STRONG DOUBLING CONDITIONS

Stephen M. Buckley

Abstract. We show that the class of strong doubling measures depends essentially on the parameter t, and that the measure of the boundary layer of a QHBC domain decays geometrically, if the measure is suitably strong doubling.

0. Introduction

Various areas of analysis utilize doubling measures, i.e. positive Borel measures on $\mathbb{R}^n$ satisfying (some variation of) the condition:

$$\mu(B(x,r)) \leq C\mu(B(x,r/2)), \qquad \text{for all } x \in \mathbb{R}^n,\ r > 0. \tag{0.1}$$

For instance, Chapter I of [S2] investigates many questions in harmonic analysis within a general framework involving a measure that satisfies a doubling condition relative to a set of generalized balls in $\mathbb{R}^n$, and [HKM] develops the potential theory of a certain class of degenerate elliptic partial differential equations that involve admissable weights, where a weight w is admissable if the measure $w\,dx$ satisfies certain conditions including (0.1).

Much of this analysis takes place on an open subset Ω of $\mathbb{R}^n$, rather than on all of $\mathbb{R}^n$ (for instance, this is often the case for PDE-related analysis). Some such results require only a local doubling condition for balls $B(x,2r) \subset \Omega$, for instance, but often a stronger form of doubling is required. It is then quite common to assume that the measure is defined on all of $\mathbb{R}^n$ and satisfies (0.1); this, for example, is the approach adopted in [HKM] for the definition of an admissable weight. However, there exist rather nice measures defined on an open set Ω which are not restrictions of global doubling measures, e.g. *power-weight measures* $d\mu = \delta_\Omega^a\,dx$ for certain domains Ω, where $\delta_\Omega(x)$ is the distance from x to $\partial\Omega$, and $a > 0$. The author wishes to thank Paul MacManus for kindly providing an explicit example of this type (given at the end of Section 1).

One doubling condition applicable to measures on Ω is the *boundary doubling condition*:

$$\mu(B(x,r)\cap\Omega) \leq C\mu(B(x,r/2)\cap\Omega), \qquad \text{for all } x \in \Omega,\ r > 0. \tag{0.2}$$

1991 *Mathematics Subject Classification.* Primary 28A75; Secondary 42B25.

Key words and phrases. Doubling condition, QHBC domain, maximal functions, covering lemmas.

The author was partially supported by Forbairt.

This condition, however, places restrictions on Ω as well as on μ, since even Lebesgue measure does not always satisfy (0.2) (see the proof of Theorem 1.1). The concept of a *strong doubling measure*, employed in [BKL] and [BO] to prove inequalities of Poincaré and Trudinger type, is an attractive intermediate option; there are actually a family of such strong doubling conditions indexed by a parameter $1 < t < \infty$ (see Section 1). These conditions are strong enough to do some non-local analysis, but weaker than boundary doubling. Additionally, they are all satisfied by the measure $\delta_\Omega^a dx$, $a \geq 0$, no matter how bad the geometry of the domain Ω.

In Section 1, we determine how strong doubling conditions relate to each other and to other doubling conditions; in particular, we show that all strong doubling conditions are different, since there exist measures which are strong doubling for all parameters less than t, but not for parameter t. In Section 2, we prove that if a measure is appropriately strong doubling on a QHBC domain Ω, then the measure of the part of Ω lying within a distance ϵ of $\partial\Omega$ is dominated by a power of ϵ. This result, which generalizes a result of Smith and Stegenga on the Minkowski dimension of $\partial\Omega$, has been used in [BO, Theorem 3.10] to prove a theorem on Trudinger-type inequalities.

1. Various doubling conditions

Throughout this paper, Ω is a proper open subset of $\mathbb{R}^n$, which we may further restrict as necessary. If $B = B(x,r)$ is a ball, and $t > 0$, we write $tB = B(x,tr)$ (and so $t^{-1}B = B(x,r/t)$). We also write $\delta_\Omega(x) \equiv \operatorname{dist}(x,\partial\Omega)$, $x \in \Omega$, and define the *quasihyperbolic length* of a rectifiable path $\gamma \subset \Omega$ to be

$$k_\Omega(\gamma) = \int_\gamma \delta_\Omega(x)^{-1} ds.$$

The *quasihyperbolic distance* between $x, y \in \Omega$, $k_\Omega(x,y)$, is then defined to be the infimum of $k_\Omega(\gamma)$, as γ ranges over all paths linking x and y. There exists a *quasihyperbolic geodesic* between any pair of points $x, y \in \Omega$, i.e. a path $\gamma_{x,y}$ such that $k_\Omega(x,y) = k_\Omega(\gamma_{x,y})$; see [GO].

A (necessarily bounded) domain Ω satisfies a *quasihyperbolic boundary condition* (more briefly, Ω is *QHBC*) with respect to its *QHBC center* $x_0 \in \Omega$ if there exists a constant $C \geq 1$ such that for all $x \in \Omega$,

$$k_\Omega(x,x_0) \leq C \log\left(\frac{C}{\delta(x)}\right).$$

The *QHBC path for* x is the quasihyperbolic geodesic for x, x_0, and the *QHBC constant of* Ω, denoted C_Ω, is the smallest value of C for which the above inequality is valid.

We say that a bounded domain Ω is a *John domain* with respect to its *John center* $x_0 \in \Omega$ if there exists a constant $K \geq 1$ such that for all $x \in \Omega$, there is a path $\gamma = \gamma_x : [0,l] \to \Omega$ parametrized by arclength satisfying $\gamma(0) = x$, $\gamma(l) = x_0$, and $\delta(\gamma(t)) \geq t/K$. We call γ_x the *John path for* x, and we define K_Ω, the *John constant of* Ω, to be the smallest value of K for which the above inequality is valid.

Clearly every John domain is a QHBC domain, but it is not difficult to construct examples of non-John QHBC domains (e.g. see [BO, Section 5]). Note that the choice of center point $x_0 \in \Omega$ in the definitions of John and QHBC domains is unimportant, in the sense that if Ω is John (or QHBC) with respect to one point, it is John (or QHBC) with respect to all of its points (of course, the John/QHBC constant tends to infinity as we let x_0 approach $\partial\Omega$).

Suppose that $0 < t \leq \infty$ and that μ is a positive Borel measure on Ω. We say that μ is *t-doubling on* Ω, denoted $\mu \in D_t(\Omega)$, if there exists a constant C such that

$$\mu(B \cap \Omega) \leq C\mu(2^{-1}B \cap \Omega) < \infty$$

whenever B is a ball for which $t^{-1}B \subset \Omega$ (in the case $t = \infty$, we merely require the center of B to lie in Ω, or equivalently in $\overline{\Omega}$). We denote by $C_{\mu,t}$ the smallest such constant C for which this doubling condition is true ($0 < t \leq \infty$).

Note that the t-doubling condition imposes restrictions on the boundary behaviour of the measure precisely when $t \geq 1$. We say that a t-doubling measure μ is a *locally doubling* if $t < 1$, *strong doubling* if $t > 1$, and *boundary doubling* if $t = \infty$. Obviously, strong doubling is logically stronger than local doubling but weaker than boundary doubling. In fact, it is not difficult to construct examples of a measure that is local doubling but not strong doubling, or strong doubling but not boundary doubling. Whether or not strong doubling depends on the parameter $t \in (1, \infty)$ is a more difficult question which we now answer.

Theorem 1.1. *Suppose* $0 < t < t' \leq \infty$. *If* $t' \geq 1$, $D_{t'}(\Omega) \setminus D_t(\Omega)$ *is non-empty for some QHBC domain* $\Omega \subset \mathbb{R}^n$. *If* $t' < 1$, *then* $D_t(\Omega) = D_{t'}(\Omega)$ *for every proper open set* Ω.

Before proving this theorem, we first state a simple but useful lemma.

Lemma 1.2. *A sphere* $S \subset \mathbb{R}^n$ *of radius* $a > 0$ *can be covered by balls* $\{B_i\}_{i=1}^m$, *centered on* S *and of radius* $b \in (0, a)$, *for some* m *dependent only on* n *and* b/a.

Proof. We choose a sequence of disjoint balls $B'_1, B'_2, \ldots$, centered on S and of radius $b/3$ as long as we can continue to do so; this process must halt in a bounded number of steps since each ball covers a fixed fraction (dependent only on $b/a, n$) of the surface measure of S. If the resulting balls are $B'_1, \ldots, B'_m$, then the required balls are $B_i = 3B'_i$, $1 \leq i \leq m$. $\square$

Proof of Theorem 1.1. The equivalence of all local doubling conditions is intuitively rather obvious, but we prove it for completeness. Assume that $\mu \in D_t(\Omega)$ for some $t < 1$, and so $\mu(B) \leq C\mu(2^{-1}B)$ whenever $B = B(x, r)$, $0 < r \leq t\delta_\Omega(x)$. We fix such a ball $B(x, r)$ with $r = t\delta_\Omega$, and write $c = (2-2t)/(2-t)$. Applying Lemma 1.2 with $a = (1 - c/2)r$, $b = cr/4$, to the sphere $S = \{y : |x - y| = a\}$, we get balls

$B_1, \ldots, B_m$ covering S, where m is bounded by some number dependent only on n and t. Our choice of parameters ensures that

$$\begin{aligned} 2B_i &\subset B(x,r), \\ 4B_i &\supset B(x,(1+(c/4))r) \setminus B(x,r), \\ 4t^{-1}B_i &\subset \Omega. \end{aligned}$$

We deduce that $\mu \in D_{f(t)}$, where $f(t) = (5t - 3t^2)/(4-2t) = (1+c/4)t$. Defining $t_0 = t$ and $t_k = f(t_{k+1})$ for all $k > 0$, it follows iteratively that $\mu \in D_{t_k}$ for every $k > 0$. Note that $ct/4 < 1 - t$ and so the sequence (t_k) is increasing and bounded by 1. Since f is continuous on $(0,2)$ and 1 is the only fixed point there, we deduce that t_k tends to 1 as $k \to \infty$. Thus $\mu \in D_{t'}$ for all $t' < 1$, as required.

Letting Ω be the unit ball in $\mathbb{R}^n$, it is easy to find $\mu \in D_t(\Omega) \setminus D_1(\Omega)$ whenever $t < 1$. For example, $d\mu = (1-|x|)^{-1}\,dx$ is such a measure. Alternatively, we could take $d\mu = [\log(2/(1-|x|))]^{-2}(1-|x|)^{-1}\,dx$; in this latter case, $\mu(\Omega) < \infty$.

In the remaining cases, we give only planar counterexamples to equality; these are easily modified to give counterexamples in any larger dimension. The domains we use will consist of a central square with small narrow pieces attached. It is convenient for us to take as our central square

$$Q_0 = \{(x,y) \in \mathbb{R}^2 \,:\, -1 < x < 0, 0 < y < 1\}.$$

We first consider the case $t' = \infty$ (even though it follows from the case $t' < \infty$), because we can produce a counterexample here with μ equal to Lebesgue measure. Note first that Lebesgue measure lies in $D_t(\Omega)$ for all $t < \infty$, regardless of the domain Ω. We define Ω to be the union of Q_0 and the rectangles

$$R_k = \{(x,y) \in \mathbb{R}^2 \,:\, 0 \le x < 2^{-k}, 1 - 2^{-k}(1+1/k) < y < 1 - 2^{-k}\}.$$

Then $|\cdot|$ is not boundary doubling because $|2^{-1}B_k \cap \Omega| \approx |B_k \cap \Omega|/k$, where B_k is the ball whose center is the same as the center of R_k and whose radius is 2^{-k}. The only possible obstacle to Ω being QHBC is the narrowness of the rectangles R_k. Since the length-to-width ratio of R_k, i.e. k, is dominated by the logarithm of the reciprocal of R_k's diameter, this is not a genuine obstacle, and it is easy to check that Ω is QHBC.

To prove the remaining cases, it suffices to find, for all $1 < t < t' < \infty$, a QHBC domain Ω and a measure μ such that $\mu \in D_t(\Omega) \setminus D_{t'}(\Omega)$. By elementary geometry, we note that if B is a ball inscribed in the cone $K_a = \{(x,y) \,:\, |y| < ax\}$, $a > 0$, then the dilate sB, $s > 1$, contains the vertex of K_a if and only if $a > f(s) = [s^2-1]^{-1/2}$. We shall define Ω to be a union of Q_0 and a sequence of diamond-shaped sets S_k. First, we define the preliminary diamond-shaped sets

$$S'_k = \{(x,y) \in \mathbb{R}^2 \,:\, 0 \le |x| < x_k, |y| < f(t')|x - x_k|\},$$

where $x_k = 2^{-k}c$, and $c = \min\{1, [4f(t')]^{-1}\}$. We then write $y_k = 1 - 2^{-k}$ and define S_k to be the translate of S'_k by the vector $(x_k - x_k^2, y_k)$. Note that the sets S_k have a small overlap with Q_0 but are disjoint from each other. The sets S_k are of a fixed length-to-width ratio, but there is a new potential obstacle to Ω being a QHBC domain: each S_k is attached to Q_0 by a narrow neck whose width is proportional to x_k^2. However, it is a routine exercise to check that if "satellite pieces" (such as S_k) are adjoined to the main part of the domain via bottlenecks of width proportional to a fixed power of the length of the satellite, then this does not destroy the QHBC condition. Consequently, Ω is QHBC.

Let us denote by U_k and V_k the vertices $(-x_k^2, y_k)$ and $(2x_k - x_k^2, y_k)$, respectively, of S_k. Defining

$$\begin{aligned} g_k(x) &= x_k - |x - x_k + x_k^2|, \quad x \in \mathbb{R}, \\ w_s(x,y) &= [x_k^2/g_k(x)]^s, \quad (x,y) \in S_k, \\ d\mu_s &= w_s(x,y)\, dx\, dy, \quad (x,y) \in S_k, \end{aligned}$$

we see that $\mu_s(S_k) < \infty$ for $0 < s < 2$, but not for $s = 2$. Furthermore, as $s \to 2^-$, more and more of the μ_s-mass of S_k is concentrated closer and closer to U_k and V_k. More precisely,

$$\lim_{s \to 2^-} \frac{\mu_s(\{X \in S_k \,:\, \min(|X - U_k|, |X - V_k|) < (2-s)x_k\})}{\mu_s(S_k)} = 1.$$

By a routine calculation, this last limit reduces to the fact that $\lim_{t \to 0^+} t^t = 1$.

We are now ready to define a measure $\mu \in D_t(\Omega) \setminus D_{t'}(\Omega)$. Specifically, we take $d\mu \equiv w(x,y)\, dxdy$, where

$$w(x,y) = \begin{cases} 1, & (x,y) \in Q_0, \\ [x_k^2/g_k(x)]^{2-2/k}, & (x,y) \in S_k \setminus Q_0. \end{cases}$$

Note that w is continuous across the necks of the sets S_k (i.e. at $x = 0$) and, by the above considerations, most of the μ-measure of S_k is concentrated very near V_k if k is large. Considering balls inscribed in S_k near this vertex, we deduce that any t'-doubling condition is violated for sufficiently large k. By contrast, μ is t-doubling for all $0 < t < t'$. To see this, note that balls centered in S_k satisfy a t-doubling condition (because their t^{-1}-dilates stay away from V_k), and that balls centered in Q_0 actually satisfy an ∞-doubling condition (because the average value of w on S_k is bounded, as can easily be checked). □

In the above proof, we chose Ω to be the unit ball when defining a locally doubling measure on Ω which is not 1-doubling. By contrast, the fact that, for $1 < t \le \infty$, the D_t-conditions are all distinct, made use of a domain which, although QHBC, was nevertheless rather nasty. We now show that such nastiness is in fact unavoidable.

Proposition 1.3. *If Ω is a John domain, then $D_t(\Omega) = D_\infty(\Omega)$ for all $t \geq t_0$, where t_0 depends only on K_Ω, the John constant of Ω.*

Proof. Let x_0 be the John center of Ω. We fix a ball $B = B(x,r)$, $x \in \Omega$. If $x_0 \in B$, then either $B \subset \Omega$, or B contains a ball of radius $\delta_\Omega(x_0)/2$. In both cases, the required estimate
$$\mu(B \cap \Omega) \leq C\mu(2^{-1}B \cap \Omega) < \infty$$
follows easily from the assumption that $\mu \in D_t(\Omega)$ for sufficiently large $t = t(K_\Omega)$. Thus we may assume that $x_0 \notin B$. We choose any point y on the John path for x with respect to x_0 which lies in the annulus $B(x,r/3) \setminus B(x,r/6)$. The John condition ensures that $B' = B(y,r') \subset \Omega$ where $r' = r/6K_\Omega$. Since $B' \subset 2^{-1}B$ and $8K_\Omega B' \supset B$, it follows that $\mu(B \cap \Omega) \leq C\mu(2^{-1}B \cap \Omega)$ if $\mu \in D_{8K_\Omega}(\Omega)$. $\square$

We end this section by giving an example, essentially due to Paul MacManus, of a strong doubling measure which is not the restriction of a global doubling measure. Let us fix $s > 0$ and define $d\mu_\Omega = \delta_\Omega^s \, dx$ for any proper non-empty open subset Ω of $\mathbb{R}^n$. Note that $\mu_\Omega \in D_t(\Omega)$ for every proper open subset Ω of $\mathbb{R}^n$, with doubling constant dependent only on s, n, and t. We define Ω_k to consist of the interval $(0,2)$ with the points i/k removed, $1 \leq i \leq k$. Suppose that μ_{Ω_k} is a restriction of a global doubling measure μ_k. Since the measure of a countable set is zero (for any global doubling measure), $\mu_k(1,2)/\mu_k(0,1) \to \infty$ as $k \to \infty$. By piecing together sets like Ω_k, it is thus easy to define a set Ω such that μ_Ω is not the restriction of a global doubling measure. We could for instance take Ω to be the bounded open set given by
$$\Omega = \{2^{-k-1}x + 1 - 2^{-k+1} \,:\, x \in \Omega_k, k \in \mathbb{N}\}.$$
One can even define a domain $D \subset \mathbb{R}^n$, $n > 1$, such that $\delta_D^s \, dx$ is strong doubling but not the restriction of a global doubling measure. For instance, if Ω is as above, then $D = \Omega \times (0,1) \cup (-1,0] \times (0,1)$ is one such domain. Note that here we need the rather well-known fact that line segments are null sets for all doubling measures on $\mathbb{R}^2$; this fact is, for example, an easy corollary of Theorem 2.4).

2. Geometric decay of the measure of a QHBC boundary layer

In this section, we shall prove that the measure of the boundary layer of a QHBC domain decays like a power of its thickness if the measure is appropriately strong doubling. We begin, though, with some preliminary definitions and lemmas. If p is an exponent and S is a set, we write $p' = p/(p-1)$, and χ_S for the characteristic function of S. If Ω is a bounded domain, we denote by $\operatorname{diam}(\Omega)$ and $\operatorname{inrad}(\Omega)$ its diameter and inradius (the latter being the radius of the largest ball that fits inside Ω). If $t > 0$ and $f \in L^1_{\text{loc}}(\Omega)$, we define the maximal function
$$M_t f(x) \equiv M_{t;\Omega,\mu} f(x) = \sup_{x \in B \subset \Omega} \frac{1}{\mu(tB)} \int_{tB \cap \Omega} |f| \, d\mu,$$
where the supremum is taken over all balls B satisfying the indicated conditions.

Our first lemma is both a generalization of the well-known Besicovitch Covering Theorem, and a special case of a theorem of Morse [M] (also stated in [G]), and consequently needs no proof.

Lemma 2.1. *Suppose that $0 < s < 1$, that $A \subset \mathbb{R}^n$, and that $\mathcal{F}$ is a family of balls of bounded radius. If for every $x \in A$, $\mathcal{F}$ contains a ball B_x of radius at most R such that $x \in sB_x$, then there exist subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_k \subset \mathcal{F}$ such that each $\mathcal{F}_i$ is a pairwise disjoint collection of balls, $\bigcup_{i=1}^k \mathcal{F}_i$ covers A, and $k \le N$ for some N dependent only on n and s.*

The following lemma belongs to the large family of results that state that various maximal operators are bounded on L^p, $1 < p \le \infty$.

Lemma 2.2. *If Ω is a bounded domain in $\mathbb{R}^n$, and μ is a positive Borel measure on $\mathbb{R}^n$, then $M_{t;\Omega,\mu}$ is bounded on $L^p(\Omega,\mu)$ for all $1 < p \le \infty$, $1 < t$. Furthermore, its operator norm is bounded by Cp', for some constant C dependent only on n and t.*

Note that we do not assume that μ satisfies any doubling assumption. If we assumed that $\mu \in D_{5t}(\Omega)$, then the alternative "5-covering lemma" (see e.g. [S1, Section 1.1]) could be used in place of Lemma 2.1 in the following proof sketch; additionally, the lemma would be true for all $t > 0$, and not just $t > 1$.

Sketch of proof of Lemma 2.2. As usual for results of this type, the proof consists of an interpolation between the (obvious) boundedness of M_t on $L^\infty(\Omega,\mu)$, and its boundedness from $L^1(\Omega,\mu)$ to the Lorentz (or "weak-type") space $L^{1,\infty}(\Omega,\mu)$. Such weak-type boundedness results are always proved by means of a covering theorem (see, for example, [S2, Section I.3.1]). Here, we take $f \in L^1(\Omega,\mu)$, fix a cut-off value $\alpha > 0$ and, for each x such that $A = \{x : M_t f(x) > \alpha\}$, we associate a ball B'_x such that $x \in B'_x \subset \Omega$, and such that the μ-average of $|f|$ on $B_x \equiv tB'_x$ exceeds α. By applying Lemma 2.1 with $s = 1/t$ to the family $\{B_x : x \in A\}$, weak boundedness follows in the usual manner. $\square$

The next lemma is also a variant of a rather well-known lemma (e.g. see [Bo]); we include a proof for completeness. In its proof and later, we use $A \lesssim B$ if $A \le CB$ for some constant C dependent only on allowed parameters. In particular, we stress that C is not allowed to depend on p in this lemma.

Lemma 2.3. *Suppose that $1 \le p < \infty$, $1 < t$, $\Omega \subset \mathbb{R}^n$, and $\mu \in D_t(\Omega)$. Let $\mathcal{F}$ be a family of balls contained in Ω, and let a_B be a non-negative number for each $B \in \mathcal{F}$. Then*

$$\|\sum_{B\in\mathcal{F}} a_B \chi_{tB}\|_{L^p(\Omega,\mu)} \le Cp \|\sum_{B\in\mathcal{F}} a_B \chi_B\|_{L^p(\Omega,\mu)}.$$

where C depends only on n, t, and $C_{\mu,t}$.

Proof. Let g be a non-negative function in $L^{p'}(\Omega,\mu)$. Since μ is t-doubling,

$$A \equiv \int_\Omega \left(\sum_{B\in\mathcal{F}} a_B \chi_{tB}\right) g\, d\mu \lesssim \sum_{B\in\mathcal{F}} a_B \left[\frac{1}{\mu(tB)} \int_{tB\cap\Omega} g\, d\mu\right] \cdot \mu(B).$$

We now use the fact that the bracketed quantity is dominated by $M_t g(x)$ for every $x \in B$, together with Hölder's inequality and Lemma 2.2, to get

$$\begin{aligned} A &\lesssim \sum_{B\in\mathcal{F}} a_B \int_B M_t g \, d\mu = \int_\Omega M_t g \cdot \sum_{B\in\mathcal{F}} a_B \chi_B \, d\mu \\ &\le \|M_t g\|_{L^{p'}(\Omega,\mu)} \cdot \| \sum_{B\in\mathcal{F}} a_B \chi_B \|_{L^p(\Omega,\mu)} \\ &\lesssim \|g\|_{L^{p'}(\Omega,\mu)} \cdot \| \sum_{B\in\mathcal{F}} a_B \chi_B \|_{L^p(\Omega,\mu)} \end{aligned}$$

Taking a supremum over all $g \ge 0$ in the unit ball of $L^{p'}(\Omega, \mu)$, the required result follows by duality. □

In [SS], Smith and Stegenga prove that if $\Omega \subset \mathbb{R}^n$ is a QHBC domain, then the Minkowski dimension d of $\partial\Omega$ is bounded away from n, i.e. the Lebesgue measure of the "boundary layer" decays geometrically; for more on Minkowski content and the decay of the Lebesgue measure of boundary layers of sets, we refer the reader to [MV]. In the planar simply-connected case, Smith and Stegenga's result follows, with a sharp estimate of d, from the results in [JM]. Koskela and Rohde [KR], reproved Smith and Stegenga's result, in the process getting the sharp estimate of d in all dimensions. The next theorem generalizes this boundary layer decay to the setting of strong doubling measures; our proof is based on the method of [KR].

Theorem 2.4. *Suppose that Ω is QHBC and that $\mu \in D_t(\Omega)$, for some $t > t_0$, where $t_0 \in (1, \infty)$ is dependent only on n and C_Ω. Then there exist $C, \alpha > 0$ dependent only on n, C_Ω, and $C_{\mu,t}$, such that*

$$\mu(\Omega_r) \le C(r/\operatorname{diam}(\Omega))^\alpha \mu(\Omega) < \infty, \quad \textit{for all } r > 0.$$

In the above statement, recall that C_Ω is the QHBC constant of Ω and $C_{\mu,t}$ is the t-doubling constant of μ. The QHBC condition is necessary in the above theorem—just take μ to be Lebesgue measure, and $\Omega \subset \mathbb{R}^n$ to be any domain whose boundary has Minkowski dimension n. It is also necessary to assume a $D_t(\Omega)$ condition for sufficiently large t. For instance, if $\Omega \subset \mathbb{R}^2$ consists of all points in the unit disk whose argument is at most $\theta \in (0, \pi)$, the measure $d\mu(x) = (|x| \log^2(2/x))^{-1} dx$ does not satisfy the conclusion of the theorem even though $\mu \in D_t(\Omega)$ for $t < \sec^{-1}\theta$ (note that both $\sec^{-1}\theta$ and C_Ω tend to infinity as $\theta \to 0$).

Proof of Theorem 2.4. Assuming $t \ge t_1 \equiv \operatorname{diam}(\Omega)/\operatorname{inrad}(\Omega)$, the doubling condition ensures that $\mu(\Omega) < \infty$; note also that t_1 is bounded above by a constant dependent only on C_Ω. Without loss of generality, we normalize Ω so that $\operatorname{diam}(\Omega) = 1$, and μ so that $\mu(\Omega) = 1$. Let $\epsilon = 1/C_\Omega$ and $c = 1/10$. For each $x \in \partial\Omega$, and $n > 0$, we define

$$A_n(x) = \{y \in \mathbb{R}^n \, : \, (1+\epsilon)^{-n} < |x-y| < (1+\epsilon)^{-n+1}\},$$
$$\chi_n(x) = \begin{cases} 1, & \text{if } \exists\, y \in \Omega \cap A_n(x) : \, d(y,\partial\Omega) > c\epsilon|x-y| \\ 0, & \text{otherwise,} \end{cases}$$
$$\sigma_n(x) = \sum_{k=1}^{n} \chi_k(x).$$

Koskela and Rohde [KR] prove that the boundary of a QHBC domain is what they term an *ϵ-mean porous set* (with auxiliary constant $c = 1/10$, as here). This means that there exists a number n_0, depending only on C_Ω, such that $\sigma_n(x) > n/2$ for all $n \geq n_0$ (actually, the mean porosity of a set only implies the existence of certain holes in its complement, but an examination of the proof of Theorem 5.1 in [KR] reveals that one can assume that these holes are contained in the domain itself, as we do here).

It follows, as in Theorem 2.1 of [KR], that we can find a collection $\mathcal{F}$ of pairwise disjoint open balls and constants $t_2 > 1$, $j_0 \geq 1$, $c' > 0$, all dependent only on n and C_Ω, such that

$$\sum_{B\in\mathcal{F}} \chi_{t_2 B}(x) > c'j, \qquad x \in \Omega_{2^{-j}},\, j \geq j_0.$$

Note that in [KR], an initial reduction argument (which we do not use here) gives $n_0 = 1$, and hence $j_0 = 1$. We define $t_0 = \max\{t_1, t_2\}$.

Writing $u(x) = \sum_{B\in\mathcal{F}} \chi_{t_2 B}(x)$ for all $x \in \Omega$, we have $\exp(au(x)) > \exp(ac'j)$ for all $x \in \Omega_{2^{-j}}$, $j > j_0$, and $a > 0$. It therefore suffices to find a constant $a = a(n, C_\Omega, C_{\mu,t}) > 0$ such that

$$\int_{\Omega_{2^{-j}}} e^{au(x)}\, d\mu(x) \lesssim \mu(\Omega_1).$$

Now,

$$\int_{\Omega_{2^{-j}}} e^{au}\, d\mu \leq \sum_{k\geq 0} \int_{\Omega_1} \frac{(au)^k}{k!}\, d\mu \leq \mu(\Omega_1) + \sum_{k>0} \frac{a^k}{k!} \int_{\Omega_1} \left(\sum_{B\in\mathcal{F}} \chi_{t_2 B} \right)^k d\mu.$$

Since $\mu \in D_t(\Omega) \subset D_{t_2}(\Omega)$, we may use Lemma 2.3 to get

$$\begin{aligned} \int_{\Omega_{2^{-j}}} e^{au(x)}\, d\mu(x) &\leq \mu(\Omega_1) + \sum_{k>0} \frac{(aCk)^k}{k!} \int_{\Omega_1} \left(\sum_{B\in\mathcal{F}} \chi_B \right)^k d\mu \\ &\lesssim \mu(\Omega_1) \left(1 + \sum_{k>0} \frac{(aCk)^k}{k!} \right). \end{aligned}$$

This last series converges for all $a < 1/Ce$, and so we are done. $\square$

References

[Bo] B. Bojarski, *Remarks on Sobolev imbedding inequalities*, Proc. of the conference on Complex Analysis, Joensuu 1987, Lecture Notes in Math. 1351, Springer-Verlag, Berlin, 1989, pp. 52–68.

[BKL] S.M. Buckley, P. Koskela, and G. Lu, *Subelliptic Poincaré inequalities: the case $p < 1$*, Publ. Mat. **39** (1995), 313–334.

[BO] S.M. Buckley and J. O'Shea, *Weighted Trudinger type inequalities*, preprint.

[GO] F.W. Gehring and B. Osgood, *Uniform domains and the quasihyperbolic metric*, J. Analyse Math. **36** (1979), 50–74.

[G] M. de Guzmán, *Differentiation of integrals in $\mathbb{R}^n$*, Lecture Notes in Math. 481, Springer-Verlag, Berlin, 1975.

[HKM] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Univ. Press, Oxford, 1993.

[JM] P.W. Jones and N.G. Makarov, *Density properties of harmonic measure*, Ann. of Math. (2) **142** (1995), 427–455.

[KR] P. Koskela and S. Rohde, *Hausdorff dimension and mean porosity*, Math. Ann. **309** (1997), 593–609.

[MV] O. Martio and M. Vuorinen, *Whitney cubes, p-capacity, and Minkowski content*, Expo. Math. **5** (1987), 17–40.

[M] A.P. Morse, *Perfect blankets*, Trans. Amer. Math. Soc. **6** (1947), 418–442.

[SS] W. Smith and D.A. Stegenga, *Exponential integrability of the quasihyperbolic metric in Hölder domains*, Ann. Acad. Sci. Fenn. Ser. A I. Math. **16** (1991), 345–360.

[S1] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.

[S2] E.M. Stein, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, Princeton, 1993.

Department of Mathematics, National University of Ireland, Maynooth, Co. Kildare, Ireland.

E-mail address: sbuckley@maths.may.ie