mathrm{H}^+(\mathbb{S})$ is uniformly perfect, meaning that there is a positive integer N such that each element of $\mathrm{H}^+(\mathbb{S})$ can be expressed as a composite of N or fewer commutators. Since it is easy to express the rotation by π as a commutator, and each involution in $\mathrm{H}^+(\mathbb{S})$ is conjugate to the rotation by π, it follows from Theorem 1.2 that $\mathrm{H}^+(\mathbb{S})$ is uniformly perfect with $N = 3$. In fact, Eisenbud, Hirsch, and Neumann [1] proved that $\mathrm{H}^+(\mathbb{S})$ is uniformly perfect with $N = 1$.

5. Proof of Theorem 1.3

For the remainder of this document we work in the full group of homeomorphisms of the circle. In §2 we showed that all reversible maps in $\mathrm{H}^+(\mathbb{S})$ have rotation number either 0 or $\frac{1}{2}$. This is not the case in $\mathrm{H}(\mathbb{S})$ because if h is an orientation reversing map, and f an orientation preserving map, then $\rho(hfh^{-1}) = -\rho(f)$. Since also $\rho(f^{-1}) = -\rho(f)$, the rotation number tells us nothing about reversibility by orientation reversing homeomorphisms. In fact, since all rotations are strongly reversible in $\mathrm{H}(\mathbb{S})$ by reflections, there are strongly reversible maps in $\mathrm{H}(\mathbb{S})$ with any given rotation number.

We divide our analysis of strongly reversible maps in $\mathrm{H}(\mathbb{S})$ between three cases corresponding to when the rotation number is 0, rational, or irrational. The first case is Theorem 1.3 (i). We need a preliminary lemma.

Lemma 5.1. *If f is a fixed point free homeomorphism of an open proper arc A in $\mathbb{S}$, then f is conjugate to f^{-1} on A by an orientation reversing involution of A.*

Proof. The situation is topologically equivalent to the situation when f is a fixed point free homeomorphism of the real line. Such maps f are conjugate to non-trivial translations, and translations are reversible by the orientation reversing involution $x \mapsto -x$. $\square$

Proof of Theorem 1.3 (i). If $\tau f \tau = f^{-1}$ for an involution τ, then $\Delta_f = -\deg(\tau) \cdot \Delta_f \circ \tau$, by Lemma 2.1. For the converse, we are given a homeomorphism h that satisfies $\Delta_f = -\deg(h) \cdot \Delta_f \circ h$. Either h preserves orientation and satisfies $\rho(h) = \frac{1}{2}$, in which case the result follows from Theorem 3.2, or else h reverses orientation.

In the latter case, by Lemmas 2.1 and 3.1 there is an orientation preserving homeomorphism k that fixes the fixed points of f, and satisfies $k^{-1}h^{-1}fhk = f^{-1}$. Define $s = k^{-1}h^{-1}$. Let s have fixed points p and q. Let a denote the point in $\text{fix}(f)$ that is clockwise from p, and closest to p. Possibly $a = p$. Define b to be the point in $\text{fix}(f)$ that is anticlockwise from p and closest to p. Let $I = (a, b)$. Similarly we define an interval J about q. If we ignore the trivial case in which $\text{fix}(f)$ has only one component then I and J only intersect, if at all, in their end-points. Now, f fixes I so we can, by Lemma 5.1, choose an orientation reversing involution τ_I of I such that $\tau_I f \tau_I(x) = f^{-1}(x)$ for $x \in I$. Similarly we define τ_J. From the equation $sfs^{-1} = f^{-1}$ we deduce that s fixes I and J. Hence we can define

$$\mu(x) = \begin{cases} \tau_I(x) & \text{if } x \in I, \\ \tau_J(x) & \text{if } x \in J, \\ s(x) & \text{if } x \in [p,q] \setminus (I \cup J), \\ s^{-1}(x) & \text{if } x \in [q,p] \setminus (I \cup J). \end{cases} \tag{5.1}$$

One can check that s is an orientation reversing involution that satisfies $sfs = f^{-1}$. $\square$

To prove Theorem 1.3 (ii) we use a lemma that enables us to deal with rational rotation numbers of the form $1/n$, rather than m/n. Recall that n_f denotes the minimal period of f.

Lemma 5.2. *Let f be an element of $\mathrm{H}^+(\mathbb{S})$ with a periodic point. If f^d is strongly reversible for an integer d in $\{1, 2, \ldots, n_f - 1\}$ that is coprime to n_f, then f is strongly reversible.*

Proof. There exist integers u and t such that $dt = 1 + un_f$. Let q be the positive integer between 0 and n_f, and coprime to n_f, such that $\rho(f) = q/n_f$. Observe that

$$\rho(f^{dt}) = (1 + un_f)\rho(f) = \rho(f) + uq = \rho(f) \pmod 1.$$

Recall that a map g in $\mathrm{H}^+(\mathbb{S})$ with fixed points is conjugate in $\mathrm{H}^+(\mathbb{S})$ to all its powers. Let $g = f^{n_f}$. Then g is conjugate to g^{dt}. In other words, f^{n_f} is conjugate to $(f^{dt})^{n_f}$. Apply Theorem 2.3 to the maps f and f^{dt} to see that these two maps are conjugate. The second map f^{dt} is strongly reversible, because f^d is strongly reversible. Conjugacy preserves strong reversibility, therefore f is also strongly reversible. $\square$

We first prove a special case of Theorem 1.3 (ii).

Lemma 5.3. *Let f be an orientation preserving homeomorphism of $\mathbb{S}$ with rotation number $1/n$, for a positive integer n. Suppose that there is an open interval J such that the intervals $J, f(J), \ldots, f^{n-1}(J)$ are pairwise disjoint, and such that $\text{fix}(f^n)$ is the* *complement of $J\cup f(J)\cup\cdots\cup f^{n-1}(J)$. Then f is strongly reversible by an orientation reversing involution that maps $f^k(J)$ to $f^{-k+1}(J)$ for each integer k.*

Proof. Let R denote an anticlockwise rotation by $2\pi/n$. After conjugating f suitably, the function f and interval J may be adjusted so that $f^k(J)=R^k(J)$ for all integers k. Let τ denote reflection in a line ℓ through the origin that bisects J. Thus τ fixes J. Let σ denote reflection in a line through the origin that is π/n anticlockwise from ℓ. Then $R=\sigma\tau$. Orient J in an anticlockwise sense, and choose an increasing homeomorphism ϕ of J, without fixed points, such that $\tau\phi\tau=\phi^{-1}$. This is possible because we can, by conjugation, consider J to be the real line and consider τ still to be a reflection, in which case ϕ can be chosen to be a translation. Now define an orientation preserving circle homeomorphism g to satisfy $g(x)=f(x)$, for $x\in \operatorname{fix}(f^n)$, and $g(x)=R^{k+1}\phi R^{-k}(x)$, for $x\in R^k(J)$. This means that g^n has the same signature on all of the intervals $R^k(J)$ (either all -1 or all 1). By Theorem 2.3, g is conjugate to f. Also, one can check that $\sigma g\sigma=g^{-1}$ and $\sigma(g^k(J))=g^{-k+1}(J)$ for each integer k. Since strong reversibility is preserved under conjugation, the result follows. □

Proof of Theorem 1.3 (ii). If f is strongly reversible by an involution τ then f^{n_f} is also strongly reversible by τ. If τ is orientation preserving then, by Theorem 1.1, f is an involution. Therefore f^{n_f} is the identity, and as such it is reversible by any orientation reversing involution.

Conversely, suppose that there is an orientation reversing involution τ such that $\tau f^{n_f}\tau=f^{-n_f}$. Let p be a fixed point of f^{n_f}. There is an integer d that is coprime to n_f such that the distinct points $p, f^d(p),\dots,f^{(n_f-1)d}(p)$ occur in that order anticlockwise around $\mathbb{S}$. The function $(f^d)^{n_f}$ is strongly reversible because it equals $(f^{n_f})^d$. If we can deduce that f^d is strongly reversible then it follows from Lemma 5.2 that f is strongly reversible. In other words, it is sufficient to prove the theorem when the points $p, f(p),\dots,f^{n_f-1}(p)$ occur in that order around $\mathbb{S}$.

The map τf has a fixed point q which, by replacing τ with $f^k\tau f^{-k}$ for an appropriate integer k, we can assume that it lies in the interval $(f^{-1}(p),p]$. Either q is a fixed point of f^{n_f} (that is, a periodic point of f) or it is not. In the former case let $I=[q,f(q)]$ and define, for each integer k,

$$\mu(x)=f^k\tau f^k(x),\quad x\in f^{-k}(I). \tag{5.2}$$

This is a well defined homeomophism because $f^{n_f}\tau f^{n_f}=f^{-n_f}$. One can check that μ is an involution and satisfies $\mu f\mu=f^{-1}$. In Figure 2 the action of μ on certain f iterates of q is shown in the case $n_f=2m$. If q is not a fixed point of f^{n_f} then it lies in a unique component $J=(s,t)$ in the complement of $\operatorname{fix}(f^{n_f})$. The interval J is contained in $(f^{-1}(p),p)$, and both s and t lie in $\operatorname{fix}(f^{n_f})$. From the equation $\tau f^{n_f}\tau=f^{-n_f}$, we can deduce that τ maps J to another open interval component in $\mathbb{S}\setminus\operatorname{fix}(f^{n_f})$. Also, $f(J)$ is a component of $\mathbb{S}\setminus\operatorname{fix}(f^{n_f})$. But $\tau(J)$ and $f(J)$ both contain the point $\tau(q)=f(q)$; therefore $\tau(J)=f(J)$. This means that $\tau(s)=f(t)$ and $\tau(t)=f(s)$.

We define an involution μ in a similar fashion to (5.2), but this time we do not, yet, define μ on the intervals $J, f(J),\dots,f^{n_f-1}(J)$. Specifically, let $I=[t,f(s)]$ and define, for each integer k,

$$\mu(x)=f^k\tau f^k(x),\quad x\in f^{-k}(I).$$

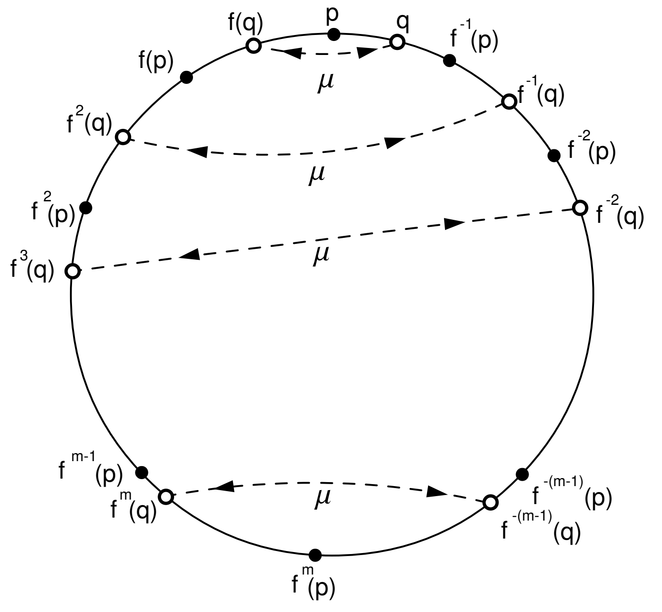

FIGURE 2.

We can extend μ to $J, f(J), \dots, f^{n_f-1}(J)$ using Lemma 5.3. The now fully defined map μ is an orientation reversing involutive homeomorphism of $\mathbb{S}$ that satisfies $\mu f \mu = f^{-1}$. □

To prove Theorem 1.3 (iii) we need a lemma. We prove the lemma explicitly, although it can be deduced quickly from Lemma 5.3.

Lemma 5.4. *Let J and J' be two disjoint non-trivial closed intervals in $\mathbb{S}$, and let g be an orientation preserving homeomorphism from J to J'. Then there exists an orientation reversing homeomorphism γ from J to J' such that $\gamma g^{-1} \gamma = g$.*

Proof. Let a and b be points such that $J = [a, b]$. Choose a point q in (a, b). Choose an orientation reversing homeomorphism α from $[a, q]$ to $[g(q), g(b)]$. The map γ defined by

$$\gamma(x) = \begin{cases} \alpha(x) & \text{if } x \in [a, q], \\ g\alpha^{-1}g(x) & \text{if } x \in [q, b], \end{cases}$$

has the required properties. □

Proof of Theorem 1.3 (iii). If f is strongly reversible (by an orientation reversing involution) then, since $I_{f^{-1}} = I_f$, we see from Theorem 2.4 that $w_f(I_f)$ has a reflectional symmetry.

Conversely, suppose that there is a reflection τ in a line through the origin of $\mathbb{R}^2$ that fixes $w_f(I_f)$. We define an involution μ from I_f to I_f by the equation $\mu(x) = w_f^{-1} \tau w_f(x)$. In this equation, w_f^{-1} is the inverse of the function $w_f : I_f \to w_f(I_f)$. For $x \in I_f$ we have

$$\mu f \mu(x) = (w_f^{-1} \tau w_f)(w_f^{-1} R_\theta w_f)(w_f^{-1} \tau w_f)(x) = f^{-1}(x).$$

Recall that K_f is the complement in $\mathbb{S}$ of a countable collection of disjoint open intervals (a_i, b_i), and I_f is the complement in $\mathbb{S}$ of the union of the intervals $[a_i, b_i]$. We can extend the definition of μ to K_f by defining $\mu(a_i)$ to be the limit of $\mu(x_n)$, where x_n is a sequence in I_f that converges to a_i. Similarly for b_i. The extended map μ is a homeomorphism from K_f to itself. Notice that μ has the property that for each integer i there is an integer j such that μ interchanges a_i and b_j, and also interchanges a_j and b_i.

We can extend the definition of μ to the whole of $\mathbb{S}$ by introducing, for each i, an orientation reversing homeomorphism $\phi_i : [a_i, b_i] \to [a_j, b_j]$ (where $\mu(a_i) = b_j$ and $\mu(b_i) = a_j$), and defining $\mu(x) = \phi_i(x)$ for $x \in [a_i, b_i]$. The resulting map μ will be a homeomorphism. It remains only to show how to choose particular maps ϕ_i such that μ is an involution that satisfies $\mu f \mu = f^{-1}$.

Let $J = (a_i, b_i)$ and let $J' = (a_j, b_j)$. We have two collections

$$\{\dots, f^{-1}(J), J, f(J), f^2(J), \dots\}, \quad \{\dots, f^{-1}(J'), J', f(J'), f^2(J'), \dots\}, \tag{5.3}$$

each consisting of pairwise disjoint intervals. These two collections either share no common members, or else they coincide. In the first case, choose an arbitrary orientation reversing homeomorphism γ from J to J'. In the second case, there is an integer m such that $f^m(J) = J'$. Apply Lemma 5.4 with $g = f^m$ to deduce the existence of an orientation reversing homeomorphism γ from J to J' satisfying $\gamma f^{-m} \gamma = f^m$. In each case we define, for each $n \in \mathbb{Z}$,

$$\mu(x) = \begin{cases} f^n \gamma f^n(x) & \text{if } x \in f^{-n}(J), \\ f^n \gamma^{-1} f^n(x) & \text{if } x \in f^{-n}(J'). \end{cases}$$

One can check that μ is well-defined for points x in one of the intervals of (5.3), and that $\mu^2(x) = x$ and $\mu f \mu(x) = f^{-1}(x)$. In this manner μ can be defined on each of the intervals (a_k, b_k). The resulting map is a homeomorphism of $\mathbb{S}$ that is an involution and satsifies $\mu f \mu = f^{-1}$. □

It follows from Theorem 1.3 and the results on conjugacy in §2 that an orientation preserving circle homeomorphism is reversible by an orientation *reversing* involution if and only if it is reversible by an orientation *reversing* homeomorphism. We now sketch the details of an example to show that there are orientation preserving circle homeomorphisms that are reversible by orientation preserving homeomorphisms, but are *not* strongly reversible in $\mathcal{H}$. This means that the concepts of reversibility and strong reversibility are not equivalent in either $\mathcal{H}$ or $\mathcal{H}^+$.

Let a be the point on $\mathbb{S}$ with co-ordinates $(0, 1)$ and let b be the point with co-ordinates $(0, -1)$. Let $\cdots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of points in (b, a) that accumulates only at a and b. Let g be an orientation preserving homeomorphism from $[b, a]$ to $[b, a]$ that fixes only the points a_i, a, and b. We construct a doubly infinite sequence s consisting of 1s and -1s as follows. Let u represent the string of six numbers $1, 1, 1, -1, -1, 1$. Let $-u$ represent the string $-1, -1, -1, 1, 1, -1$. Then s is given by $\dots, u, -u, u, -u, \dots$. We say that two doubly infinite sequences (x_n) and (y_n) are equal if and only if there is an integer k such that $x_{n+k} = y_n$ for all n. Our sequence s has been constructed such that $s = -s$, and the sequence formed by reversing s is distinct from s.

Suppose that g is defined in any fashion on the intervals

$$\dots, (a_{-2}, a_{-1}), (a_{-1}, a_0), (a_0, a_1), (a_1, a_2), \dots$$

such that the signature of g on these intervals is determined by s. For $x \in (a, b)$ we define $g(x) = R_\pi g R_\pi(x)$, where R_π is the rotation by π. A diagram of the homeomorphism g is shown in Figure 3. Since we can embed g in a flow, we can certainly choose a square-root

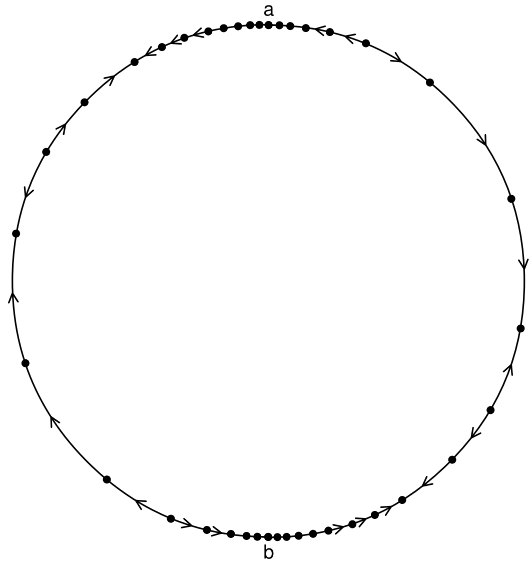

FIGURE 3.

$\sqrt{g}$ of g (which shares the same signature as g). Define another circle homeomorphism f by the formula

$$f(x) = \begin{cases} R_\pi \sqrt{g}(x) & \text{if } x \in [b, a], \\ \sqrt{g} R_\pi(x) & \text{if } x \in (a, b). \end{cases}$$

This map f has rotation number $\frac{1}{2}$ and it satisfies $f^2 = g$. The map f is not reversible by an orientation preserving involution, by Theorem 3.3, because it is not an involution. Nor is f reversible by an orientation reversing involution; for it were then g would also be reversed by the same orientation reversing involution, and from Theorem 1.3 one can deduce that this would mean that the sequence s coincides with the reversed sequence s. Finally, f is reversible since g is reversible, by Theorem 2.3; a conjugation from g to g^{-1} can be constructed that maps a_i to a_{i+6} for each i.

6. PROOF OF THEOREM 1.4

Proof of Theorem 1.4. If f is strongly reversible then it is expressible as a composite of two involutions, one of which, τ, must reverse orientation. From the equation $\tau f \tau = f^{-1}$ we see that $\tau f^2 \tau = f^{-2}$, and that τ preserves the pair of fixed points of f as a set. If τ fixes each of the fixed points of f then τf is an orientation preserving involution with fixed points. Hence it is the identity map. Therefore $f = \tau$. The alternative is that τ interchanges the fixed points of f, which is the condition stated in Theorem 1.4.

For the converse, suppose that a and b are the two fixed points of f. By Theorem 1.3 (i) we can construct from h an orientation reversing involution μ such that $\mu f^2 \mu = f^{-2}$. We require that μ, like h, interchanges a and b; this is not given by the statement of Theorem 1.3, however, it is immediate from the definition of μ in (5.1) (where, in that equation, s is our current homeomorphism h). Now define

$$\tau(x) = \begin{cases} \mu(x) & \text{if } x \in [a, b], \\ f\mu f(x) & \text{if } x \in [b, a]. \end{cases}$$

Then τ is an involution in $\mathrm{H}^-(\mathbb{S})$ and $\tau f \tau = f^{-1}$. □

The hypothesis that h interchanges the pair of fixed points of f cannot be dropped from Theorem 1.4: one can construct examples of orientation reversing homeomorphisms f and involutions τ such that $\tau f^2 \tau = f^{-2}$ even though f is not strongly reversible. There are also examples of orientation reversing circle homeomorphisms that are reversible, but not strongly reversible.

References

[1] D. Eisenbud, U. Hirsch, and W. Neumann. Transverse foliations of seifert bundles and self-homeomorphisms of the circle. *Comment. Math. Helv.*, 56:638–660, 1981.

[2] N. J. Fine and G. E. Schweigert. On the group of homeomorphisms of an arc. *Ann. of Math.*, 62(2):237–253, 1955.

[3] E. Ghys. Groups acting on the circle. *Enseign. Math.*, 47(3-4):329–407, 2001.

[4] W. Jarzcyk. Reversible interval homeomorphisms. *J. Math. Anal. Appl.*, 272:473–479, 2002.

[5] N. G. Markley. Homeomorphisms of the circle without periodic points. *Proc. London Math. Soc.*, 20(3):688–698, 1969.

[6] A. G. O'Farrell. Conjugacy, involutions, and reversibility for real homeomorphisms. *Irish Math. Soc. Bulletin*, 54:41–52, 2004.

[7] S. Young. The representation of homeomorphisms on the interval as finite compositions of involutions. *Proc. Amer. Math. Soc*, 121:605–610, 1994.