The cosmological constant and black hole thermodynamic potentials

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August 18, 2011

Abstract

The thermodynamics of black holes in various dimensions are described in the presence of a negative cosmological constant which is treated as a thermodynamic variable, interpreted as a pressure in the equation of state. The black hole mass is then identified with the enthalpy, rather than the internal energy, and heat capacities are calculated at constant pressure not at constant volume. The Euclidean action is associated with a bridge equation for the Gibbs free energy and not the Helmholtz free energy. Quantum corrections to the enthalpy and the equation of state of the BTZ black hole are studied.

Report No. DIAS-STP-10-10 PACS nos: 04.60.-m; 04.70.Dy

1 Introduction

The thermodynamics of black-holes has been an active and fascinating area of research ever since the early papers of Beckenstein and Hawking's derivation of the temperature associated with the event horizon, [1, 2]. In most treatments of black-hole thermodynamics the cosmological constant, Λ, is treated as a fixed parameter (possibly zero) but it has been considered as a dynamical variable in [3, 4] and it has further been suggested that it is better to consider Λ as a thermodynamic variable, [5, 6, 7, 8, 9]. Physically Λ is interpreted as a thermodynamic pressure in [9], consistent with the observation in [7] that the conjugate thermodynamic variable is proportional to a volume. This naturally leads to a slightly different interpretation of the black-hole mass than is usual in thermodynamic treatments, the black-hole mass is equated with enthalpy, H, in [9] rather than the internal energy E, as is more usual.

A black-hole with a positive cosmological constant has both a cosmological event horizon and a black-hole event horizon, these have different Hawking temperatures associated with them in general which necessarily complicates any thermodynamical treatment. We shall therefore focus on the case of a negative cosmological constant in this work though, many of the conclusions are applicable to the positive Λ case. The negative Λ case is of course of interest for studies on AdS/CFT correspondence and the considerations here are likely to be relevant to current attempts to model condensed matter systems in $2+1$ dimensions using the boundary of $3+1$ dimensional anti-de Sitter space, see *e.g.* [10]. In particular quantities calculated at constant Λ correspond physically to constant pressure: specific heats, for example, are specific heats at constant pressure and not specific heats at constant volume and it is the former that are more relevant to solid state applications.

In this work the idea that Λ is a thermodynamic pressure, and the conjugate variable a thermodynamic volume, is elaborated on and the thermodynamical structure developed further. The relation between various thermodynamical potentials and the black-hole equation of state in anti-de Sitter space is explored in detail. §2 deals with the thermodynamic potentials in the 4-dimensional case; §3 discusses the black-hole equation of state and §4 the partition function. In §5 the discussion is extended to arbitrary dimensions while §6 deals specifically with the 3-dimensional case of the BTZ black-hole, for which quantum corrections to the thermodynamical potential are known, at least perturbatively. Finally §6 summarizes the main points and conclusions.

2 Enthalpy

Consider a black hole with mass M in the presence of a cosmological constant Λ. The cosmological constant generates a pressure

$$P = -\frac{\Lambda}{8\pi G_N} \tag{1}$$

and has an energy density ϵ associated with it with $\varepsilon + P = 0$, *i.e.* the enthalpy density associated with Λ is zero. If the black hole has a volume V the total energy contained in V is

$$E = M + \varepsilon V = M - PV \qquad \Rightarrow \qquad M = E + PV,$$

hence M is most naturally associated with the enthalpy H of the black hole

$$H = E + PV.$$

It is not obvious what the volume of a black hole should be. The naïve identification of V with the volume of a sphere with the radius of the event horizon r_h is too simplistic since the radial co-ordinate is time-like inside the horizon and $\frac{4\pi r_h^3}{3}$ is not the volume of any space-like section of space-time inside the horizon. It is suggested in [9] that V be identified with the volume excluded by the black hole horizon from a spatial slice exterior to the black hole, giving the naïve result $V = \frac{4\pi r_h^3}{3}$ but from a more physically acceptable perspective. For the moment we shall leave V unspecified and determine it below from thermodynamic considerations.

The natural variables for enthalpy are entropy and pressure, so we should view M as a function of S and P,

$$M = H(S, P).$$

The functional form of H is determined by the geometry together with the Hawking relation, that entropy is one quarter of the horizon area

$$S = \frac{\pi r_h^2}{\hbar G_N} = \frac{\pi r_h^2}{\ell^2}, \tag{2}$$

where $l = \sqrt{\hbar G_N}$ is the Planck length.[1]

The metric of four-dimensional space-time is given by

$$d^2 s = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2,$$

with

$$f(r) = 1 - \frac{2G_N M}{r} - \frac{\Lambda}{3} r^2, \tag{3}$$

and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ the solid angle area element. The event horizon is defined by $f(r_h) = 0$,

$$\frac{\Lambda}{3} r_h^3 - r_h + 2G_N M = 0. \tag{4}$$

One can solve the cubic equation to find $r_h(M, \Lambda)$ analytically, but the explicit form will not be needed in the following. For $\Lambda > 0$ and $0 < 3M\sqrt{\Lambda}G_N < 1$ there are two event horizons and the region of space-time outside the black hole horizon but inside the de Sitter horizon lies between them. They coincide when $3M\sqrt{\Lambda}G_N = 1$. Each event horizon has a different Hawking temperature associated with it and the system is not in thermal equilibrium. In order to ensure thermal equilibrium we shall assume $\Lambda < 0$ in the following.

Equation (3) gives the mass as

$$M = \frac{r_h}{2G_N}\left(1 - \frac{\Lambda}{3} r_h^2\right). \tag{5}$$

Identifying M with the enthalpy and using (1) and (2) gives

$$H(S, P) = \frac{1}{2G_N}\left(\frac{\ell^2 S}{\pi}\right)^{\frac{1}{2}}\left(1 + \frac{8G_N \ell^2 SP}{3}\right). \tag{6}$$

The usual thermodynamic relations can now be used to determine the temperature and the volume,

$$T = \left(\frac{\partial H}{\partial S}\right)_P \quad \Rightarrow \quad T = \frac{\hbar}{4\pi}\left(\frac{\pi}{\ell^2 S}\right)^{\frac{1}{2}}(1 + 8PG_N\ell^2 S) = \frac{\hbar(1 - \Lambda r_h^2)}{4\pi r_h} \tag{7}$$

$$V = \left(\frac{\partial H}{\partial P}\right)_S \quad \Rightarrow \quad V = \frac{4}{3}\frac{(\ell^2 S)^{\frac{3}{2}}}{\sqrt{\pi}} = \frac{4\pi r_h^3}{3}. \tag{8}$$

[1] In the ensuing analysis we shall make factors of G_N and $\hbar$ explicit in order to exhibit clearly which aspects of the physics are classical and which are quantum. Note that $\ell^2 S = \pi r_h^2$ is a classical quantity, S diverges in the classical limit while $\ell^2 S$ remains finite.

The Hawking temperature (7) comes as no surprise since it follows from the usual formula relating the temperature to the surface gravity κ at the event horizon

$$T = \frac{\hbar\kappa}{2\pi} = \frac{\hbar f'(r_h)}{4\pi}.$$

Varying (4) gives

$$df = -\frac{2G_N}{r_h}dM - \frac{r_h^2}{3}d\Lambda + f'(r_h)dr_h = 0,$$

so

$$\left.\frac{\partial M}{\partial r_h}\right|_\Lambda = \frac{r_h}{2G}f'(r_h) \qquad \Rightarrow \qquad \left.\frac{\partial M}{\partial S}\right|_P = \frac{\hbar}{4\pi}f'(r_h),$$

which is equation (7). Equation (8) suggests that the "naïve" volume is indeed the correct one to use in thermodynamic relations.

Legendre transforming (6) gives the internal energy

$$E(S,V) = H(S,P) - PV = \frac{1}{2G_N}\sqrt{\frac{\ell^2 S}{\pi}}, \tag{9}$$

but the Legendre transform is not invertible: because $H(S,P)$ is linear in P, $E(S,V)$ is independent of V and so the pressure cannot be determined from a knowledge of $E(S,V)$ alone. For the same reason $T = \left.\frac{\partial E}{\partial S}\right|_V$ gives the wrong answer for the temperature if the pressure is non-zero.

Nevertheless we can still use (6) to determine the heat capacity at constant pressure using the standard thermodynamic relation

$$C_P = \frac{T}{\left.\frac{\partial T}{\partial S}\right|_P}.$$

One finds

$$C_P = 2S\left(\frac{8G_N P\ell^2 S + 1}{8G_N P\ell^2 S - 1}\right). \tag{10}$$

The heat capacity at constant volume

$$C_V = \frac{T}{\left.\frac{\partial T}{\partial S}\right|_V} = T\left.\frac{\partial S}{\partial T}\right|_V$$

vanishes, if we use the Hawking formula (2), since the variation of the entropy is necessarily zero when the volume, and hence r_h, is fixed.

It could have been anticipated in advance that $C_V = 0$, from (9): E depends only on S and $S = \frac{\pi}{\ell^2}\left(\frac{3V}{4\pi}\right)^{\frac{2}{3}}$ is a function of V alone so E is constant if V is held fixed as the temperature is varied.

Local stability requires that $C_P > 0$ so $8G_N P\ell^2 S > 1$, or equivalently

$$-\Lambda r_h^2 > 1,$$

so Λ must be negative. The fact that black holes can be thermodynamically stable in anti-de Sitter space-time is well known [12]. Physically this condition for stability can be understood as follows. For $\Lambda < 0$ the vacuum energy density $\varepsilon < 0$ so the black hole contains negative vacuum energy. As it radiates at constant pressure, and hence constant ε, the volume decreases and the vacuum energy it contains increases (becoming less negative). At the same time its temperature increases hence the energy can go up as the temperature goes up, if $|\Lambda|$ is large enough the heat capacity is positive and the black hole is stabilised by the negative vacuum energy.

Local stability therefore implies a minimum temperature

$$T_{min} = \hbar\sqrt{\frac{2G_N P}{\pi}} \tag{11}$$

below which the black hole is not stable, corresponding to the divergence in C_P when $\left.\frac{\partial T}{\partial S}\right|_P = 0$. As is well known, this is below the Hawking-Page temperature, $T_{HP} = \hbar\sqrt{\frac{8G_N P}{3\pi}}$, below which pure AdS space, with no black hole and $M = 0$, has a lower free energy than that of a black hole with the same Λ and $M > 0$ which occurs for $r_h = \sqrt{\frac{3}{|\Lambda|}}$, [12].

3 The black hole equation of state

Writing equation (7) in terms of V and P gives the black hole equation of state

$$T(V,P) = \frac{\hbar}{4\pi}\left\{\left(\frac{3V}{4\pi}\right)^{-\frac{1}{3}} + 8\pi G_N P\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}\right\}. \tag{12}$$

For a given pressure there is a minimum volume at T_{min},

$$V(T_{min}) = \frac{4\pi}{3}\left(\frac{1}{8\pi G_N P}\right)^{\frac{3}{2}}. \tag{13}$$

Figure 1 shows $T(V)$ for various pressures and figure 2 shows the black hole indicator diagram, $P(V)$, for various temperatures. The temperature as a function of entropy, at constant pressure, is shown in figure 3, for comparison with the $J = Q = 0$ case in figure 1 of [5].

4 The partition function

A key concept in understanding black hole thermodynamics is the relation between the Euclidean path integral and the black hole partition function [11]. Defining the Euclidean action requires a regularisation procedure as the volume of space-time is infinite and the Ricci scalar is non-zero [12]. A regularised Euclidean action can be obtained by adding surface terms at large r to cancel the infinities arising from taking $r \to \infty$ in the bulk integrals, [13, 15]. Two terms are necessary, one corresponding the extrinsic curvature of the sphere at large radius, involving the unit normal n^μ, and one simply proportional to the area of the sphere,

$$\begin{aligned} I &= -\frac{1}{16\pi G_N}\int_{\mathcal{M}}(R-2\Lambda)\sqrt{-g}\,d^4x \\ &\quad +\frac{1}{8\pi G_N}\int_{\partial\mathcal{M}}\gamma^{\mu\nu}\nabla_\mu n_\nu\sqrt{-\gamma}\,d^3x - \frac{1}{2\pi G_N L}\int_{\partial\mathcal{M}}\sqrt{-\gamma}\,d^3x. \end{aligned} \tag{14}$$

In [11] the integral is taken over $r_h < r < \infty$ with γ the three-dimensional metric on the asymptotic boundary $\partial\mathcal{M}$, $r \to \infty$. In particular the event horizon is not considered to be part of $\partial\mathcal{M}$. L is the AdS length scale, $\Lambda = -\frac{3}{L^2}$, and the Euclidean time parameter x^0 is periodic with $0 < x^0 < \frac{1}{T}$. Performing the integrals gives

$$I = \frac{r_h}{4G_N T}\left(1 + \frac{\Lambda r_h^2}{3}\right). \tag{15}$$

In the Euclidean approach to quantum gravity, [12], this is related to the partition function $Z = e^{-I}$ through the bridge equation

$$TI = -T\ln Z, \tag{16}$$

and hence TI is identified with the free energy

$$\mathcal{F} = \frac{r_h}{4G_N}\left(1 + \frac{\Lambda r_h^2}{3}\right). \tag{17}$$

But is this the Helmholtz free energy $F(T,V)$ or the Gibbs free energy $G(T,P)$?

The functional integral is performed with fixed T and Λ, so $\mathcal{F}(T,\Lambda)$ should be though of as a function of T and Λ and, using (7), it is readily shown that

$$d\mathcal{F} = -\frac{\pi r_h^2}{G_N \hbar} dT - \frac{r_h^3}{6G_N} d\Lambda. \tag{18}$$

Hence

$$-\left.\frac{\partial \mathcal{F}}{\partial T}\right|_\Lambda = \frac{\pi r_h^2}{G_N \hbar} = S \qquad \text{and} \qquad \left.\frac{\partial \mathcal{F}}{\partial P}\right|_T = -8\pi G_N \left.\frac{\partial \mathcal{F}}{\partial \Lambda}\right|_T = \frac{4\pi r_h^3}{3} = V. \tag{19}$$

These are the thermodynamic relations associated with the Gibbs free energy, $G(T,P)$, and not the Helmholtz free energy. It is natural therefore to identify $\mathcal{F} = G(T,P)$ with the Gibbs free energy. The enthalpy is the Legendre transform of the Gibbs free energy $H = G + TS$ and a simple calculation shows that $H = M$ is the black-hole mass.

Euler's equation for thermodynamic potentials follows from dimensional analysis. Equation (4) is invariant under the rescalings $r_h \to \eta r_h$, $\Lambda \to \eta^{-2}\Lambda$ and $M \to \eta M$ (keeping G_N fixed). Hence

$$\eta M(S,P) = M(\eta^2 S, \eta^{-2} P) \qquad \Rightarrow \qquad M = 2(TS - PV),$$

which is easily checked. This is Smarr's formula [16] treated from the same point of view as in [9]. A simple consequence of this scaling argument is that $M(S,P)$ must have the functional form

$$M = \sqrt{S}\,\Phi(SP)$$

for some function $\Phi(SP)$: in fact (6) shows that $\Phi(SP)$ is a linear function. Corrections to the entropy can modify this simple scaling analysis.

5 Higher dimensional black holes and different event horizon geometries

In D space-time dimensions the AdS-Schwarzschild line element is

$$d^2 s = -f(r)dt^2 + f^{-1}(r)dr^2 + r^d d\Omega^2_{(d)}, \tag{20}$$

where $d = D - 2$ and $d\Omega^2_{(d)}$ is the line element on a d-dimensional sphere of unit radius. Denoting the volume of the unit sphere by

$$\Omega_{(d)} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

the D-dimensional Einstein equations give the function $f(r)$ to be

$$f(r) = 1 - \frac{16\pi G_N}{\Omega_{(d)}\, d}\frac{M}{r^{d-1}} - \frac{2\Lambda}{d(d+1)}r^2, \tag{21}$$

with $\Lambda < 0$ for AdS, [17].

A more general line element is possible if surfaces of constant r are taken to be any Einstein space with constant curvature, such as flat space or d-dimensional hyperbolic spaces. Replacing the spherical line element $d\Omega^2_{(d)}$ in (20) by the appropriate constant curvature line element in each case we denote the line element by $d\Omega^2_{(d,k)}$ and the volume by $\Omega_{(d,k)}$. For example $k = +1$ for spheres, -1 for hyperbolic space and 0 for flat space. Einstein's equations are now solved by replacing the function $f(r)$ with[2]

$$f(r) = k - \frac{16\pi G_N}{\Omega_{(d,k)}\, d}\frac{M}{r^{d-1}} - \frac{2\Lambda}{d(d+1)}r^2. \tag{22}$$

For a general constant curvature Einstein space with d-dimensional Ricci tensor

$$R_{ij} = \lambda g_{ij},$$

where λ is a constant and $i, j = 1, \ldots, d$, the d-dimensional Ricci scalar is $R_{(d)} = \lambda d$ and (20) solves Einstein's equations with cosmological constant Λ provided

$$k = \frac{\lambda}{d-1} = \frac{R_{(d)}}{d(d-1)},$$

and $d\Omega^2_{(d,k)}$ corresponds to the line element of the Einstein metric of the event horizon. Note that $\Omega_{(d,k)}$ is dimensionless in the conventions adopted here, the event horizon has area $r_h^d\Omega_{(d,k)}$ and $R_{(d)}$ is a dimensionless constant fixed by the geometry of the event horizon, in particular it does not depend on r_h.

[2]For flat space the volume can be taken to be finite by making periodic identifications to get the topology of a torus. For negatively curved spaces the same procedure gives more complicated topologies, *e.g.* higher genus surfaces for $d = 2$.

The analysis of §2 is easily repeated with minor modifications. In D dimensions the Planck length is given by $l^d = \hbar G_N$ and the entropy is

$$S = \frac{\Omega_{(d,k)}}{4}\frac{r_h^d}{l^d}. \tag{23}$$

Equating M with the enthalpy and $\Lambda = -\frac{16\pi G_N}{d} P$ results in

$$H(S,P) = \frac{\hbar S}{4\pi}\left\{\frac{R_{(d)}}{d-1}\left(\frac{4l^d S}{\Omega_{(d,k)}}\right)^{-\frac{1}{d}} + \frac{16\pi G_N P}{d+1}\left(\frac{4l^d S}{\Omega_{(d,k)}}\right)^{\frac{1}{d}}\right\}, \tag{24}$$

from which all thermodynamic quantities can be calculated.

The thermodynamic volume is the naïve result

$$V = \frac{\Omega_{(d,k)} r_h^{d+1}}{d+1}, \tag{25}$$

and the equation of state is

$$T = \frac{\hbar}{4\pi d}\left\{R_{(d)}\left(\frac{(d+1)V}{\Omega_{(d,k)}}\right)^{-\frac{1}{d+1}} + 16\pi G_N P\left(\frac{(d+1)V}{\Omega_{(d,k)}}\right)^{\frac{1}{d+1}}\right\}. \tag{26}$$

For positive k there is a minimum temperature and volume for any given fixed pressure

$$T_{min} = \frac{2\hbar}{d}\sqrt{\frac{R_{(d)} G_N P}{\pi}}, \qquad V(T_{min}) = \frac{\Omega_{(d,k)}}{d+1}\left(\frac{R_{(d)}}{16\pi G_N P}\right)^{\frac{d+1}{2}}$$

and the heat capacity is

$$C_P = Sd\left\{\frac{16\pi G_N P\left(\frac{4l^d S}{\Omega_{(d,k)}}\right)^{\frac{2}{d}} + R_{(d)}}{16\pi G_N P\left(\frac{4l^d S}{\Omega_{(d,k)}}\right)^{\frac{2}{d}} - R_{(d)}}\right\},$$

which diverges at T_{min} and is negative for $T < T_{min}$. There is thus a Hawking-Page phase transition for any positive curvature event horizon in $D \geq 4$ dimensions, for spatially flat event horizons the specific heat $C_P = Sd$ is always positive while for $R_{(d)} < 0$ there is no minimum temperature, but there is still a minimum value of $|\Lambda|$ below which C_P is negative. In all cases one must have

$$|\Lambda| > \frac{|R_{(d)}|}{2r_h^2}$$

for a black hole to be stable in anti-de Sitter space-time.

6 The BTZ black hole

It is worthwhile studying the special case of the $2+1$-dimensional BTZ black hole, not only because it is conceptually and mathematically simpler than its higher dimensional cousins but also because higher order corrections to the entropy are easier to calculate and not as uncertain. For a review of BTZ black holes see [18].

The BTZ black hole has line element

$$d^2s = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\phi^2.$$

For a non-rotating BTZ black hole

$$f(r) = -8G_N M + \frac{r^2}{L^2},$$

with cosmological constant $\Lambda = -\frac{2}{L^2}$ giving a pressure $P = \frac{1}{8\pi G_N L^2}$.

The black hole radius

$$r_h = \sqrt{8G_N M}L \tag{27}$$

is immediate. The event horizon is a circle and the entropy is one-quarter of the circumference in Planck units,

$$S = \frac{\pi r_h}{2l},$$

where $l = \hbar G_N$ in three dimensions. Identifying the mass with the enthalpy then gives

$$H(S,P) = \frac{4\ell^2}{\pi} S^2 P, \tag{28}$$

from which

$$T = \frac{8\ell^2}{\pi} SP = \frac{\hbar r_h}{2\pi L^2} \tag{29}$$

$$V = \frac{4\ell^2}{\pi} S^2 = \pi r_h^2, \tag{30}$$

the standard results (for uniformity of notation the symbol V is used for the thermodynamic "volume", even though it is an area in three space-time dimensions). The Gibbs free energy is

$$G = H - TS = \frac{4l^2 S^2 P}{\pi} - 2M = -M.$$

C_P is easily calculated, since T is linear in S we have $\left.\frac{\partial T}{\partial S}\right|_P = \frac{T}{S}$ so

$$C_P = \frac{T}{\left.\frac{\partial T}{\partial S}\right|_P} = S > 0.$$

There is no local instability, there is however a global instability since ordinary 3-dimensional AdS, with $f(r) = 1 + \frac{r^2}{L^2}$ corresponding to $M = -\frac{1}{8G_N}$, has $T = 0$ giving $G = H = -\frac{1}{8G_N}$, and so has lower Gibbs free energy than the $M = 0$ black-hole, making it more stable. This suggests a phase transition from a black-hole AdS state to pure AdS_3 when the Gibbs free energies are equal, which happens for $M = \frac{1}{8G_N}$ at a temperature $T = \hbar\sqrt{\frac{2G_N P}{\pi}}$, [19, 20].

Using (30) to express S in terms of V we derive the BTZ equation of state

$$PV^{\frac{1}{2}} = \frac{\sqrt{\pi}}{4\ell^2}T.$$

From (28) one finds $H = PV$ so Legendre transforming gives $E = 0$, the BTZ internal energy vanishes classically.

The partition function for the BTZ black-hole, including quantum corrections to all orders in perturbation theory, is given in [21] (earlier attempts at calculating corrections to the BTZ black-hole entropy can be found in [18, 22, 23, 24]). To understand the structure fully it is necessary to start with the rotating black-hole, with metric [18]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\phi - \frac{4G_N J}{r^2}dt\right)^2, \tag{31}$$

where

$$f(r) = \left(-8G_N M + \frac{r^2}{L^2} + \frac{16G_N^2 J^2}{r^2}\right)$$

and J is the angular momentum, bounded above by $J \leq ML$. There are now two event horizons, and inner and an outer horizon at r_+ and r_- respectively, with

$$r_\pm^2 = 4G_N M L^2 \left\{1 \pm \left[1 - \left(\frac{J}{ML}\right)^2\right]^{\frac{1}{2}}\right\}. \tag{32}$$

The Hawking temperature associated with (31) is

$$T = \frac{f'(r_+)}{4\pi} = \frac{(r_+^2 - r_-^2)\hbar}{2\pi L^2 r_+}. \tag{33}$$

The entropy corrections described in [21] are best discussed in the Euclidean formalism. Wick rotating the time parameter sends $t \to -it_E$ and the angular momentum also rotates, $J \to iJ_E$. The right hand side of (32) then translates to

$$4G_N ML^2 \left\{ 1 \pm \left[1 + \left(\frac{J_E}{ML} \right)^2 \right]^{\frac{1}{2}} \right\}. \tag{34}$$

in the Euclidean sector and we see that $r_+ \to r_{E,+}$ while $r_- \to ir_{E,-}$ where

$$r_{E,\pm}^2 = 4G_N ML^2 \left\{ \left[1 + \left(\frac{J_E}{ML} \right)^2 \right]^{\frac{1}{2}} \pm 1 \right\}. \tag{35}$$

The partition function is elegantly described in terms of the dimensionless complex parameter

$$\tau = \frac{r_{E,-} + ir_{E,+}}{L} \tag{36}$$

which has $Im(\tau) > 0$. The inverse Hawking temperature in the Euclidean formalism is given by

$$\frac{1}{2\pi T} = \frac{r_{E,+}}{(r_{E,+}^2 + r_{E,-}^2)} \frac{L^2}{\hbar} = \frac{L}{\hbar} \left\{ Im\left(-\frac{1}{\tau} \right) \right\}^{-1}. \tag{37}$$

The BTZ partition function given in [21], including all perturbative quantum corrections, is most succinctly written by defining $q = e^{2\pi i\tau}$ in terms of which

$$Z_{BTZ} = (q\bar{q})^{-\frac{L}{16\hbar G_N}} \prod_{n=2}^{\infty} |1 - q^n|^{-2}. \tag{38}$$

Equation (38) does not include non-perturbative quantum corrections, but it suffices to illustrate this discussion of corrections to thermodynamic quantities. (Note that τ in our notation is $-\frac{1}{\tau}$ in the notation of [21], and Z_{BTZ} here is denoted $Z_{1,0}$ there).

We now specialise to the case of zero angular momentum, when

$$T = \frac{r_{E,+}}{2\pi} \frac{\hbar}{L^2} = \frac{r_+}{2\pi} \frac{\hbar}{L^2}, \qquad \tau = \frac{2\pi i TL}{\hbar} = i\frac{r_+}{L} \qquad \text{and} \qquad q = e^{-4\pi^2 \frac{LT}{\hbar}}.$$

The partition function in this case is

$$Z_{BTZ} = e^{\frac{\pi^2 TL^2}{2\hbar^2 G_N}} \prod_{n=2}^{\infty} \left(1 - e^{-4\pi^2 n \frac{TL}{\hbar}} \right)^{-2}. \tag{39}$$

Thermodynamic functions can immediately be read off. Defining $x = \frac{TL}{\hbar} = \frac{r_+}{2\pi L}$ the Gibbs free energy is

$$G(T, P) = -T \ln Z_{BTZ} = -\frac{\pi^2 x^2}{2G_N} + 2T \sum_{n=2}^{\infty} \ln\Big(1 - e^{-4\pi^2 nx}\Big), \qquad (40)$$

where the first term on the right hand side is the classical result.

The entropy was calculated in [21] and the enthalpy can be determined using the standard formula $H(S, P) = G + TS$, giving

$$H = \frac{\pi^2 x^2}{2G_N} - 8\pi^2 xT \sum_{n=2}^{\infty} \frac{n}{e^{4\pi^2 nx} - 1}, \qquad (41)$$

but this is not expressed explicitly in terms of the natural variables S and $P = \frac{1}{8\pi G_N L^2}$, the S dependence is only implicit.

The quantum corrections embodied in the logarithmic terms of (40) modify the thermodynamic volume,

$$V = \frac{\partial G}{\partial P}\bigg|_T = \pi r_h^2 \left[1 - 8\pi \left(\frac{G_N \hbar}{L}\right) \sum_{n=2}^{\infty} \frac{n}{e^{4\pi^2 nx} - 1}\right]. \qquad (42)$$

We see that the quantum corrections serve to reduce the volume below its classical value. Figure 4 plots the PV-diagram, quantum effects reduce the volume to zero at finite pressure, the effect being more pronounced at lower temperatures.

7 Conclusions

Some consequences of the suggestion in [9], that the correct thermodynamic interpretation of black-hole mass in the presence of a negative cosmological constant is that it should be associated with enthalpy rather than the more usual interpretation of internal energy, have been explored and expanded upon. The cosmological constant is treated as a thermodynamic variable proportional to the pressure and the black-hole mass is identified with the enthalpy rather than the internal energy. The interpretation of many thermodynamic quantities is modified in this approach: specific heats, for example, are naturally calculated as specific heats at constant pressure rather than at constant volume and the Euclidean action gives the Gibbs free energy and not the Helmholtz free energy.

Black-hole solutions of Einstein's equations in any dimension, and with any Einstein manifold with constant scalar curvature as event horizon, can easily be constructed and the classical equation of state determined. The Hawking-Page transition is manifest as a change in sign the specific heat and is present for black-holes with an event horizon with positive curvature in any space-time dimension greater than three.

Quantum corrections to the thermodynamics relations for the BTZ black-hole in three dimensions have been derived from the partition function in [21], which includes corrections to all orders in perturbation theory but does not include non-perturbative corrections. These corrections reduce the volume at a given pressure and temperature, with a finite pressure giving zero volume.

The considerations presented here will have important implications for AdS/CFT approaches to condensed matter systems, in which the specific heat at constant pressure has greater significance than the specific heat constant volume.

It is a pleasure to thank the Perimeter Institute, Waterloo, Canada for hosting a visit during which this work was initiated. This work was partly funded by the EU Research Training Network in Noncommutative Geometry (EU-NCG).

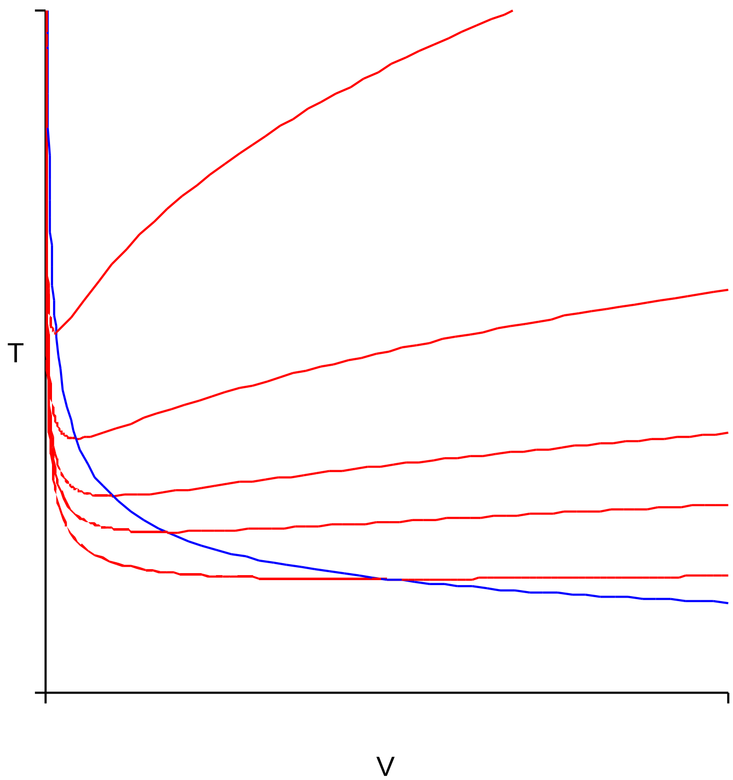

Figure 1: Black hole T-V diagram, showing curves of constant pressure in AdS. The blue line shows the stability limit, the region to the left and below the blue curve is unstable.

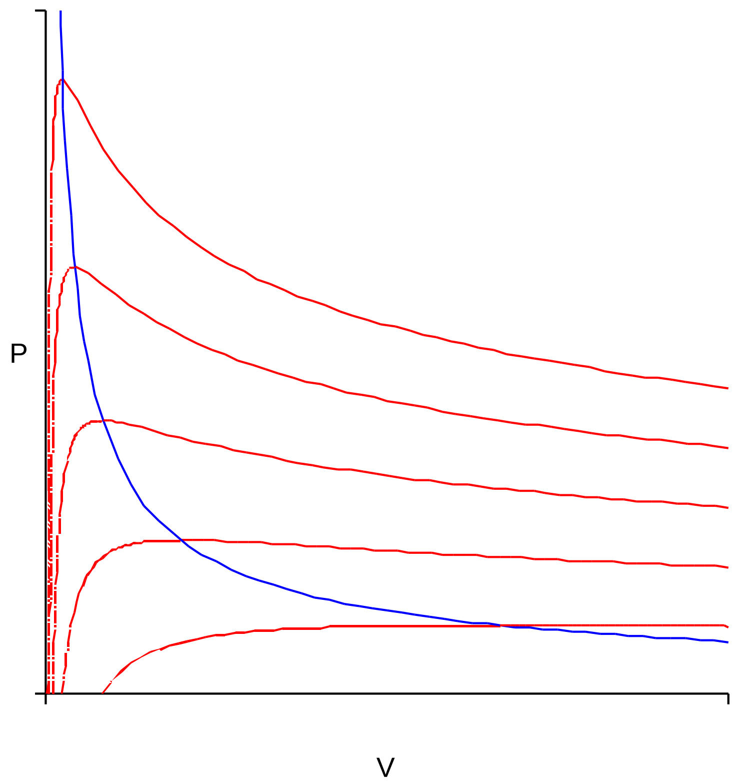

Figure 2: Black hole P-V diagram, showing curves of constant temperature in AdS. The blue line shows the stability limit, the region to the left and below the blue curve is unstable.

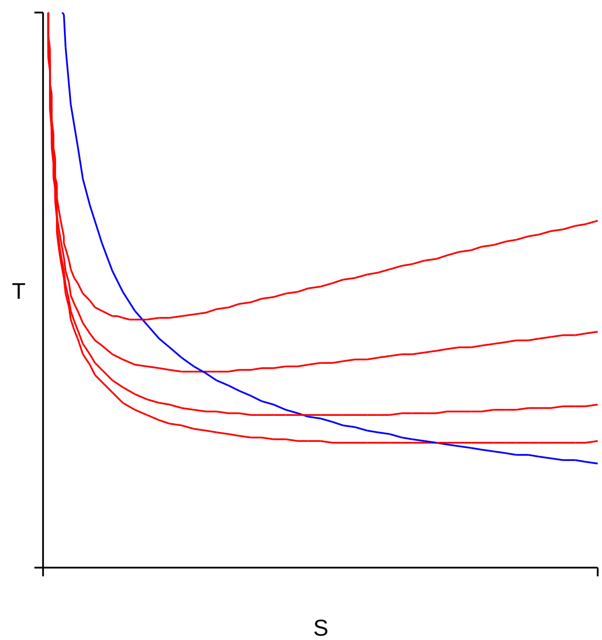

Figure 3: T-S diagram, showing curves of constant pressure. The region left of the blue line is unstable, temperature is an increasing function of entropy in the stable region.

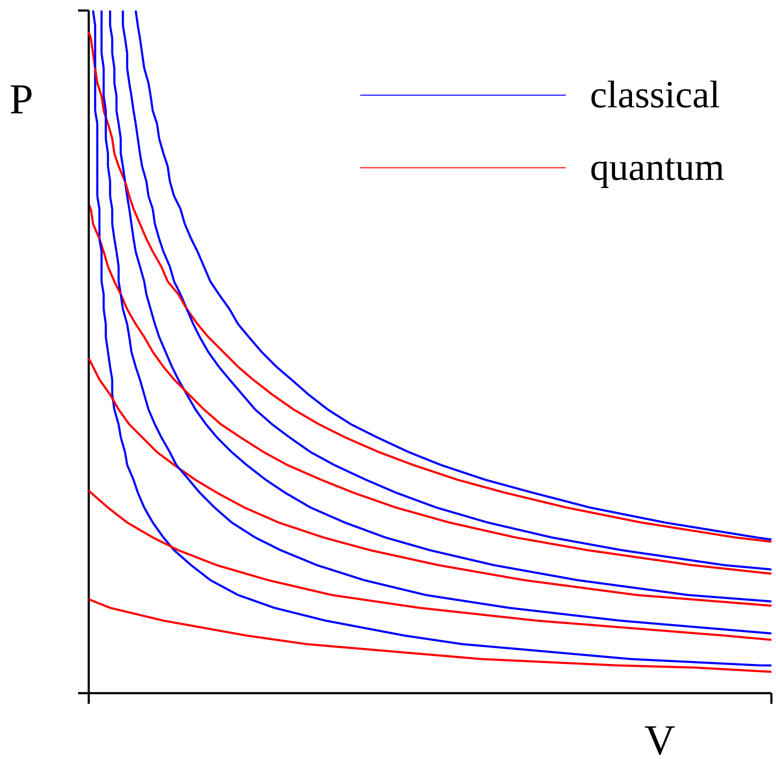

Figure 4: Black hole P-V diagram, showing curves of constant temperature in AdS_3. The blue lines show the classical curves, $P \propto \frac{1}{\sqrt{V}}$. Red lines show the quantum corrected equation of state. The region left of the blue line is unstable, temperature is an increasing function of entropy in the stable region.

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