Meromorphic Scaling Flow of $N = 2$ Supersymmetric $SU(2)$ Yang-Mills with Matter

Brian P. Dolan

Dept. of Mathematical Physics,
National University of Ireland, Maynooth
and
Dublin Institute for Advanced Studies,
10, Burlington Rd., Dublin, Ireland
`bdolan@thphys.nuim.ie`

ABSTRACT: β-functions are derived for the flow of $N = 2$ SUSY $SU(2)$ Yang-Mills in 4-dimensions with massless matter multiplets in the fundamental representation of the gauge group. The β-functions represent the flow of the couplings as the VEV of the Higgs field is lowered and are modular forms of weight -2. They have the correct asymptotic behaviour at both the strong and weak coupling fixed points. Corrections to the massless β-functions when masses are turned on are discussed.

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Contents

1. Introduction

In a previous paper [1] the flow of $N = 2$ SUSY pure $SU(2)$ Yang-Mills, with no matter fields, was analysed and a meromorphic β-function was constructed which is finite at both weak and strong coupling. Up to a constant factor this β-function reproduces the correct 1-loop Callan-Symanzik flow at both strong and weak coupling and interpolates between them analytically, although there is no a priori reason to interpret it as Callan-Symanzik β-function away from the region of the fixed points. This analysis modified previous suggestions in the literature concerning the β-functions for $N = 2$ SUSY [2, 3, 4, 5] and evades the criticisms in [6].

In the present paper the analysis is extended to include massless matter fields in the fundamental representation of $SU(2)$ with $N_f = 1, 2, 3$ flavours. The construction uses the gauge invariant flow parameter $u = tr < \varphi^2 >$, where φ is the Higgs field whose VEV is a free parameter, and the fact that the β-functions are modular forms of weight -2 of a sub-group of the full modular group, $\Gamma(1) \approx Sl(2, \mathbf{Z})/\mathbf{Z}_2$, depending on N_f. The significance of the parameter u was emphasised in [7] where is was shown that u is the Legendre transform of the pre-potential.

Following [8, 9] a convenient choice of modular parameter for $N_f > 0$ is

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}, \tag{1.1}$$

where θ is the usual topological parameter labelling θ-vacua and g is the Yang-Mills coupling constant. In terms of τ the relevant sub-groups of $Sl(2, \mathbf{Z})$ for determining the β-functions are: $\Gamma^0(2)$ for $N_f = 0$; $\Gamma(1)$ for $N_f = 1$; $\Gamma_0(2)$ for $N_f = 2$; and $\Gamma_0(4)$ for $N_f = 3$.[1] For $N_f = 0$ and $N_f = 2$ these are larger than the monodromy group. The $N_f = 1$ case realises the full modular group, so the self-dual point $\tau = i$ is a fixed point of the element sending $\tau \to -\frac{1}{\tau}$ which is in the monodromy group. All the β-functions discussed here do still have singularities somewhere in the fundamental domain of the relevant sub-group of $\Gamma(1)$, they must do since any modular form of weight -2 must have at least one singularity within, or on the boundary of, the fundamental domain, but these singularities are off the real axis and, with the exception of $N_f = 1$, correspond to repulsive fixed points in both directions of the flow. For the $N_f = 1$ β-functions there are two types of singularities off the real axis: one at $\tau = i$ and its images under $\Gamma(1)$ which is repulsive in both directions of the flow; and one at $\tau = e^{i\pi/3}$ and its images under $\Gamma(1)$ which is attractive in the direction of decreasing Higgs VEV.

The strategy is to use the technique of [2] where the Seiberg-Witten curves describing the various theories [9] are written both in terms of the 'bare' $\tau = i\infty$ and the renormalised finite τ and the co-efficients compared to extract $\tau(u)$. An important tool in the analysis is the modular symmetry derived in [8, 9] where it was shown that $N = 2$ SUSY Yang-Mills has an infinite hierarchy of vacua with massless BPS states and the modular group relates these vacua to each other. The value of τ in one vacuum is related to that of another by

$$\tau \to \gamma(\tau) = \frac{a\tau + b}{c\tau + d} \tag{1.2}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \Gamma(1)$ with Γ a sub-group of the full modular group $\Gamma(1) \cong PSl(2, \mathbf{Z})$. We therefore have, under a variation $\delta\tau$ of τ,

$$\delta\gamma(\tau) = \frac{1}{(c\tau + d)^2}\delta\tau, \tag{1.3}$$

since $ad - bc = 1$, so we expect that

$$\beta(\gamma(\tau)) = \frac{1}{(c\tau + d)^2}\beta(\tau). \tag{1.4}$$

If β is meromorphic, it will be a modular form of weight -2 and this fact proves to be a powerful analytical tool.

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[1]The notation is that of [10] and is summarised in the appendix for ease of reference.

In §2 the β-function for $N_f = 0$ discussed in [1] is re-derived in terms of (1.1), which differs from the normalisation in [1]. The cases $N_f = 2$, $N_f = 3$ and $N_f = 1$ are treated in sections 3, 4 and 5 respectively where meromorphic β-functions are proposed which vanish at all the strong coupling fixed points and are constant at the weak coupling fixed point.

In principle the β-functions for different values of N_f should be related by holomorphic decoupling but the inclusion of non-zero masses for the matter multiplets makes analytic calculations much harder and cannot be pushed through using the techniques of the present analysis. In section 6 a perturbative approach is presented, using a strong coupling expansion and turning on a mass for one of the matter fields. It is shown that the limits $\tau_D = -1/\tau \to i\infty$ and $m \to 0$ do not commute, but nevertheless a β-function for the massive theory with acceptable behaviour near $\tau_D = i\infty$ can be constructed.

Section 7 contains our conclusions. Two appendices give a summary of the conventions concerning Jacobi ϑ-functions and the technical aspects of the strong coupling instanton expansion used for the analysis in section 6.

2. $N_f = 0$

This case was treated in [1] using the normalisation appropriate to the adjoint representation of $SU(2)$, $\tilde{\tau} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$, but when matter in the fundamental representation of $SU(2)$ is included it is better to define $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$, [9]. In order to set the notation and illustrate the method for $N_f \neq 0$ the derivation of the β-function in [1] is given here using the original techniques of [2], adapted to the present notation.

First recall the monodromy of the $N_f = 0$ theory [9]. It is generated by

$$\mathcal{M}_0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \mathcal{M}_\infty = \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix} \tag{2.1}$$

with

$$\mathcal{M}_\infty \mathcal{M}_0 = \mathcal{M}_2^{-1} \qquad \text{where} \qquad \mathcal{M}_2 = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}, \tag{2.2}$$

[9] ($\mathcal{M}_0$ leaves $\tau = 0$ invariant, $\mathcal{M}_2$ leaves $\tau = 2$ invariant and $\mathcal{M}_\infty$ leaves $\tau = i\infty$ invariant). The two matrices $\mathcal{M}_\infty$ and $\mathcal{M}_0^{-1}$,

$$\tau \to \tau + 4, \qquad \tau \to \frac{\tau}{\tau + 1}, \tag{2.3}$$

generate $\Gamma^0(4)$ and therefore β will be a modular form for $\Gamma^0(4)$ of weight -2. To determine its explicit form we start by following the method of [2]. The massless $N_f = 4$ curve can be written

$$y^2 = x^4 - 2\bar{u}F(\tau)x^2 + \bar{u}^2, \tag{2.4}$$

where

$$F(\tau) = \frac{\vartheta_3^4(\tau) + \vartheta_2^4(\tau)}{\vartheta_4^4(\tau)} \tag{2.5}$$

and the $N_f = 0$ curve of [8, 9] can be written as

$$y^2 = x^4 - 2ux^2 + u^2 - \Lambda_0^4. \tag{2.6}$$

Note that $F(\tau)$ is an invariant function under $\Gamma(2)$.

Equation (2.4) is scale invariant in the sense that τ does not depend on $\bar{u}$. Equating co-efficients and eliminating $\bar{u}$ gives

$$F(\tau) = \frac{u}{\sqrt{u^2 - \Lambda_0^4}} \tag{2.7}$$

hence

$$u\frac{d\tau}{du}F'(\tau) = -F(F^2 - 1). \tag{2.8}$$

where $F' = \frac{dF}{d\tau}$. Using (A.11) to evaluate $F'(\tau)$ from (2.5) gives

$$F' = 2\pi i \frac{\vartheta_3^4(\tau)\vartheta_2^4(\tau)}{\vartheta_4^4(\tau)} \tag{2.9}$$

finally leading to

$$u\frac{d\tau}{du} = \frac{2i}{\pi}\frac{\vartheta_3^4(\tau) + \vartheta_2^4(\tau)}{\vartheta_4^8(\tau)} \tag{2.10}$$

which is the result of [2], after taking account of the difference of notation (the variable τ here is 2τ in [2]). The asymptotic behaviour of (2.10) is

$$u\frac{d\tau}{du} \xrightarrow[\tau \to i\infty]{} \frac{2i}{\pi} \tag{2.11}$$

which is the correct asymptotically free behaviour. However it was pointed out in [6] that (2.10) has a pathology in that it is singular at strong coupling, in particular at $\tau = 0$ ($u = \Lambda_0^2$) and $\tau = 2$ ($u = -\Lambda_0^2$).

A remedy was proposed in [1], based on [4]. The idea is to tame the singularities at strong coupling, without disturbing the asymptotic properties at $u \to \infty$, by defining

$$\beta(\tau) = -\frac{(u - \Lambda_0^2)^m(u + \Lambda_0^2)^n}{u^{m+n}}u\frac{d\tau}{du}, \tag{2.12}$$

where m and n are positive integers (the sign is chosen so the direction of flow is the same as in [2]). The correct behaviour at strong coupling then forces $m = n = 1$.

The motivation behind (2.12) relies on a theorem that any modular form of weight -2 for a sub-group $\Gamma \subset \Gamma(1)$ can always be written as

$$\beta(\tau) = \frac{P(f)}{Q(f)f'} \tag{2.13}$$

where $f(\tau)$ is a particular invariant function for Γ (essentially one with the smallest number of zeros plus number of poles, counting multiplicity) and $P(f)$ and $Q(f)$ are polynomials in f (see *e.g* [11], page 111, the idea of applying this theorem to $N = 2$ SUSY was first proposed in [4]). Now $u(\tau)$ is just such an invariant function for $\Gamma^0(4)$ — the explicit form of $u(\tau)$ follows from (2.5) and (2.7)

$$\frac{u}{\Lambda_0^2} = \frac{\vartheta_3^4(\tau) + \vartheta_2^4(\tau)}{2\vartheta_3^2(\tau)\vartheta_2^2(\tau)}, \tag{2.14}$$

and it can be verified explicitly, using (A.5) and (A.6), that u is invariant under $\Gamma^0(4)$. Choosing the zeros of P and Q so as to get the correct asymptotic behaviour at $\tau = 0, 2$ and $i\infty$ without introducing unnecessary zeros or poles, one is led to (2.12) and, using $m = n = 1$ in conjunction with (2.10) and (2.14), this leads uniquely to

$$\beta(\tau) = -\frac{(u - \Lambda_0^2)(u + \Lambda_0^2)}{u} \frac{d\tau}{du} = \frac{2}{\pi i} \frac{1}{\vartheta_3^4(\tau) + \vartheta_2^4(\tau)}. \tag{2.15}$$

The asymptotic behaviour is

$$\beta(\tau) \rightarrow \frac{2}{\pi i} \quad \text{as} \quad \tau \rightarrow i\infty$$
$$\beta(\tau) \rightarrow -\frac{1}{\pi i}\tau^2 \quad \text{as} \quad \tau \rightarrow 0 \tag{2.16}$$
$$\beta(\tau) \rightarrow -\frac{1}{\pi i}(\tau - 2)^2 \quad \text{as} \quad \tau \rightarrow 2.$$

In particular the behaviour $\beta \approx \frac{i}{\pi}\tau^2$ as $\tau \rightarrow 0$ implies that $\beta(\tau_D) \approx \frac{i}{\pi}$ as $\tau_D \rightarrow i\infty$, where $\tau_D = -1/\tau$ is the dual coupling. Had any value of n been used in (2.12) other than unity, the result near $\tau \approx 0$ would have been

$$\beta(\tau_D) \propto \left(e^{2i\pi\tau_D}\right)^{n-1} \frac{i}{\pi} \tag{2.17}$$

which is not the correct asymptotic behaviour. An exactly parallel argument applies at $\tau = 2$ with n replaced by m.

Near both the weak and the strong coupling fixed points this β-function can be interpreted as a Callan-Symanzik β-function for $N_f = 0$ SUSY $SU(2)$ Yang-Mills [1] because u is a mass squared at weak coupling, where

$$\beta \approx -u\frac{d\tau}{du}, \tag{2.18}$$

while at strong coupling near $\tau \approx 0$, where $u - \Lambda_0^2 \approx \Lambda_0 a_D$ with a_D proportional to the mass of the BPS monopole, we have

$$\beta \approx -2(u - \Lambda_0^2)\frac{d\tau}{du} \approx -2a_D\frac{d\tau}{da_D}. \tag{2.19}$$

The flow is shown in figure 1. As the Higgs VEV is lowered there are attractive fixed points for all even integral τ and repulsive fixed points for all odd integral τ. There are repulsive fixed points in both flow directions at $\tau = k+i$ for all odd k ($\beta(\tau)$ diverges at these points because $\vartheta_3^4(\tau) = -\vartheta_2^4(\tau)$ there). From the transformation properties of the ϑ-functions in the appendix (A.5) and (A.6), the β-function in (2.15) is a modular form of weight -2 for the group generated by

$$\tau \to \tau + 2, \qquad \tau \to \frac{\tau}{\tau + 1}, \tag{2.20}$$

which is the group $\Gamma^0(2)$. $\Gamma^0(4)$ is a sub-group of $\Gamma^0(2)$ and the fact that the β-function has a larger symmetry than that of the monodromy group is due to the $\mathbf{Z}_2$ symmetry of β under $u \to -u$, enforced by choosing $m = n$ in (2.12). This $\mathbf{Z}_2$ is the anomaly-free residue of global R-symmetry [9] which is not a symmetry of the effective action. Although u is not invariant under $\Gamma^0(2)$, it changes sign under $\tau \to \tau + 2$, u^2 is invariant.

There are attractive fixed points at all the images of $\tau = 0$ under $\Gamma^0(2)$ and repulsive fixed points at all the images of $\tau = 1$ under $\Gamma^0(2)$: at strong coupling all rational values $\tau = q/m$ with q even are attractive fixed points and those with q odd are repulsive (with q and m mutually prime) as the Higgs VEV is decreased.

The β-functions in [1], using $\tilde{\tau} = \tau/2$ rather than τ, are the same as (2.15) above as can be checked using (A.7) from which

$$\beta(\tilde{\tau}) = \frac{1}{2}\beta(\tau) = \frac{2}{\pi i} \frac{1}{\vartheta_3^4(\tilde{\tau}) + \vartheta_4^4(\tilde{\tau})}. \tag{2.21}$$

These are modular forms for $\Gamma_0(2)$ which is generated by

$$\tilde{\tau} \to \tilde{\tau} + 1, \qquad \tilde{\tau} \to \frac{\tilde{\tau}}{2\tilde{\tau} + 1}, \tag{2.22}$$

equivalent to

$$\tau \to \tau + 2, \qquad \tau \to \frac{\tau}{\tau + 1} \tag{2.23}$$

with $\tau = 2\tilde{\tau}$. The details of the flow generated by (2.21) were analysed in [1].

3. $N_f = 2$

The results of the last section can be immediately be used to guess the form of the β-function for $N_f = 2$, which can then be verified explicitly. The monodromy for the $N_f = 2$ case is generated by [9]

$$\mathcal{M}_0 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad \mathcal{M}_\infty = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}, \tag{3.1}$$

with

$$\mathcal{M}_\infty \mathcal{M}_0 = \mathcal{M}_1^{-1} \qquad \text{where} \qquad \mathcal{M}_1 = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \qquad (3.2)$$

($\mathcal{M}_0$ leaves $\tau = 0$ invariant, $\mathcal{M}_1$ leaves $\tau = 1$ invariant and $\mathcal{M}_\infty$ leaves $\tau = i\infty$ invariant). The two matrices $\mathcal{M}_\infty$ and $\mathcal{M}_0^{-1}$,

$$\tau \to \tau + 2, \qquad \tau \to \frac{\tau}{2\tau + 1}, \qquad (3.3)$$

generate $\Gamma(2)$. If a β-function is constructed which is invariant under the anomaly-free $\mathbf{Z}_2$ acting on the u-plane, $u \to -u$, along the same lines as before then we expect it will have a further symmetry under

$$\tau \to \tau + 1, \qquad (3.4)$$

and so will be a modular form of $\Gamma_0(2)$ of weight -2. Demanding the correct behaviour at $\tau = i\infty$, $\tau = 0$ and $\tau = 1$ leaves only one possibility and that is (2.21) with $\tilde{\tau}$ replaced by τ, namely

$$\beta(\tau) = \frac{2}{\pi i} \frac{1}{\vartheta_3^4(\tau) + \vartheta_4^4(\tau)}, \qquad (3.5)$$

the normalisation being determined by the asymptotic condition

$$\beta(\tau) \xrightarrow[\tau \to i\infty]{} \frac{1}{\pi i}. \qquad (3.6)$$

We now verify (3.5) by explicit calculation using the same technique as in the previous section. The analysis initially parallels that of [2]: the massless $N_f = 2$ curve is

$$y^2 = x^4 - 2(u + 3\Lambda_2^2/8)x^2 + (u - \Lambda_2^2/8)^2. \qquad (3.7)$$

Equating co-efficients with (2.4) gives

$$F(\tau) = \frac{u + 3\Lambda_2^2/8}{u - \Lambda_2^2/8} \qquad (3.8)$$

or equivalently

$$u(\tau) = \frac{\Lambda_2^2}{8} \frac{\vartheta_3^4(\tau)}{\vartheta_2^4(\tau)}. \qquad (3.9)$$

Using (2.5) and (2.9), now leads to

$$u \frac{d\tau}{du} = -\frac{(F-1)(F+3)}{4F'} = \frac{i}{2\pi} \frac{\vartheta_3^4(\tau) + \vartheta_4^4(\tau)}{\vartheta_3^4(\tau)\vartheta_4^4(\tau)} \qquad (3.10)$$

as in [2]. There are singularities at $\tau = 0$ (where $u = \Lambda_2^2/8$) and $\tau = 1$ (where $u = -\Lambda_2^2/8$) which can be modified, along lines similar to §2, to give a β-function with the correct strong coupling behaviour,

$$\beta(\tau) = -\frac{\left(u^2 - \frac{\Lambda_2^4}{64}\right)}{u^2} u \frac{d\tau}{du} = \frac{2}{\pi i} \frac{1}{\vartheta_3^4(\tau) + \vartheta_4^4(\tau)}, \qquad (3.11)$$

confirming (3.5). This flow is essentially the same as that of the $N_f = 1$ case treated in §2, except that τ is rescaled by a factor of $1/2$, and the flow shown in figure 2.

4. $N_f = 3$

The monodromy for $N_f = 3$ is generated by

$$\mathcal{M}_0 = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}, \qquad \mathcal{M}_\infty = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \tag{4.1}$$

with

$$\mathcal{M}_0 \mathcal{M}_\infty = \mathcal{M}_{-1/2}^{-1} \qquad \text{where} \qquad \mathcal{M}_{-1/2} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \tag{4.2}$$

($\mathcal{M}_0$ leaves $\tau = 0$ invariant, $\mathcal{M}_{-1/2}$ leaves $\tau = -1/2$ invariant and $\mathcal{M}_\infty$ leaves $\tau = i\infty$ invariant), [9]. The two matrices $\mathcal{M}_\infty$ and $\mathcal{M}_0^{-1}$,

$$\tau \to \tau + 1, \qquad \tau \to \frac{\tau}{4\tau + 1}, \tag{4.3}$$

generate $\Gamma_0(4)$. But it is not obvious what the full symmetry of the β-functions might be as $\mathbf{Z}_2$ does not play any role in the $N_f = 3$ theory [9]. We must therefore perform the explicit calculation. Our starting point this time is the massless $N_f = 4$ curve in the original form of [9], namely

$$y^2 = x^3 - \frac{1}{4}g_2(\tau)\bar{u}^2 x - \frac{1}{4}g_3(\tau)\bar{u}^3 \tag{4.4}$$

with

$$g_2(\tau) = \frac{2}{3}\left(\vartheta_2^8(\tau) + \vartheta_3^8(\tau) + \vartheta_4^8(\tau)\right) \tag{4.5}$$

and

$$g_3(\tau) = \frac{4}{27}\left(\vartheta_3^4(\tau) + \vartheta_4^4(\tau)\right)\left(\vartheta_2^4(\tau) + \vartheta_3^4(\tau)\right)\left(\vartheta_4^4(\tau) - \vartheta_2^4(\tau)\right), \tag{4.6}$$

the combination

$$\frac{g_2^3(\tau)}{g_2^3(\tau) - 27g_3^2(\tau)} = \frac{(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8)^3}{54\vartheta_2^8\vartheta_3^8\vartheta_4^8} = J(\tau) \tag{4.7}$$

being Klein's absolute invariant which is invariant under the action of $\Gamma(1)$ on τ.

The massless $N_f = 3$ curve is

$$y^2 = \left(x^2 - \frac{\Lambda_3^2}{64}(x - u)\right)(x - u). \tag{4.8}$$

Without loss of generality we can set[2] $\Lambda_3^2 = 64$ and then eliminate the quadratic term in (4.8) by shifting $x \to x + (u+1)/3$ giving

$$y^2 = x^3 - \frac{1}{3}(u^2 - 4u + 1)x - \frac{1}{27}(2u - 1)(u^2 + 8u - 2). \tag{4.9}$$

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[2]This is equivalent to defining the dimensionless variable $\tilde{u} = 64u/(\Lambda_3^2)$ and then dropping the tilde.

There are singular points were the roots of this equation coincide, at $u = 0$, $u = 1/4$ and $|u| = \infty$, corresponding to $\tau = 0$, $\tau = -1/2$ and $\tau = i\infty$ respectively.

Equating co-efficients between (4.4) and (4.9) and eliminating $\bar{u}$ yields

$$J(\tau) = \frac{4}{27} \frac{(u^2 - 4u + 1)^3}{u^4(1 - 4u)}. \tag{4.10}$$

To extract the β-function from this equation we will need an explicit expression for $u(\tau)$. To this end define

$$Y(u) = \frac{u^2}{1 - 4u} \qquad \text{and} \qquad X(\tau) = \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)} = \frac{2}{F(\tau) + 1} \tag{4.11}$$

in terms of which (4.10) reads

$$\frac{(1 + Y)^3}{Y^2} = \frac{(1 - X + X^2)^3}{(1 - X)^2 X^2} \tag{4.12}$$

with the three roots

$$Y_1 = -X(1 - X), \qquad Y_2 = \frac{(1 - X)}{X^2} \qquad \text{and} \qquad Y_3 = \frac{X}{(1 - X)^2}. \tag{4.13}$$

Comparing with the three asymptotic forms

$$
\begin{aligned}
\tau \to i\infty, & \qquad X \to 1, & \qquad |u| \to \infty, & \qquad Y \to \infty, \\
\tau \to 0, & \qquad X \to 0, & \qquad u \to 0, & \qquad Y \to 0, \\
\tau \to -1/2, & \qquad X \to 1, & \qquad u \to 1/4, & \qquad Y \to \infty,
\end{aligned} \tag{4.14}
$$

only Y_3 has the correct asymptotic behaviour at the three singular points and so we must choose

$$\frac{u^2}{1 - 4u} = \frac{X}{(1 - X)^2} = \frac{\vartheta_3^4 \vartheta_4^4}{\vartheta_2^8}, \tag{4.15}$$

which is an invariant function for $\Gamma_0(2)$ [11]. Solving for u the asymptotic conditions pick out the unique solution

$$u(\tau) = -\frac{\vartheta_3^2 \vartheta_4^2}{(\vartheta_3^2 - \vartheta_4^2)^2} \tag{4.16}$$

which is, of course, an invariant function for $\Gamma_0(4)$.

Differentiating this equation with respect to τ, and employing (A.11), yields

$$u\frac{d\tau}{du} = \frac{2i}{\pi} \frac{1}{(\vartheta_3^2(\tau) + \vartheta_4^2(\tau))^2}. \tag{4.17}$$

This has the correct asymptotic form as $\tau \to i\infty$,

$$u\frac{d\tau}{du} \xrightarrow[\tau \to i\infty]{} \frac{i}{2\pi}, \tag{4.18}$$

and is well behaved at $\tau = 0$

$$u\frac{d\tau}{du} \xrightarrow[\tau \to 0]{} -\frac{2i}{\pi}\tau^2, \tag{4.19}$$

but diverges at $\tau = -1/2$.

The method used in the previous sections for $N_f = 0$ and $N_f = 2$ is not immediately applicable here since one of the fixed points is at $u = 0$: to eliminate the singularity at $u = 1/4$ we would have to multiply by $(u - 1/4)^m/u^m$ for some positive m which would introduce another singularity at $u = 0$. One strategy is to use $m = 1$ to remove the singularity at $u = 1/4$ and shift the other fixed point away from the origin by shifting u: thus let $u' = u - \epsilon$ for some constant ϵ and define

$$\beta(\tau) = -\frac{(u' + \epsilon - 1/4)(u' + \epsilon)}{u'}\frac{d\tau}{du'}. \tag{4.20}$$

This does not disturb the behaviour at $|u| = \infty$. Clearly this is equivalent to

$$\beta(\tau) = -\frac{(u - 1/4)}{(u - \epsilon)}u\frac{d\tau}{du} \tag{4.21}$$

which preserves the good behaviour at $\tau = i\infty$ and $\tau = 0$ and gives the correct behaviour at $\tau = -1/2$.

In terms of ϑ-functions

$$\beta(\tau) = \frac{1}{2\pi i}\frac{1}{\vartheta_3^2(\tau)\vartheta_4^2(\tau) + \epsilon(\vartheta_3^2(\tau) - \vartheta_4^2(\tau))^2}, \tag{4.22}$$

and has the following asymptotic forms

$$\begin{aligned}
\beta(\tau) &\xrightarrow[\tau \to \infty]{} \frac{1}{2\pi i}, \\
\beta(\tau) &\xrightarrow[\tau \to 0]{} -\frac{2}{\pi i}\frac{\tau^2}{\epsilon}, \\
\beta(\tau) &\xrightarrow[\tau \to -1/2]{} -\frac{2}{\pi i}\frac{(\tau + 1/2)^2}{(1 - 4\epsilon)}.
\end{aligned} \tag{4.23}$$

If ϵ is real and $0 < \epsilon < 1/4$, the β-function is finite at both $\tau = 0$ and $\tau = -1/2$ and flows in the right direction, i.e. in towards the fixed points as the Higgs VEV is lowered.

In fact just such a constant shift of u was found to be necessary in instanton calculations performed to check the validity of the Seiberg-Witten curve [12, 13]. These instanton calculations give $\epsilon = 4/27$ when $\Lambda_3/8$ is set to one as we have done here. The point is that, since $N_f = 3$ has no discrete symmetry acting on the u-plane, there is no a priori way, using the techniques in [9], to determine where the origin of the u-plane should lie and a constant finite shift of u does not affect the

weak coupling physics. Seiberg and Witten chose $u = 0$ to correspond to $\tau = 0$ but the instanton calculations show that $u' = -4/27$ is a more natural choice of origin.

The flow (4.22) with $\epsilon = 4/27$ is plotted in figure 3. At first glance it looks very like the flow for $N_f = 0$ and $N_f = 2$, with τ rescaled, but closer examination reveals subtle differences. Figure 3 is not symmetric under $\tau \to \tau + 1/2$, it is slightly distorted and the repulsive fixed point close to $\tau = (1 + i)/4$ is not exactly at $\tau = (1 + i)/4$, it is displaced away from the top of the semi-circular arch by a small amount. The flow looks like a distorted version of the $N_f = 2$ flow with τ rescaled by a factor of 2, $\tau_{N_f=2} = 2\tau_{N_f=3}$. This is because $\Gamma^0(4)$ acting on τ is equivalent to $\Gamma(2)$ acting on 2τ: the $N_f = 3$ β-functions for 2τ are therefore modular forms of $\Gamma(2)$. Modular forms of $\Gamma(2)$ can be obtained by distorting modular forms of $\Gamma_0(2)$. Provided $0 < \epsilon < 1/4$ the unstable fixed point of the $N_f = 3$ flow lies on the semi-circle in the upper-half τ-plane spanning the two points $\tau = 0$ and $\tau = 1/2$ on the real axis. The special case $\epsilon = 1/8$ has a higher symmetry than other values because this corresponds to the unstable fixed point being at the top of the semi-circular arch at $\tau = (1 + i)/4$ and this gives $\Gamma_0(2)$ symmetry acting on 2τ. Other values of ϵ have the lower symmetry of $\Gamma(2)$, as in figure 3. Flow diagrams like figure 3 have been postulated for the quantum Hall effect when the electron spins are poorly split [14].

5. $N_f = 1$

The $N_f = 1$ case has been left till last because it is more involved than the three cases already considered, though paradoxically it has a higher symmetry — the monodromy generates the full modular group $\Gamma(1)$. The monodromy is calculated in [9]: there are four singular points, at $\tau = 0$, $\tau = 1$, $\tau = 2$ and $\tau = i\infty$, with monodromies

$$\mathcal{M}_0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \mathcal{M}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix},$$
$$\mathcal{M}_2 = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}, \qquad \mathcal{M}_\infty = \begin{pmatrix} -1 & -3 \\ 0 & -1 \end{pmatrix} \tag{5.1}$$

respectively. Now

$$\mathcal{M}_0^2 \mathcal{M}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.2}$$

and

$$\mathcal{M}_0 \mathcal{M}_1 \mathcal{M}_0^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{5.3}$$

so the two operations

$$\tau \to -1/\tau \qquad \text{and} \qquad \tau \to \tau + 1 \tag{5.4}$$

are in the group generated by (5.1), which is therefore the full modular group $\Gamma(1)$.

Again the discussion starts along the lines of [2]. We take the massless $N_f = 4$ curve

$$y^2 = x^3 - \frac{1}{4}g_2(\tau)\bar{u}^2 x - \frac{1}{4}g_3(\tau)\bar{u}^3 \tag{5.5}$$

with

$$\frac{g_2^3(\tau)}{g_2^3(\tau) - 27g_3^2(\tau)} = \frac{(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8)^3}{54\vartheta_2^8\vartheta_3^8\vartheta_4^8} = J(\tau), \tag{5.6}$$

and compare this with the massless $N_f = 1$ curve

$$y^2 = x^2(x - u) - \Lambda_1^6/64. \tag{5.7}$$

First eliminate the x^2 term in the $N_f = 1$ curve by shifting $x \to x + u/3$,

$$y^2 = x^3 - \frac{u^2}{3}x - \left(\frac{2}{27}u^3 + \frac{\Lambda_1^6}{64}\right). \tag{5.8}$$

Equating co-efficients with (5.5) and eliminating $\bar{u}$ then produces

$$J(\tau) = -\frac{1}{4}\frac{u^6}{u^3 + 1}, \tag{5.9}$$

where the scale has been set by choosing $27\Lambda_1^6/256 = 1$. As $\tau \to i\infty$, $J \to \infty$ and this corresponds to $u \to -\infty$. But J is invariant under $\Gamma(1)$ and so $J \to \infty$ at the three points $\tau = 0$, $\tau = 1$ and $\tau = 2$ as well and these correspond to the three roots of $u^3 = -1$. Differentiating equation (5.9) then leads to

$$u\frac{d\tau}{du} = 3\left(\frac{u^3 + 2}{u^3 + 1}\right)\frac{J}{J'} \tag{5.10}$$

and (A.11) gives

$$\frac{J}{J'} = \frac{1}{2\pi i}\frac{\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8}{(\vartheta_3^4 + \vartheta_2^4)(\vartheta_2^4 - \vartheta_4^4)(\vartheta_3^4 + \vartheta_4^4)}. \tag{5.11}$$

As $\tau \to i\infty$,

$$J/J' \to \frac{i}{2\pi} \quad \text{and} \quad u\frac{d\tau}{du} = \frac{3i}{2\pi}, \tag{5.12}$$

which is the correct asymptotic behaviour. But J/J' has the same value at $\tau = 0, 1$ and 2 as at $\tau = i\infty$ and $u^3 = -1$ at these 3 points, so (5.10) diverges at strong coupling. Following the same procedure as before the singularities in (5.10) can be eliminated, without disturbing the behaviour at $u \approx -\infty$, by using

$$\left(\frac{u^3 + 1}{u^3}\right)u\frac{d\tau}{du} = 3\left(\frac{u^3 + 2}{u^3}\right)\frac{J}{J'}. \tag{5.13}$$

This is not a modular form for $\Gamma(1)$ however, since u is not invariant. Solving (5.9) for u gives

$$u^3 = -2\left(J \pm \sqrt{J(J-1)}\right). \tag{5.14}$$

and both roots are necessary: the upper sign for $u \to -\infty$ and the lower sign for $u^3 \approx -1$. Eliminating u^3 from (5.13) then gives an ambiguity in the direction of flow

$$\left(\frac{u^3 + 1}{u^3}\right) u \frac{d\tau}{du} = \pm 3 \frac{\sqrt{J(J-1)}}{J'} = \frac{\pm 3}{i\pi \sqrt{2(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8)}}. \tag{5.15}$$

The resolution of this problem is that the true β-function, which should be a modular form for $\Gamma(1)$, must have poles or zeros that are not accounted for in (5.13). This equation was derived by making the minimal modification of (5.10) that would remove the infinities at $u^3 = -1$ yet not disturb the asymptotic behaviour at $u = -\infty$. For $N_f = 1, 2$ and 3 this minimalist approach worked. For $N_f = 1$ it does not, there must be another pole or zero somewhere. In fact (5.15) already has a singularity at $\tau = e^{i\pi/3}$, where $\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8 = 0$ corresponding to $u = 0$, and its images. This is a fixed point of $\Gamma(1)$, meaning that there is at least one element $\gamma \in \Gamma(1)$ that leaves it invariant. [3] The point $\tau = i$, where $\vartheta_2^2 = \vartheta_4^2$, is also a fixed point of $\Gamma(1)$ and, if the flow commutes with $\Gamma(1)$, then $\tau = i$ should be a fixed point of the flow also. The next simplest assumption that can be made is that the only fixed points above the real axis are at $\tau = i$ and $\tau = e^{i\pi/3}$ and their images under $\Gamma(1)$, and we shall therefore assume that these are the only places above the real axis where a pole or a zero of the β-function is allowed. We shall further assume that the total number of poles plus zeros (counting multiplicity) is the smallest number compatible with these requirements, since this was our experience for $N_f = 0, 1$ and 3 and was an assumption in [8]. Jacobi's invariant function takes the values $J = 0$ at $\tau = e^{i\pi/3}$ and $J = 1$ at $\tau = i$. Therefore, with the above assumption, the β-function must be of the form

$$\beta(\tau) = -3 \frac{J^m (J-1)^n}{J'} \tag{5.16}$$

with m and n integers. In order that the β-function has the correct asymptotic form as $\tau \to i\infty$ it must be the case that $m + n = 1$ and minimising the total number of zeros and poles leaves only two possibilities $m = 0$, $n = 1$ or $m = 1$, $n = 0$. Examining these two possibilities the first gives

$$\frac{J-1}{J'} = \frac{2}{\pi i} \frac{(\vartheta_3^4 + \vartheta_2^4)(\vartheta_2^4 - \vartheta_4^4)(\vartheta_3^4 + \vartheta_4^4)}{(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8)^2} \tag{5.17}$$

which is singular when $\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8 = 0$, that is at $\tau = e^{i\pi/3}$ and its images, and the second gives

$$\frac{J}{J'} = \frac{1}{2\pi i} \frac{\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8}{(\vartheta_3^4 + \vartheta_2^4)(\vartheta_2^4 - \vartheta_4^4)(\vartheta_3^4 + \vartheta_4^4)} \tag{5.18}$$

which is singular when $(\vartheta_3^4 + \vartheta_2^4)(\vartheta_2^4 - \vartheta_4^4)(\vartheta_3^4 + \vartheta_4^4) = 0$, that is at $\tau = i$ and its images.[4] The latter has a milder singularity, and clearly has a smaller number of

---

[3] Actually there two such elements: $\gamma = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

[4] $\vartheta_2^4 = \vartheta_4^4$ at $\tau = i$, $\vartheta_3^4 = -\vartheta_4^4$ at $\tau = (1 + i)/2$ and $\vartheta_3^4 = -\vartheta_2^4$ at $\tau = 1 + i$.

poles plus zeros than the former, and so is the unique choice that fits the criteria. So we conjecture that the analytic β-function for $N_f = 1$ is

$$\beta(\tau) = -\left(\frac{u^3 + 1}{u^3 + 2}\right) u \frac{d\tau}{du} = -\frac{3}{2\pi i} \frac{(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8)}{(\vartheta_3^4 + \vartheta_2^4)(\vartheta_2^4 - \vartheta_4^4)(\vartheta_3^4 + \vartheta_4^4)}, \tag{5.19}$$

which differs from (5.13) only by using $u^3 + 2$ in the denominator, rather than u^3. This flow is plotted in figure 4, the pole at $\tau = i$, corresponding to $u^3 = -2$, renders this a repulsive fixed point in both flow directions while the zero at $\tau = e^{i\pi/3}$, corresponding to $u = 0$, is an attractive fixed point in the direction of decreasing Higgs VEV.

6. Massive Matter Multiplets

When the matter fields in the fundamental representation of $SU(2)$ have non-zero mass the analysis is much harder. For massive matter multiplets the Seiberg-Witten curves determine $\tau(u, m_i, \Lambda_{N_f})$ with $i = 1, \ldots, N_f$ and the $N_f - 1$ case can be determined from the N_f case by holomorphic decoupling [9]: set $\Lambda_{N_f-1}^{5-N_f} = m_{N_f} \Lambda_{N_f}^{4-N_f}$ and send $m_{N_f} \to \infty$ and $\Lambda_{N_f} \to 0$ keeping Λ_{N_f-1} finite. In principle therefore it ought to be possible to distort figure 3 for example, by turning on one mass, and recover figure 2 as the mass goes to infinity. This would correspond to a family of β-functions obtained by differentiating $\tau(u, m_i, \Lambda_{N_f})$ with respect to u and considering the m_i to parameterise different β-functions: near the fixed points this corresponds to defining β-functions by varying the $W^{\pm}$-boson, and therefore also the gluino, masses while keeping the quark masses fixed.

In this section we address the question of quark masses using a strong coupling expansion, taking the $N_f = 3$ case for illustrative purposes. The details of the analysis are rather technical and so are relegated to appendix B, from which we quote the relevant formulae here.

For $N_f = 3$ with three different masses m_1, m_2 and m_3 the Seiberg-Witten curve is uniquely determined by m_1, m_2, m_3 and the $\Gamma_0(4)$ invariant parameter u. With the term quadratic in x eliminated the curve is

$$\begin{aligned} y^2 = x^3 &- \frac{1}{3}\left(u^2 - 4u\tilde{\Lambda}_3^2 + \tilde{\Lambda}_3^4 + 3(m_1^2 + m_2^2 + m_3^2)\tilde{\Lambda}_3^2 - 6\tilde{\Lambda}_3 m_1 m_2 m_3\right) x \\ &- \frac{1}{27}\left\{(2u - \tilde{\Lambda}_3^2)(u^2 + 8u\tilde{\Lambda}_3^2 - 2\tilde{\Lambda}_3^4 - 9(m_1^2 + m_2^2 + m_3^2)\tilde{\Lambda}_3^2)\right. \\ &\left. - 18(u + \tilde{\Lambda}_3^2)\tilde{\Lambda}_3 m_1 m_2 m_3 + 27\tilde{\Lambda}_3^2(m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2)\right\}, \end{aligned} \tag{6.1}$$

where, in the notation of [9], $\tilde{\Lambda}_3 = \Lambda_3/8$. In principle this curve determines $\tau(u, m_i)$ and there are four different masses that could be varied to define β-functions. To connect with the β-functions of §4 we shall keep m_i fixed and vary only u.

In the massless case the strategy was to find an explicit expression for $u(\tau)$, in terms of Jacobi ϑ-functions, and then use the properties of the ϑ-functions to

determine $\frac{du}{d\tau}$. The problem in the massive case is to find $u(\tau, m_i)$ and this is much harder. We shall first simplify the problem and only consider one non-zero mass $m_1 = m$, $m_2 = m_3 = 0$. Even then one cannot hope for a closed form solution. However it is shown in appendix B that, at strong coupling where $\tau_D = -1/\tau \to i\infty$, it is appropriate to expand in $\tilde{q}_D := e^{i\pi\tau_D/2}$. The details are somewhat technical and left to the appendix but for one non-zero mass there are singularities at $\tau = 0$ when $u = \pm m\tilde{\Lambda}_3$. Near $u = m\tilde{\Lambda}_3$ the expansion

$$u(\tau_D) = m\tilde{\Lambda}_3 \left(1 + \alpha(m)e^{i\pi\tau_D} + \cdots\right) \tag{6.2}$$

is derived in appendix B where $\alpha(m)$ is an unknown function of $m/\tilde{\Lambda}_3$ which cannot be determined without further assumptions, but it diverges like $\sim 1/m^2$ as $m \to 0$ and $\alpha(m) \to 16$ as $m \to \infty$.

For u infinitesimally close to m the β-function will be of the form

$$\beta(\tau) \propto -(u - m)\frac{\partial\tau}{\partial u}, \tag{6.3}$$

assuming the BPS monopoles have mass $\propto u - m$. Using

$$\frac{\partial u}{\partial\tau_D} \approx i\pi m\tilde{\Lambda}_3\alpha(m)e^{i\pi\tau_D} \approx i\pi(u - m), \tag{6.4}$$

gives, for u near m,

$$\beta(\tau_D) \propto -(u - m)\frac{\partial\tau_D}{\partial u} \approx \frac{i}{\pi}, \tag{6.5}$$

which is perfectly well behaved, even in the massless limit $m \to 0$. Indeed

$$\beta(\tau) \propto \frac{i\tau^2}{\pi} \tag{6.6}$$

which, up to a constant, is the same behaviour as equation (4.23) even though it is clear from (6.2) that the limits $\tau_D \to i\infty$ and $m \to 0$ do not commute, since $\alpha(m)$ behaves as $1/m^2$ as $m \to 0$.

7. Conclusions

Explicit expressions have been proposed for the β-functions of $N = 2$ SUSY $SU(2)$ Yang-Mills with massless matter fields in the fundamental representation. Asymptotically close to the strong and weak coupling fixed points they co-incide with the 1-loop Callan-Symanzik β-functions, up to a constant factor.

The β-functions are modular forms of sub-groups of $\Gamma(1)$ for each value of N_f:

$$N_f = 0 \quad \Gamma^0(2)$$
$$N_f = 1 \quad \Gamma(1)$$

$$N_f = 2 \quad \Gamma_0(2)$$
$$N_f = 3 \quad \Gamma_0(4), \tag{7.1}$$

for $N_f = 1$ and $N_f = 3$ the group is the same as the monodromy group and for $N_f = 0$ and $N_f = 2$ it is larger, due to the $\mathbf{Z}_2$ action on the u-plane.

The β-functions are determined by demanding that they have the correct asymptotic behaviour at both weak and strong coupling fixed points. The relevant flows, in the direction of decreasing Higgs VEV, are shown in figures 1,4,2 and 3 respectively.

These functions differ from previous expressions in the literature in that they have the correct behaviour at all the strong coupling fixed points. In all cases the β-functions have a singularity in the interior, or on the boundary, of the fundamental domain, corresponding to an unstable fixed point which is repulsive in both directions of flow. This is a necessary consequence of their being modular forms of weight -2. For the $N_f = 1$ case, for which the monodromy group is the full modular group this repulsive fixed point is at the self-dual point $\tau = i$ and its images and in this case there is also a fully attractive fixed point at $\tau = e^{i\pi/3}$ and its images. The physical significance, if any, of this attractive fixed point is not clear.

The case of finite masses is probably intractable using the methods developed here, though it may well be possible to make progress with other gauge groups by using the methods in [2]. Nevertheless it has been possible to show, in a strong coupling expansion, that turning on one quark mass in the $N_f = 3$ case still allows for a β-function with the correct asymptotic behaviour near $\tau = 0$. Unfortunately it is not possible to say anything about the β-function away from $\tau = 0$ because of the limitations of the technique.

A. Appendix: properties of Jacobi ϑ-functions

We collect together some useful properties of ϑ-functions. The definitions are those of [15] and most of the formulae here are proven in that reference. The three Jacobi ϑ-functions used in the text are defined as

$$\vartheta_2(\tau) = 2 \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} = 2q^{\frac{1}{4}} \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n})^2, \tag{A.1}$$

$$\vartheta_3(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1})^2, \tag{A.2}$$

$$\vartheta_4(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1})^2, \tag{A.3}$$

where $q := e^{i\pi\tau}$.

These three ϑ-functions are not independent but are related by

$$\vartheta_3^4(\tau) = \vartheta_2^4(\tau) + \vartheta_4^4(\tau). \tag{A.4}$$

The following relations can be used to determine their properties under modular transformations:

$$\vartheta_2(\tau + 1) = e^{i\pi/4}\vartheta_2(\tau), \qquad \vartheta_3(\tau + 1) = \vartheta_4(\tau), \qquad \vartheta_4(\tau + 1) = \vartheta_3(\tau), \tag{A.5}$$

$$\begin{aligned}
\vartheta_2(-1/\tau) &= \sqrt{-i\tau}\ \vartheta_4(\tau), \\
\vartheta_3(-1/\tau) &= \sqrt{-i\tau}\ \vartheta_3(\tau), \\
\vartheta_4(-1/\tau) &= \sqrt{-i\tau}\ \vartheta_2(\tau).
\end{aligned} \tag{A.6}$$

The duplication formulae

$$\begin{aligned}
\vartheta_2^2(2\tau) &= \frac{1}{2}\left((\vartheta_3^2(\tau) - \vartheta_4^2(\tau)\right), \\
\vartheta_3^2(2\tau) &= \frac{1}{2}\left(\vartheta_3^2(\tau) + \vartheta_4^2(\tau)\right), \\
\vartheta_4^2(2\tau) &= \vartheta_3(\tau)\vartheta_4(\tau),
\end{aligned} \tag{A.7}$$

and

$$\vartheta_3(4\tau) = \frac{1}{2}\left(\vartheta_3(\tau) + \vartheta_4(\tau)\right), \quad \vartheta_2(4\tau) = \frac{1}{2}\left(\vartheta_3(\tau) - \vartheta_4(\tau)\right) \tag{A.8}$$

are also useful.

At the special points $\tau = e^{i\pi/2}$ and $\tau = e^{i\pi/3}$ the ϑ-functions have the values

$$\vartheta_3^2(e^{i\pi/2}) = \sqrt{2}\vartheta_2^2(e^{i\pi/2}) = \sqrt{2}\vartheta_4^2(e^{i\pi/2}) = \frac{2}{\pi}K\left(\sin\left(\frac{\pi}{4}\right)\right), \tag{A.9}$$

$$e^{-i\pi/4}\vartheta_2^2(e^{i\pi/3}) = e^{-i\pi/12}\vartheta_3^2(e^{i\pi/3}) = e^{i\pi/12}\vartheta_4^2(e^{i\pi/3}) = \frac{2}{\pi}K\left(\sin\left(\frac{\pi}{12}\right)\right),$$

where $K(k)$ is the complete elliptic of the second kind: $K(\sin(\pi/4)) = \frac{1}{4\sqrt{\pi}}(\Gamma(1/4))^2$, with $\Gamma(1/4) \approx 3.6256$ the Euler Γ-function evaluated at $1/4$, and $K(\sin(\pi/12)) \approx 1.5981$.

The ϑ-functions have the following asymptotic forms

$$\tau \to i\infty \ : \quad \vartheta_2(\tau) \approx 2\,e^{\frac{i\pi\tau}{4}} \to 0, \quad \vartheta_3(\tau) \to 1, \quad \vartheta_4(\tau) \to 1; \tag{A.10}$$

$$\tau \to 0 \ : \quad \vartheta_2(\tau) \approx \sqrt{\frac{i}{\tau}}, \quad \vartheta_3(\tau) \approx \sqrt{\frac{i}{\tau}}, \quad \vartheta_4(\tau) \approx 2\sqrt{\frac{i}{\tau}}\,e^{-\frac{i\pi}{4\tau}} \to 0.$$

In addition they satisfy the following differential equations (see [11], p.231, equation (7.2.17)),

$$\begin{aligned}
\frac{\vartheta_3'}{\vartheta_3} - \frac{\vartheta_4'}{\vartheta_4} &= \frac{i\pi}{4}\vartheta_2^4, \\
\frac{\vartheta_2'}{\vartheta_2} - \frac{\vartheta_3'}{\vartheta_3} &= \frac{i\pi}{4}\vartheta_4^4, \\
\frac{\vartheta_2'}{\vartheta_2} - \frac{\vartheta_4'}{\vartheta_4} &= \frac{i\pi}{4}\vartheta_3^4.
\end{aligned} \tag{A.11}$$

In the text $\Gamma_0(N)$ consists of matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\Gamma(1) \approx PSl(2, \mathbf{Z})$ with $c \equiv 0 \bmod N$, sometimes written

$$\gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N, \tag{A.12}$$

$\Gamma^0(N)$ consists of matrices with $b \equiv 0 \bmod N$,

$$\gamma \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \bmod N \tag{A.13}$$

and $\Gamma(N)$ consists of matrices with

$$\gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N. \tag{A.14}$$

B. Appendix: Massive Matter Fields for $N_f = 3$

The starting point is the curve for four massive matter multiplets in the fundamental representation [9],

$$
\begin{aligned}
y^2 &= (x^2 - c_2^2 \bar{u}^2)(x - c_1 \bar{u}) - c_2^2(x - c_1 \bar{u})^2 \bar{A} - c_2^2(c_1^2 - c_2^2)(x - c_1 \bar{u})\bar{B} \\
&\quad + 2c_2(c_1^2 - c_2^2)(c_1 x - c_2^2 \bar{u})\bar{C} - c_2^2(c_1^2 - c_2^2)^2 \bar{D},
\end{aligned}
\tag{B.1}
$$

with

$$c_1(\tau) = \frac{1}{2}(\vartheta_3^4(\tau) + \vartheta_4^4(\tau)), \qquad c_2(\tau) = \frac{1}{2}(\vartheta_3^4(\tau) - \vartheta_4^4(\tau)), \tag{B.2}$$

and

$$\bar{A} = \sum_i^4 \bar{m}_i^2, \qquad \bar{B} = \sum_{i<j} \bar{m}_i^2 \bar{m}_j^2, \qquad \bar{C} = \bar{m}_1 \bar{m}_2 \bar{m}_3 \bar{m}_4 \qquad \bar{D} = \sum_{i<j<k} \bar{m}_i^2 \bar{m}_j^2 \bar{m}_k^2. \tag{B.3}$$

First eliminate the quadratic term in x by shifting $x \to x + (c_1 \bar{u} + c_2^2 \bar{A})/3$, giving

$$y^2 = x^3 - \frac{1}{3}P(\tau, \bar{u}, \bar{m}_i)x - \frac{1}{27}Q(\tau, \bar{u}, \bar{m}_i), \tag{B.4}$$

where

$$P(\tau, \bar{u}, \bar{m}_i) = (c_1^2 + 3c_2^2)\bar{u}^2 - 4c_1 c_2^2 \bar{A}\bar{u} + 3c_2(c_1^2 - c_2^2)(c_2 \bar{B} - 2c_1 \bar{C}) + c_2^4 \bar{A}^2, \tag{B.5}$$

is quadratic in $\bar{u}$ and

$$
\begin{aligned}
Q(\tau, \bar{u}, m_i) &= 2c_1(c_1^2 - 9c_2^2)\bar{u}^3 + 3c_2^2(5c_1^2 + 3c_2^2)\bar{A}\bar{u}^2 \\
&\quad - 2\left(6c_1 c_2^4 \bar{A}^2 + 9c_1 c_2^2(c_1^2 - c_2^2)\bar{B} + 9c_2(c_1^2 - c_2^2)(c_1^2 - 3c_2^2)\bar{C}\right)\bar{u} \\
&\quad + 2c_2^6 \bar{A}^3 + 27c_2^2(c_1^2 - c_2^2)^2 \bar{D} + 9c_2^3(c_1^2 - c_2^2)(c_2 \bar{B} - 2c_1 \bar{C})\bar{A}
\end{aligned}
\tag{B.6}
$$

is cubic in $\bar{u}$.

Next this curve can be reduced to the $N_f = 3$ curve using the holomorphic decoupling of [9]: send $\bar{m}_4 \to \infty$ and take the 'bare' coupling $\tau \to i\infty$, so $c_2 \to 0$ and $c_1 \to 1$, keeping $c_2\bar{m}_4 = \Lambda_3/8 := \tilde{\Lambda}_3$ finite. Taking this limit, and dropping the bars on u, m_1, m_2 and m_3, gives the massive $N_f = 3$ curve,

$$y^2 = x^3 - \frac{1}{3}(u^2 - 4u\tilde{\Lambda}_3^2 + \tilde{\Lambda}_3^4 + 3B\tilde{\Lambda}_3^2 - 6C\tilde{\Lambda}_3)x \tag{B.7}$$
$$- \frac{1}{27}(2u - \tilde{\Lambda}_3^2)(u^2 + 8u\tilde{\Lambda}_3^2 - 2\tilde{\Lambda}_3^4 - 9B\tilde{\Lambda}_3^2) + \frac{2}{3}(u + \tilde{\Lambda}_3^2)C\tilde{\Lambda}_3 - D,$$

with

$$B = m_1^2 + m_2^2 + m_3^2, \qquad C = m_1m_2m_3, \qquad D = m_1^2m_2^2 + m_2^2m_3^2 + m_3^2m_1^2. \tag{B.8}$$

This reduces to equation (4.9) when the three masses are set to zero and $\tilde{\Lambda}_3 = 1$. Following [2] this curve is now compared to (B.4) with $\bar{m}_4 = 0$, $\bar{u}$ independent of τ, that is equation (B.4)-(B.6) with $\bar{C} = 0$ and

$$\bar{A} = \bar{m}_1^2 + \bar{m}_2^2 + \bar{m}_3^2, \qquad \bar{B} = \bar{m}_1^2\bar{m}_2^2 + \bar{m}_2^2\bar{m}_3^2 + \bar{m}_3^2\bar{m}_1^2, \qquad \bar{D} = \bar{m}_1^2\bar{m}_2^2\bar{m}_3^2. \tag{B.9}$$

Equating co-efficients leads to

$$(c_1^2 + 3c_2^2)\bar{u}^2 - 4c_1c_2^2\bar{A}\bar{u} + 3c_2^2(c_1^2 - c_2^2)\bar{B} + c_2^4\bar{A}^2 = u^2 - 4u\tilde{\Lambda}_3^2 + \tilde{\Lambda}_3^4 + 3B\tilde{\Lambda}_3^2 - 6C\tilde{\Lambda}_3 \tag{B.10}$$

$$2c_1(c_1^2 - 9c_2^2)\bar{u}^3 + 3c_2^2(5c_1^2 + 3c_2^2)\bar{A}\bar{u}^2 - 2\left(6c_1c_2^4\bar{A}^2 + 9c_1c_2^2(c_1^2 - c_2^2)\bar{B}\right)\bar{u}$$
$$+ 2c_2^6\bar{A}^3 + 27c_2^2(c_1^2 - c_2^2)^2\bar{D} + 9c_2^4(c_1^2 - c_2^2)\bar{B}\bar{A} \tag{B.11}$$
$$= (2u - \tilde{\Lambda}_3^2)(u^2 + 8u\tilde{\Lambda}_3^2 - 2\tilde{\Lambda}_3^4 - 9B\tilde{\Lambda}_3^2) - 18(u + \tilde{\Lambda}_3^2)C\tilde{\Lambda}_3 + 27D.$$

In the massless case we can eliminate $\bar{u}$ from these two equations and determine $u(\tau)$, but now we want to determine $u(\tau, m_i)$ with $\bar{u}$, $\bar{A}$, $\bar{B}$ and $\bar{D}$ all unknown functions so there is not enough information to solve the problem completely. Nevertheless we can still get information from these equations. The symmetry group for $N_f = 3$ is $\Gamma_0(4)$ and u, m_i and $\tilde{\Lambda}_3$ should be invariants of $\Gamma_0(4)$ so the right hand sides of (B.10) and (B.11) are also invariants. Using the transformation properties of the ϑ-functions, (A.5) and (A.7), this implies that $\bar{u}$, $\bar{A}$, $\bar{B}$, and $\bar{D}$ are modular forms of weights -2, -4, -8 and -12 respectively, so $\bar{m}_i$ have weight -2.

For simplicity we shall focus on the case of a single mass, $m_1 = m$, $m_2 = m_3 = 0$ so

$$\bar{A} = \bar{m}^2, \qquad \bar{B} = 0, \qquad \bar{D} = 0 \tag{B.12}$$

$$B = m^2, \qquad D = 0, \qquad C = 0 \tag{B.13}$$

and the equations simplify to

$$(c_1^2 + 3c_2^2)\bar{u}^2 - 4c_1c_2^2\bar{m}^2\bar{u} + c_2^4\bar{m}^4 = u^2 - 4u\tilde{\Lambda}_3^2 + \tilde{\Lambda}_3^4 + 3m^2\tilde{\Lambda}_3^2 \tag{B.14}$$

$$2c_1(c_1^2 - 9c_2^2)\bar{u}^3 + 3c_2^2(5c_1^2 + 3c_2^2)\bar{m}^2\bar{u}^2 - 12c_1c_2^4\bar{m}^4\bar{u} + 2c_2^6\bar{m}^6$$
$$= (2u - \tilde{\Lambda}_3^2)(u^2 + 8u\tilde{\Lambda}_3^2 - 2\tilde{\Lambda}_3^4 - 9m^2\tilde{\Lambda}_3^2). \tag{B.15}$$

We now want to eliminate $\bar{u}$ and $\bar{m}$ to get $u(\tau, m)$ but there is still not enough information. However, knowing that $\bar{u}$ and $\bar{m}^2$ are modular forms of weight -2 and -4 respectively, we can say something about their functional form at strong coupling. To see how this works let us first look at the case $\bar{m} = 0$, where the explicit solution is given in §4. Using the details there one finds

$$\bar{u}(\tau) = -\frac{\tilde{\Lambda}_3^2}{(\vartheta_3^2(\tau) - \vartheta_4^2(\tau))^2} \tag{B.16}$$

which is indeed a modular form for $\Gamma_0(4)$ of weight -2 and it vanishes at strong coupling, $\tau \to 0$. Now $\tau \to -1/\tau = \tau_D$ is not in $\Gamma_0(4)$, so $\bar{u}$ is not a modular form under this transformation, rather

$$\bar{u}(\tau_D) = \frac{1}{\tau_D^2} \frac{\tilde{\Lambda}_3^2}{(\vartheta_3^2(\tau_D) - \vartheta_2^2(\tau_D))^2}. \tag{B.17}$$

Writing $\tilde{q}_D = e^{i\pi\tau_D/2}$ we have the strong coupling expansions

$$\bar{u}(\tau_D) = \frac{\tilde{\Lambda}_3^2}{\tau_D^2}\left(1 + 8\tilde{q}_D + 40\tilde{q}_D^2 + 160\tilde{q}_D^3 + \cdots\right). \tag{B.18}$$

and, from (4.16),

$$u(\tau_D) = -\frac{\vartheta_3^2(\tau_D)\vartheta_2^2(\tau_D)}{(\vartheta_3^2(\tau_D) - \vartheta_2^2(\tau_D))^2} = -4\tilde{\Lambda}_3^2\tilde{q}_D\left(1 + 8\tilde{q}_D + 44\tilde{q}_D^2 + 192\tilde{q}_D^3 + \cdots\right). \tag{B.19}$$

For non-zero m it is consistent with all we know to assume a similar form

$$\bar{u}(\tau_D) = \frac{\tilde{\Lambda}_3^2}{\tau_D^2}\left(\bar{u}_0 + \bar{u}_1\tilde{q}_D + \bar{u}_2\tilde{q}_D^2 + \bar{u}_3\tilde{q}_D^3 + \cdots\right). \tag{B.20}$$

and similarly for $\bar{m}^2$

$$\bar{m}^2(\tau_D) = \frac{\tilde{\Lambda}_3^2}{\tau_D^4}\left(\bar{a}_0 + \bar{a}_1\tilde{q}_D + \bar{a}_2\tilde{q}_D^2 + \bar{a}_3\tilde{q}_D^3 + \cdots\right), \tag{B.21}$$

where $\bar{u}_k$ and $\bar{a}_k$ are functions of $m/\tilde{\Lambda}_3$, with $\bar{a}_k$ vanishing for $m = 0$. A similar strong coupling expansion for u has no prefactor of $1/\tau_D^2$ because u has weight zero not -2,

$$\frac{u}{\tilde{\Lambda}_3^2} = u_0 + u_1\tilde{q}_D + u_2\tilde{q}_D^2 + u_3\tilde{q}_D^3 + \cdots. \tag{B.22}$$

Using these expansions in (B.14) and (B.15), together with

$$c_1(\tau) = c_1(-1/\tau_D) = -\frac{\tau_D^2}{2}\left(\vartheta_3(\tau_D)^4 + \vartheta_2(\tau_D)^4\right),$$
$$c_2(\tau) = c_2(-1/\tau_D) = -\frac{\tau_D^2}{2}\vartheta_4(\tau_D)^4, \tag{B.23}$$

we can equate powers of $\tilde{q}_D$ to obtain recurrence relations between the u_k and the $\bar{a}_k$. Without making further assumptions there is not enough information to determine $u(\tau, m)$ but we can still extract useful information about the strong coupling β-function. At zeroth order in $\tilde{q}_D$ (B.14) and (B.15) (with $\tilde{\Lambda}_3 = 1$ for simplicity) yield three possibilities:

$$u_0 = \pm m, \bar{u}_0 = 1 \mp 2m - \frac{\bar{a}_0}{4}; \qquad \text{and} \qquad u_0 = m^2 + \frac{1}{4}, \bar{u}_0 = m^2 - \frac{\bar{a}_0 + 1}{4}. \quad \text{(B.24)}$$

Only the first two are relevant for the strong coupling fixed point, $\tau = 0$ at $u = 0$, of the massless $N_f = 3$ theory (in the massless case $u = 1/4$ is associated with $\tau = -1/2$ where $\tau_D = 2$). Choosing the root $u = m$, at order $\tilde{q}_D$ we find a pair of equations linear in u_1 and $\bar{u}_1$ which are degenerate. Solving for $\bar{u}_1$ gives

$$\bar{u}_1 := -\frac{(m-2)u_1}{(2m-1)} - \frac{\bar{a}_1}{4}. \quad \text{(B.25)}$$

At order $\tilde{q}_D^2$ the pair of linear equations for for u_2 and $\bar{u}_2$ are parallel in the $u_2 - \bar{u}_2$ plane and have no solution unless they co-incide, which only happens if

$$\frac{u_1 m}{2m - 1} = 0. \quad \text{(B.26)}$$

For $m = 0$ this is automatic, but for $m \neq 0$ it forces $u_1 = 0$. Setting $u_1 = 0$ then gives one linear equation relating $\bar{u}_2$ to u_2,

$$\bar{u}_2 = -\frac{2(m-2)u_2 + \bar{a}_0(3\bar{a}_0 + 8m - 4)}{2(2m-1)} - \frac{\bar{a}_2}{4} \quad \text{(B.27)}$$

At order $\tilde{q}_D^3$ the two linear equations for u_3 and $\bar{u}_3$ are again degenerate giving only one constraint which can be used to solve for $\bar{u}_3$

$$\bar{u}_3 = -\frac{(m-2)u_3 + \bar{a}_1(3\bar{a}_0 + 4m - 2)}{(2m-1)} - \frac{\bar{a}_3}{4}. \quad \text{(B.28)}$$

At order $\tilde{q}_D^4$ one again obtains two parallel lines in the $u_4 - \bar{u}_4$ plane which do not intersect unless

$$u_2 = \pm\frac{(\bar{a}_0 + 8m - 4)^2}{2m} \quad \text{(B.29)}$$

in which case $\bar{u}_4$ can be obtained as a function of u_4, $\bar{a}_0$, $\bar{a}_2$ and m.

One can continue but for the present purposes we have gone as far as necessary. We are only really interested in u and we have

$$u(\tau_D) = m \pm \frac{(\bar{a}_0 + 8m - 4)^2}{2m}\tilde{q}_D^2 + \cdots \quad \text{(B.30)}$$

with $\bar{a}_0$ and undetermined function of m, but independent of τ_D. To get the dimensions correct we should re-instate $\tilde{\Lambda}_3$ and write

$$u(\tau_D) = m\tilde{\Lambda}_3 \left(1 + \alpha(m)e^{i\pi\tau_D} + \cdots\right) \quad \text{(B.31)}$$

where

$$\alpha(m) := \pm \frac{(\bar{a}_0 \tilde{\Lambda}_3^2 + 8m\tilde{\Lambda}_3 - 4\tilde{\Lambda}_3^2)^2}{2m^2 \tilde{\Lambda}_3^2}. \tag{B.32}$$

Note that, in order to pin down the co-efficient u_2 one has to go to order $\tilde{q}_D^4$ and examine u_4. At every value of k in the expansion one gets a pair of linear equations in u_k and $\bar{u}_k$ in terms of m and the $\bar{a}_{k'}$ with $k' \leq k$. For $k = 1$ these equations are degenerate and u_1 is not determined; for $k = 2$ the equations have no solution unless $u_1 = 0$ in which case they are again degenerate and u_2 is undetermined; for $k = 3$ the equations are again degenerate and u_3 is undetermined and for $k = 4$ the equations have no solution unless u_2 has one of the two possible values shown in (B.31).

The explicit from of $\alpha(m)$ is not needed in the analysis, but we can fix its asymptotic form as $m \to \infty$ using holomorphic decoupling [9]. For $N_f = 2$ equation (3.9) gives

$$u(\tau_D) = \frac{\Lambda_2^2}{8} \frac{\vartheta_3^4(\tau_D)}{\vartheta_4^4(\tau_D)} \approx \frac{\Lambda_2^2}{8} \left(1 + 16e^{i\pi\tau_D} + \cdots\right), \tag{B.33}$$

and this should agree with (B.31) as $m \to \infty$, $\Lambda_3 \to 0$ with $\Lambda_2^2 = m\Lambda_3 = 8m\tilde{\Lambda}_3$ fixed. Hence $\alpha(m) \to 16$ as $m \to \infty$.

The other singularity of the $N_f = 2$ theory, at $u = -\frac{\Lambda_2^2}{8}$, is obtained by holomorphic decoupling in the weak coupling limit of the $N_f = 3$ theory by expanding around $u = -m$.

Notice that the $m = 0$ expansion for u in equation (B.19) contains a term linear in $\tilde{q}_D$ while (B.31) does not. This is because, when $m \neq 0$, (B.26) forces us to set $u_1 = 0$. Since $\bar{a}_0 \to 0$ as $m \to 0$, $\alpha(m)$ diverges like $8\tilde{\Lambda}_3^2/m^2$ as $m \to 0$ and the limits $\tau_D \to i\infty$ and $m \to 0$ do not commute.

We can perform a similar expansion around $\tau = -1/2$, using $u_0 = m^2 + 1/4$, where $\tau_D = 2 + i\varepsilon$ with ε small. The analysis is simpler than the $u_0 = \pm m$ case in that at each order, at least up to order 5 in $\tilde{q}_D$ which is as far as we have gone, one simply finds a pair of linear equations in u_k and $\bar{u}_k$ which can be solved in terms of $\bar{a}_{k'}$ with $k' < k$. The details are omitted but one finds

$$u_1 = u_2 = u_3 = 0, \qquad u_4 = -4\frac{(\bar{a}_0 - 4m^2 + 1)^4}{(4m^2 - 1)^2}, \qquad u_5 = -16\frac{\bar{a}_1(\bar{a}_0 - 4m^2 + 1)^3}{(4m^2 - 1)^2}, \tag{B.34}$$

and so, for $\tau_D = 2 + i\varepsilon$ with ε small,

$$u(\tau_D) = m^2 + \frac{1}{4} - 4\frac{(\bar{a}_0 - 4m^2 + 1)^4}{(4m^2 - 1)^2}e^{-2\pi\varepsilon} + 16\frac{\bar{a}_1(\bar{a}_0 - 4m^2 + 1)^3}{(4m^2 - 1)^2}e^{-5\pi\varepsilon/2} + \cdots. \tag{B.35}$$

Re-instating $\tilde{\Lambda}_3$ now gives

$$u(\tau_D) = m^2 + \frac{\tilde{\Lambda}_3^2}{4} + \tilde{\Lambda}_3^2 \tilde{\alpha}(m) e^{-2\pi\varepsilon} \cdots. \tag{B.36}$$

where

$$\tilde{\alpha}(m) = -4\frac{(\bar{a}_0\tilde{\Lambda}_3^2 - 4m^2 + \tilde{\Lambda}_3^2)^4}{(4m^2 - \tilde{\Lambda}_3^2)^2\tilde{\Lambda}_3^4}. \tag{B.37}$$

As $m \to \infty$ this point in the u-plane goes out to infinity in the $N_f = 2$ theory.

Equations (B.31) and (B.36) were the aim of this appendix and are used in §6 in the discussion of the β-functions for the massive $N_f = 3$ theory at strong coupling.

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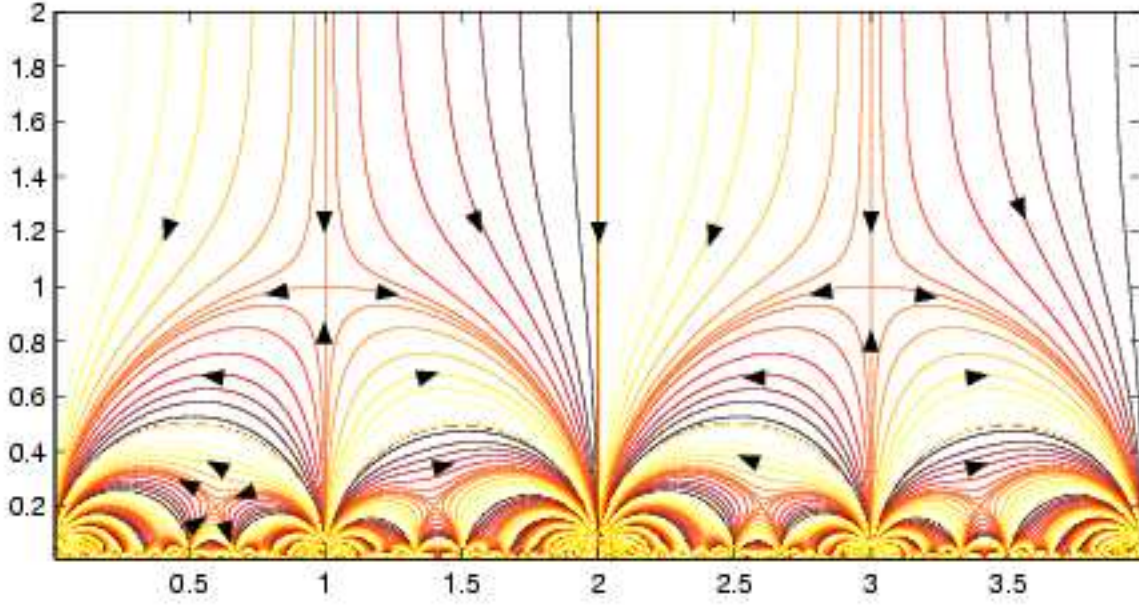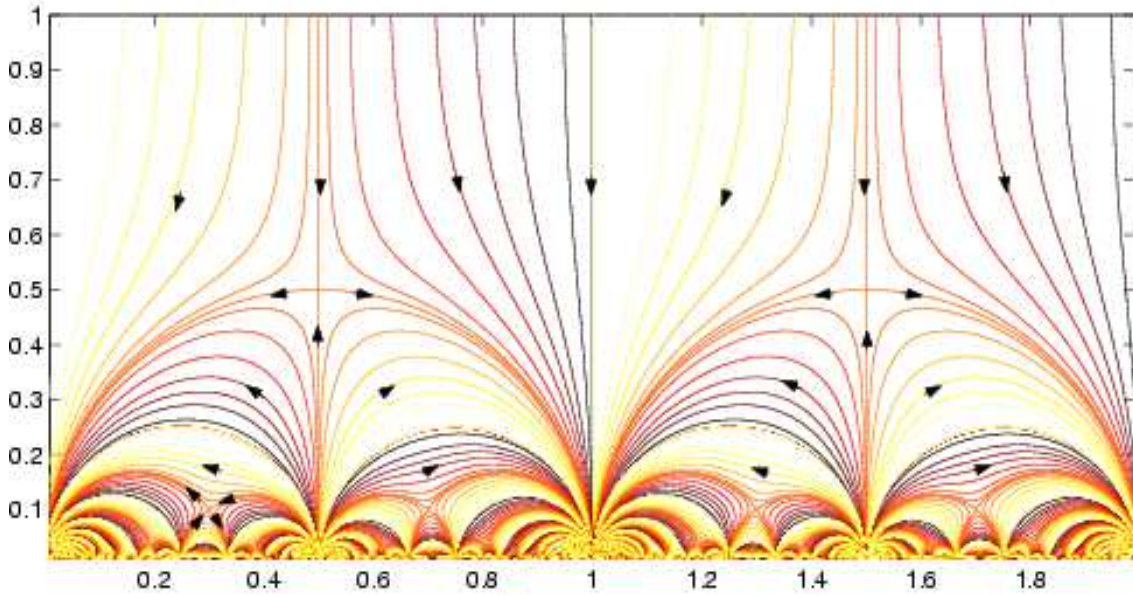[15] E.T. Whittaker G.N. and Watson, *A Course of Modern Analysis*, (1927) (CUP), 4th Edition

Fig. 1: Flow of effective coupling of $N = 2$ SUSY Yang-Mills with $N_f = 0$. The arrows indicate the direction of the flow as as the Higgs VEV is reduced. The β-functions are modular forms of $\Gamma^0(2)$ and the pattern repeats under $\tau \to \tau + 2$.

Fig. 2: Flow of effective coupling of $N = 2$ SUSY Yang-Mills with $N_f = 2$. The arrows indicate the direction of the flow as as the Higgs VEV is reduced. The β-functions are modular forms of $\Gamma_0(2)$ and the pattern repeats under $\tau \to \tau + 1$.
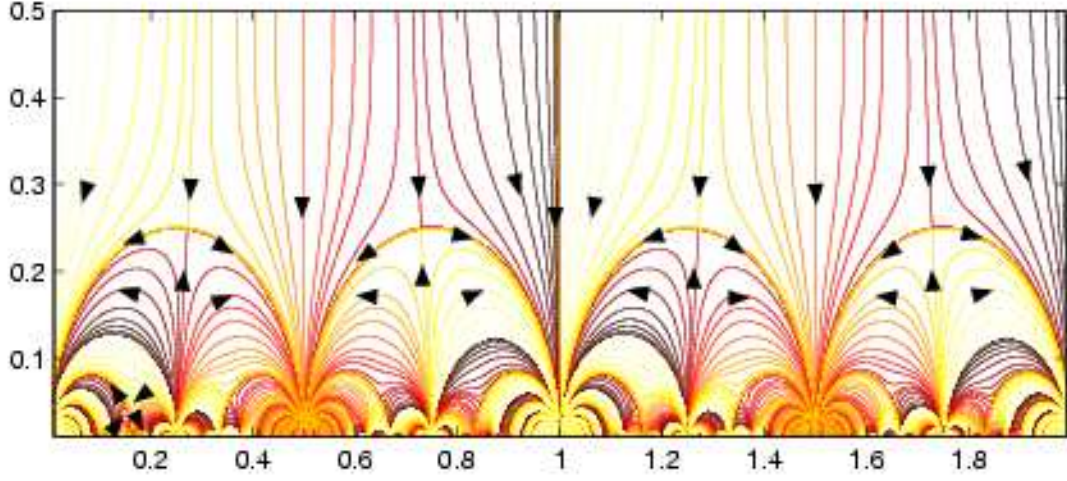
Fig. 3: Flow of effective coupling of $N = 2$ SUSY Yang-Mills with $N_f = 3$ and $\epsilon = 4/27$. The arrows indicate the direction of the flow as as the Higgs VEV is reduced. The β-functions are modular forms of $\Gamma_0(4)$ and the pattern repeats under $\tau \to \tau + 1$.
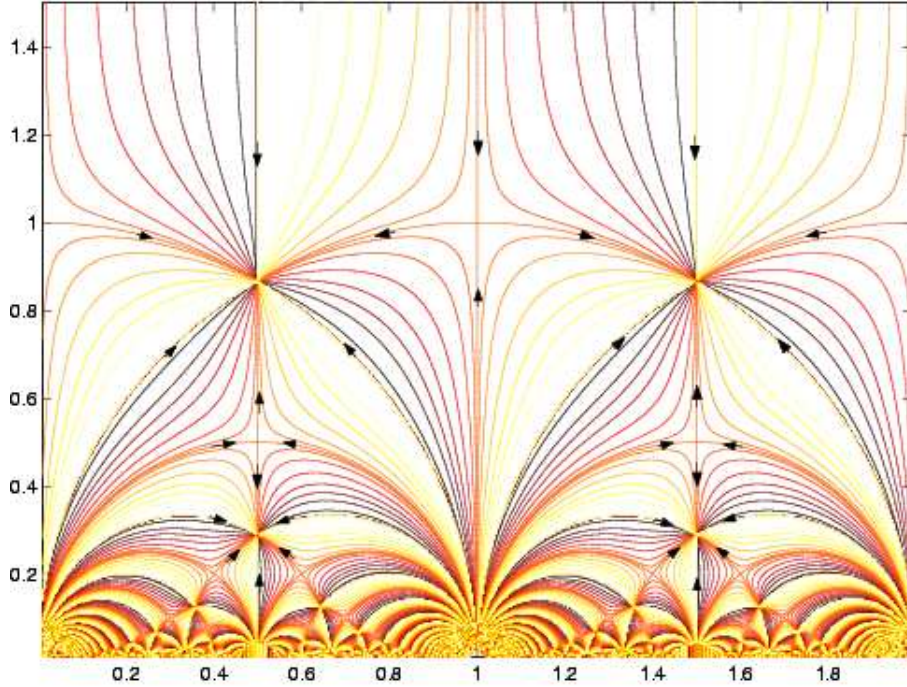

Fig. 4: Flow of effective coupling of $N = 2$ SUSY Yang-Mills with $N_f = 1$. The arrows indicate the direction of the flow as as the Higgs VEV is reduced. The β-functions are modular forms of $\Gamma(1)$ and the pattern repeats under $\tau \to \tau + 1$.