On computing the critical coupling coefficient for the Kuramoto model on a complete bipartite graph

Mark Verwoerd and Oliver Mason

Hamilton Institute, National University of Ireland

{mark.verwoerd,oliver.mason}@nuim.ie

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Abstract

We extend recent results [50] on the existence of global phase-locked states (GPLS) in the Kuramoto model on a complete graph to the case of a complete bipartite graph. In particular, we prove that, for the Kuramoto model on a complete bipartite graph, the value of the critical coupling coefficient can be determined by solving a system of two nonlinear equations that do not depend on the coupling coefficient. We show that the said system of equations can be solved using an efficient algorithm described in the paper.

1 Introduction

Coordinated or synchronized behaviour is a fundamental aspect of a remarkable array of physical, biological and engineering systems. In light of this and of the theoretical interest inherent in the topic, it is hardly surprising that the study of such behaviour occupies an important place in the general theory of dynamical systems, and that it has attracted the attention of researchers from a broad range of scientific disciplines [31, 4, 27, 11, 29].

From a historical point of view, much of the early quantitative work on synchronization was rooted in physical questions such as the analysis of networks of Josephson junctions [52, 51] or coupled pendula [3]. More recently, with the growth of activity and interest in mathematical and systems biology, there has been considerable interest in analyzing synchronization in biological systems. Examples of synchronized behaviours abound in the life sciences. Some of the more important examples to have attracted attention include: circadian rhythms [48, 2, 53], the synchronization of heart cells [24], synchronized neuronal activity [23, 49] with particular emphasis on pathological synchronization in disorders such as Parkinson's disease [14, 47, 46], and flocking behaviour in populations of animals, birds or fish [43, 6]. In addition to advancing our understanding of the world around us, a better understanding of the mechanisms of synchronization can guide the design of modern multi-agent engineering systems [28, 16, 42, 41, 38]

Of course, the development and analysis of appropriate mathematical models is crucial to clarifying the mechanisms behind coordinated and synchronized behaviour. Several general frameworks and approaches have now been proposed. These include the abstract and elegant theory of Coupled-Cell Networks [13, 12] which characterizes the possible spatiotemporal behaviours of a system in terms of symmetries in the network describing the system interactions. Much of the early work in this direction focussed on symmetries which could be captured by group structure; however recently there has been considerable effort to extend this to the more general setting provided by groupoids, which allows for asymmetric network structures. Another approach is provided by the Master Stability Function, which provides a condition for the local stability of the synchronized manifold in terms of the eigenvalues of the Laplacian of the network.

To date, one of the most widely-studied frameworks for the analysis of synchronization is the so-called *Kuramoto model*, which is a mathematical model describing the dynamics of a system of weakly coupled (nonlinear) oscillators [18, 19, 45, 44, 25, 1]. The Kuramoto model has been used in numerous applications in biology, physics and medicine; more recently it has also attracted attention from the engineering community, specifically from the systems and control community [17, 5, 30, 41, 38, 39, 40]. Indeed, recent work in the context of multi-agent systems has revealed an interesting connection between the Kuramoto model and the problem of stabilizing collective motion in the plane [41].

The basic Kuramoto model is comprised of a system of coupled oscillators, which may have different natural frequencies, where the coupling between two oscillators is given by a weighted sinusoidal function of the difference of their phases. The weights used in the model are typically taken to be the same for all pairs of oscillators and are given by the ratio of a fixed parameter, the coupling strength, to the network size.

One aspect of the Kuramoto model that has attracted a lot of attention to date is the manner in which the onset of synchronization depends on the strength of the coupling between the oscillators. It has been observed numerically and established formally, that a connected network has a particular coupling strength below which no phase-locked solutions can exist, and that increasing the coupling strength beyond this critical value can induce a phase transition in the order parameter of the system.

The first closed formula for the critical coupling coefficient was obtained by Kuramoto. This was for a setting with infinitely many oscillators, a symmetric frequency distribution and all-to-all coupling. Notwithstanding the considerable body of numerical evidence supporting the findings of Kuramoto's work, it can still be argued that Kuramoto's analysis is not entirely rigorous [44]. To develop a more rigorous theory, one possible approach is to start with a network of finite size and then prove a convergence result as the number of oscillators tends to infinity, as suggested by Kopell [44]. To the best of the authors' knowledge, such a result has yet to be reported. Recently, the authors of the present paper contributed a theorem [50, Theorem 3] towards such a result. We will review this and related work in more detail in Section 3.

Several authors have obtained estimates for the value of the critical coupling for the case of a general graph. Even if it is not clear in every case how good the respective estimate is, the key insight that has emerged from this work (see e.g. [33, 35, 34] and [17]), is that spectral properties of the Laplacian or adjacency matrix of the graphs play a key role in determining the value of the critical coupling strength. Indeed, the work described in [33] suggests that the critical coupling scales with the largest eigenvalue of the adjacency matrix, while in [17] a lower bound is given which involves the smallest non-zero eigenvalue of the Laplacian.

Related work on the Kuramoto model includes the study of desynchronization in finite populations of oscillators, which has been described in considerable detail in [21, 20]. Desynchronization is what that takes place when the coupling strength is reduced below its critical value. The authors of aforementioned work have shown that desynchronization involves a series of *frequency-splitting bifurcations*. At each such bifurcation, the ensemble of oscillators subdivides into smaller and smaller groups of oscillators with identical average frequency, until eventually all oscillators oscillate at their own intrinsic frequency. Another aspect of the Kuramoto model to have attracted attention recently is the emergence of phase chaos [22, 32] in systems of dimension four and higher. A generic feature of coupled oscillator systems, phase chaos in the Kuramoto model is most prominent in systems with relatively low dimension (comprising between ten and fifteen oscillators) [22].

Lastly, we mention closely related recent work on the spectrum and the stability of (partially) locked states [26, 25, 7, 36, 37]. In particular, we note the relation between our own work [50] and results in [25].

In this paper we consider the problem of determining the value of the critical coupling for the Kuramoto model on a complete bipartite graph. The motivation for this work is in no small part mathematical. It turns out that the ideas behind previous work on global phase-locking on complete graphs [50] carry over to the case of complete bipartite graphs, although the actual proofs are very different, as are some of the results. Bipartite graphs have applications in many different fields, including biology, sociology and engineering, to name but a few [15]. Examples of bipartite graphs include social contact networks, linking people to locations [9], or people to people [10] and metabolic networks, linking compounds (or substrates) to products.

The outline of the paper is as follows. In Section 2, we introduce some notation and terminology. In Section 3, we review relevant results on critical coupling for the case of complete graphs. In Section 4, we introduce the Kuramoto model on the complete bipartite graph, and review some of its basic properties. Here, we also give formal definitions of the notion of a global phase-locked state and critical coupling and show that a global phase-locked state always exist for sufficiently strong coupling (essentially proving that the critical coupling is a finite number). In Section 5 we derive our main result, which is a condition for criticality. To find the critical coupling we introduce an algorithm which is described in Section 6. Section 7 contains a numerical example to illustrate the results of the paper and finally, in Section 8 we present our concluding remarks.

2 Notation and terminology

Throughout the paper, $\mathbb{R}$ ($\mathbb{C}$) denotes the field of real (complex) numbers, while $\mathbb{R}^n$ ($\mathbb{C}^n$), $\mathbb{R}^{m\times n}$ ($\mathbb{C}^{m\times n}$) denote the vector spaces of n-tuples of real (complex) numbers and the space of $m \times n$ matrices with entries in $\mathbb{R}$ ($\mathbb{C}$) respectively. Frequently throughout the paper, we shall identify $\mathbb{R}^{m+n}$ with $\mathbb{R}^m \times \mathbb{R}^n$ in the obvious fashion. $\mathbb{N}$ denotes the set of positive integers. For $v \in \mathbb{R}^n$ and $1 \leq i \leq n$, v_i denotes the i^{th} component of v. Also, $\mathbb{R}_{\geq 0}$ denotes the set of all non-negative real numbers. For any finite set S, $|S|$ denotes the cardinality of S. For any real number x, $|x|$ denotes the absolute value of x and for a vector $v \in \mathbb{R}^n$, $\|v\|_\infty$ denotes the usual infinity norm given by $\max_{1\leq i\leq n} |v_i|$.

The graph theoretical terminology and notation adopted here is standard and for background on basic graph theory, the interested reader should consult [8]. As we shall only be concerned with *undirected* graphs, the term graph shall be used throughout to denote an undirected graph.

Definition 1 (Graph) *A graph G is an ordered pair (V, E), where V is a finite set of vertices and E is a set consisting of 2-element subsets of V.*

The elements of E are referred to as edges and we shall usually write uv (or vu) for the edge $\{u, v\}$ and say that u and v are neighbours in G. The *degree* of a vertex $v \in V$ is the number of neighbours of v. A particularly important class of graphs in the context of the Kuramoto model is the class of complete graphs which we define next.

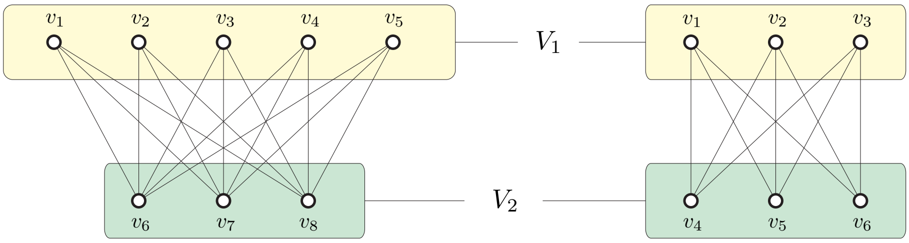
Definition 2 (Complete graph) *For any positive integer $n \geq 2$, the complete graph K_n has n vertices $V = \{v_1, \ldots, v_n\}$, and edges $E = \{\{v_i, v_j\} : 1 \leq i < j \leq n\}$.*

So, in K_n the degree of every vertex is equal to $n - 1$. In this paper, our focus shall be on a related class of graphs, namely complete bipartite graphs. First, let us recall the definition of a bipartite graph.

Definition 3 (Bipartite graph) *A graph $G = (V, E)$ is called* bipartite *if there exists a partition $V = V_1 \cup V_2$ of the vertex set so that every edge in E is of the form v_1v_2 for some $v_1 \in V_1$ and $v_2 \in V_2$.*

Definition 4 (Complete bipartite graph) *For positive integers $m, n \geq 1$, the complete bipartite graph $K_{m,n} = (V_1 \cup V_2, E)$ is a bipartite graph such that $|V_1| = m$, $|V_2| = n$ and for any two vertices $v_1 \in V_1$ and $v_2 \in V_2$, $v_1v_2 \in E$.*

Two examples of complete bipartite graphs are given in Figure 1.



(a) $K_{5,3}$ (b) $K_{3,3}$

Figure 1: Two examples of a complete bipartite graph. As illustrated, the vertex set $V := \{v_1, \ldots, v_8\}$ of the graph $K_{5,3} := (V, E)$ can be divided into subsets $V_1 := \{v_1, \ldots, v_5\}$ and $V_2 := \{v_6, v_7, v_8\}$ such that $E = V_1V_2 \cup V_2V_1$. The same holds for $K_{3,3}$, whose vertex set can be partitioned into subsets $V_1 := \{v_1, v_2, v_3\}$ and $V_2 := \{v_4, v_5, v_6\}$.

3 Global Phase-locking on complete graphs

Before considering the problem of global phase-locking on complete bipartite graphs, in this section we briefly review some relevant results on global phase-locking on complete graphs. The Kuramoto model for a system of coupled oscillators on a complete graph K_n, $n \geq 2$, is given as

$$\dot{\theta}_i \quad = \quad \omega_i + \frac{k}{n}\sum_{j=1}^{n} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, n, \tag{1}$$

where $\theta_i(\cdot) \in \mathbb{R}$ and $\omega_i \in \mathbb{R}$ denote the phase and intrinsic (or natural) frequency of oscillator i respectively, and $k \in \mathbb{R}_{\geq 0}$ denotes the coupling coefficient. One of the central problems in the literature on the Kuramoto model [18, 19, 1] has been to find conditions on the coupling strength k and the distribution of intrinsic frequencies, sometimes denoted by $g(\omega)$, under which the system (1) exhibits synchronous behaviour. This problem was essentially solved by Kuramoto himself, although subsequent to his work numerous others have contributed to a more complete solution in different ways. For an overview of classical results, see [44]; for a more recent survey, consult [1]. Although very powerful, Kuramoto's original approach has two obvious drawbacks. Firstly, it requires the frequency distribution $g(\omega)$ to be symmetric about the mean, which excludes certain distributions of practical interest. Secondly, it only works for settings with infinitely many oscillators (the so called thermodynamic limit case) as the Fokker-Planck equation describing the evolution of the oscillator density function does not, by its very nature, capture finite-size effects. The extension of classical results, on the value of the critical coupling and the instability of the incoherent state for example, to the finite-dimensional case has proven difficult, and papers on the same are few [36, 37, 25, 26]. See also [44] for a discussion of open problems in the finite-dimensional theory. In recent work [50], the authors of the present paper showed that the problem of determining the critical coupling of the Kuramoto model (1) boils down to solving a particular nonlinear equation, the solution of which can be obtained very efficiently using appropriate numerical tools. Aside from numerical implications, this enables us to prove a convergence result which may help bridge the gap between the finite-dimensional and the infinite-dimensional theory, as we hope to demonstrate in a future publication. We shall now briefly review the main contribution of the aforementioned work. But first we introduce some terminology.

Definition 5 (GPLS, complete graph) *Let $\theta^0 \in \mathbb{R}^n$ and let $\{\theta(t) : t \geq t_0\}$ be the solution of the system (1) with initial condition $\theta(t_0) = \theta^0$. We say θ^0 is a global phase-locked state (GPLS), if*

$$\theta_i(t) - \theta_j(t) \;=\; \theta_i^0 - \theta_j^0$$

for all $t \geq t_0$.

The existence of global phase-locked states in the Kuramoto model (1) is dependent on the value of the coupling strength k. The smallest value of k for which phase-locked states exist defines the critical coupling, as follows:

Definition 6 (Critical coupling, complete graph) *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\omega \in \mathbb{R}^n$ be given. Consider the Kuramoto model of coupled oscillators on the complete graph K_n (1). For this model, the critical coupling, k_c, is defined as follows:*

$$k_c \quad := \quad \min_k \left\{k \in \mathbb{R}_{\geq 0} : \textit{the system (1) admits a GPLS}\right\}.$$

Define $\Omega : \mathbb{R}^n \mapsto \mathbb{R}^n$,

$$\Omega_i(\omega) := \omega_i - \frac{1}{n}\sum_{j=1}^{n}\omega_j, \quad 1 \leq i \leq n. \tag{2}$$

We are now ready to reformulate the main result of [50]:

Theorem 1 *Let $\omega \in \mathbb{R}^n$ be given and suppose $\Omega(\omega) \neq 0$. Then the equation*

$$2\frac{1}{n}\sum_{j=1}^{n}\sqrt{1-\left(\frac{\Omega_j(\omega)}{u}\right)^2} = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{\sqrt{1-\left(\frac{\Omega_j(\omega)}{u}\right)^2}}. \tag{3}$$

has a unique solution $u^* \in (\|\Omega(\omega)\|_\infty, 2\|\Omega(\omega)\|_\infty]$, and we have that

$$k_c = \frac{u^*}{\frac{1}{n}\sum_{j=1}^{n}\sqrt{1-\left(\frac{\Omega_j(\omega)}{u^*}\right)^2}.} \tag{4}$$

The critical point u^* can be computed numerically to within user-defined precision `AbsTol` using the following algorithm.

Algorithm 1

1. $\mathtt{a} := \|\Omega\|_\infty$;
2. $\mathtt{u} := 2 \cdot \mathtt{a}$;
3. $\mathtt{AbsTol} := 10^{-6}$;
4. $\mathtt{Err} := 1$;
5. $\Delta_\mathtt{u} := \frac{1}{2}(\mathtt{u} - \mathtt{a})$;
6. `while` $|\mathtt{Err}| > \mathtt{AbsTol}$

    6.1. $\mathtt{Err} := \sum_\mathtt{j} \sqrt{1 - \left(\frac{\Omega_\mathtt{j}}{\mathtt{u}}\right)^2} - \frac{1}{2}\sum_\mathtt{j} \frac{1}{\sqrt{1-\left(\frac{\Omega_\mathtt{j}}{\mathtt{u}}\right)^2}}$;

    6.2. `if` $\mathtt{Err} \geq 0$

    $$\begin{array}{rcl} \mathtt{u} & := & \mathtt{u} - \Delta_\mathtt{u}; \\ \Delta_\mathtt{u} & := & \frac{1}{2}\Delta_\mathtt{u}; \end{array}$$

    6.3. `else`

    $$\begin{array}{rcl} \mathtt{u} & := & \mathtt{u} + \Delta_\mathtt{u}; \\ \Delta_\mathtt{u} & := & \frac{1}{2}\Delta_\mathtt{u}; \end{array}$$

    6.4. `end`

7. `end`

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4 Global phase-locking on a complete bipartite graph

Let $(m, n) \in \mathbb{N} \times \mathbb{N}$ be given. We consider a system of coupled oscillators on the complete bipartite graph $K_{m,n} = (V_1 + V_2, E)$. Without loss of generality we assume that the vertices $\{v_i\}$ are labeled in such a way that $v_1, v_2, \ldots, v_m$ belong to vertex set V_1 and $v_{m+1}, v_{m+2}, \ldots, v_{m+n}$ belong to vertex set V_2. We label oscillators according to their association with a vertex, i.e. we write oscillator i for the oscillator associated with vertex v_i. The coupling dynamics given by the Kuramoto model are as follows:

$$\dot{\theta}_i(t) = \omega_i + \frac{k}{m+n}\sum_{j=1}^{n}\sin(\phi_j(t)-\theta_i(t)), \quad i=1,\ldots,m; \tag{5}$$

$$\dot{\phi}_i(t) = \upsilon_i + \frac{k}{m+n}\sum_{j=1}^{m}\sin(\theta_j(t)-\phi_i(t)), \quad i=1,\ldots,n. \tag{6}$$

In the above notation $\theta_i(\cdot) \in \mathbb{R}$ and $\phi_j(\cdot) \in \mathbb{R}$ denote the phase of oscillator i and the phase of oscillator $m+j$, respectively. Likewise, $\omega_i \in \mathbb{R}$ and $\upsilon_j \in \mathbb{R}$ denote the intrinsic frequency of oscillators i and $m+j$, respectively. Lastly, the symbol $k \in \mathbb{R}_{\geq 0}$ denotes the coupling coefficient.

The system of equations (5)-(6) admits a range of qualitatively different solutions, depending on the value of the (bifurcation) parameter k. In this paper, we are concerned with solutions associated with the existence of global phase-locked states, which we define as follows:

Definition 7 (GPLS, complete bipartite graph) *Let $\Phi^0 \in \mathbb{R}^m \times \mathbb{R}^n$ and let $\Phi(\cdot) := \big(\theta^*(\cdot), \quad \phi^*(\cdot)\big)$, $t \geq t_0$ be a solution of (5)-(6) with initial condition $\Phi(t_0) = \Phi^0$. We say that Φ^0 is a* global phase-locked state *(GPLS) if*

$$\Phi_i(t) - \Phi_j(t) = \Phi_i^0 - \Phi_j^0, \quad 1 \leq (i,j) \leq m+n \;;\; t \geq t_0. \tag{7}$$

Note that to every global phase-locked solution x^* there corresponds a 1-dimensional manifold

$$\mathcal{M} := \{x \in \mathbb{R}^{m+n} : \; x = x^* + \mathbf{1}_{m+n}\, t, \; t \in \mathbb{R}\}$$

that is invariant under the original system dynamics (5)-(6). This correspondence is unique up to equivalence in the sense of the following definition:

Definition 8 (Equivalence of phase-locked states) *Let $x, x' \in \mathbb{R}^m \times \mathbb{R}^n$ be two global phase-locked states in the sense of Definition 7. We say that x is equivalent to x', (and write $x \simeq x'$) if there exists $c \in \mathbb{R}$ such that $x = x' + c\mathbf{1}_{m+n}$.*

From the above, and from inspection of (5)-(6) it follows easily that $(\xi^*, \nu^*) \in \mathbb{R}^m \times \mathbb{R}^n$ is a GPLS if and only if

$$\left(\sum_{j=1}^{m}\omega_j + \sum_{j=1}^{n}\upsilon_j\right) - (m+n)\,\omega_i = k\sum_{j=1}^{n}\sin(\nu_j^* - \xi_i^*), \quad i=1,\ldots,m;$$

$$\left(\sum_{j=1}^{m}\omega_j + \sum_{j=1}^{n}\upsilon_j\right) - (m+n)\,\upsilon_i = k\sum_{j=1}^{m}\sin(\xi_j^* - \nu_i^*), \quad i=1,\ldots,n.$$

Let $f : \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R}^m$, and $g : \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R}^n$ be given as

$$(m+n)\,f(\xi,\nu) := \left(\textstyle\sum_{j=1}^{n}\sin(\nu_j - \xi_1), \quad \cdots, \quad \sum_{j=1}^{n}\sin(\nu_j - \xi_m)\right)^T; \tag{8}$$

$$(m+n)\,g(\xi,\nu) := \left(\textstyle\sum_{j=1}^{m}\sin(\xi_j - \nu_1), \quad \cdots, \quad \sum_{j=1}^{m}\sin(\xi_j - \nu_n)\right)^T. \tag{9}$$

Also, let $\mathbf{1}_m$ denote the vector of length m all of whose components are 1 and define $\Omega : \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R}^m$, and $\Upsilon : \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R}^n$,

$$\Omega(\omega, \upsilon) := \omega - \frac{1}{m+n}\left(\sum_{j=1}^{m} \omega_j + \sum_{j=1}^{n} \upsilon_j\right)\mathbf{1}_m;$$

$$\Upsilon(\omega, \upsilon), := \upsilon - \frac{1}{m+n}\left(\sum_{j=1}^{m} \omega_j + \sum_{j=1}^{n} \upsilon_j\right)\mathbf{1}_n.$$

Note that $\begin{pmatrix}\Omega & \Upsilon\end{pmatrix}^T$ is a linear projection from $\mathbb{R}^{m+n}$ onto the $(m+n-1)$-dimensional linear subspace $\{x \in \mathbb{R}^{m+n} : \sum_{j=1}^{m+n} x_j = 0\}$. In this paper we present a method for computing the critical coupling associated with the system of equations (5)-(6), where the critical coupling is defined as follows.

Definition 9 (Critical coupling, complete bipartite graph) *Let $(m,n) \in \mathbb{N} \times \mathbb{N}$ and let $\omega \in \mathbb{R}^m$ and $\upsilon \in \mathbb{R}^n$ be given. Consider the Kuramoto model of coupled oscillators on the complete bipartite graph $K_{m,n}$ (5)-(6). For this model, we define the critical coupling, k_c, as follows:*

$$k_c \quad := \quad \min_k \left\{ k \in \mathbb{R}_{\geq 0} : \exists (\xi^*, \nu^*) \in \mathbb{R}^m \times \mathbb{R}^n \text{ } s.t. \begin{pmatrix}\Omega \\ \Upsilon\end{pmatrix} = -k \begin{pmatrix} f(\xi^*, \nu^*) \\ g(\xi^*, \nu^*)\end{pmatrix}\right\}$$

(omitting the argument (ω, υ) for notational convenience).

The critical coupling is thus the smallest nonnegative coupling for which the system (5)-(6) admits a global phase-locked state in the sense of Definition 7.

4.1 The order parameter

Let $\mathbb{D}$ denote the closed complex unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. For $p \in \mathbb{N}$, we define the order parameter $r^p : \mathbb{R}^p \mapsto \mathbb{D}$ as:

$$r^p(\xi) \; := \; \frac{1}{p}\sum_{j=1}^{p} e^{i\xi_j}. \tag{10}$$

Let r_0^p denote the set $\{\xi \in \mathbb{R}^p : \; r^p(\xi) = 0\}$. For the purposes of this paper, we define the arctangent as a function from $\mathbb{R}^2 \setminus \{(0,0)\}$ onto $[0, 2\pi)$, as follows:

Definition 10 *For all $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ let* $\arctan(a,b)$ *denote the unique real number $c \in [0, 2\pi)$ such that*

$$\cos(c) \; = \; \frac{a}{\sqrt{a^2+b^2}} \quad ; \quad \sin(c) \; = \; \frac{b}{\sqrt{a^2+b^2}} \tag{11}$$

Using Definition 10, we may may express $r^p(\cdot)$ in polar coordinates:

$$r^p(\xi) = \begin{cases} R^p(\xi)e^{i\psi^p(\xi)} & \xi \in \mathbb{R}^p \backslash r_0^p \\ 0 & \xi \in r_0^p \end{cases}. \tag{12}$$

Here, $R^p : \mathbb{R}^p \mapsto [0,1]$ and $\psi^p : \mathbb{R}^p \setminus r_0^p \mapsto [0, 2\pi)$, are respectively defined as

$$R^p(\xi) := \sqrt{\left(\frac{1}{p}\sum_{j=1}^{p}\cos(\xi_j)\right)^2 + \left(\frac{1}{p}\sum_{j=1}^{p}\sin(\xi_j)\right)^2}, \tag{13}$$

and

$$\psi^p(\xi) \quad := \quad \arctan(\frac{1}{p}\sum_{j=1}^{p}\cos(\xi_j), \frac{1}{p}\sum_{j=1}^{p}\sin(\xi_j)). \tag{14}$$

The following proposition summarizes some useful properties of the maps R^p and ψ^p:

Proposition 1 *The maps R^p and ψ^p defined by (13), (14) satisfy the following relations:*

1. $R^p(\xi + c\mathbf{1}_p) \; = \; R^p(\xi), \quad \forall \xi \in \mathbb{R}^p, \quad \forall c \in \mathbb{R}$*;*
2. $R^p(\xi) \; = \; \frac{1}{p}\sum_{j=1}^{p}\cos(\psi^p(\xi) - \xi_j), \quad \forall \xi \in \mathbb{R}^p \setminus r_0^p$*;*
3. $\psi^p(\xi + c\mathbf{1}_p) = \psi^p(\xi) + c \; (mod \; 2\pi) \quad \forall \xi \in \mathbb{R}^p, \quad \forall c \in \mathbb{R}$*;*
4. $\sum_{j=1}^{p}\sin(\psi^p(\xi) - \xi_j) \; = \; 0, \quad \forall \xi \in \mathbb{R}^p \setminus r_0^p$*;*

Proof: First of all, note that for all $\xi \in \mathbb{R}^p$,

$$R^p(\xi + c\mathbf{1}_p) \; := \; \left|\frac{1}{p}\sum_{j=1}^{p} e^{i(\xi_j + c)}\right| \; = \; \left|e^{ic}\right| R^p(\xi) \; = \; R^p(\xi).$$

This proves property 1. Next observe that for $\xi \in \mathbb{R}^p \setminus r_0^p$,

$$\begin{aligned} R^p(\xi) \quad &= \quad e^{-i\psi^p(\xi)}r^p(\xi) \\ &= \quad \frac{1}{p}\sum_{j=1}^{p} e^{i(\xi_j - \psi^p(\xi))}. \end{aligned} \tag{15}$$

Equating real and imaginary parts in (15), we see that for $\xi \in \mathbb{R}^p \setminus r_0^p$:

$$R^p(\xi) = \frac{1}{p}\sum_{j=1}^{p}\cos(\psi^p(\xi) - \xi_j); \tag{16}$$

$$\sum_{j=1}^{p} \sin(\psi^p(\xi) - \xi_j) = 0. \tag{17}$$

This proves properties 2 and 4. Finally, by definition of $\psi^p(\cdot)$ we have that

$$\begin{aligned}
\cos(\psi^p(\xi + c\mathbf{1}_p)) &= \frac{1}{R^p(\xi + c\mathbf{1}_p)} \frac{1}{p} \sum_{j=1}^{p} \cos(\xi_j + c) \\
&= \frac{1}{R^p(\xi)} \frac{1}{p} \sum_{j=1}^{p} \cos(\xi_j)\cos(c) - \sin(\xi_j)\sin(c) \\
&= \cos(\psi^p(\xi))\cos(c) - \sin(\psi^p(\xi))\sin(c)
\end{aligned}$$

This implies that

$$\cos(\psi^p(\xi + c\mathbf{1}_p)) = \cos(\psi^p(\xi) + c). \tag{18}$$

Similarly we have that

$$\sin(\psi^p(\xi + c\mathbf{1}_p)) = \sin(\psi^p(\xi) + c). \tag{19}$$

It follows that $\psi^p(\xi + c\mathbf{1}_p) = \psi^p(\xi) + c \pmod{2\pi}$. This proves property 3. ■

Using the definition of the order parameter, we can rewrite the functions f and g, previously defined in (8) and (9), as follows:

$$(m+n) f_i(\xi, \nu) = \begin{cases} n\, R^n(\nu)\sin(\psi^n(\nu) - \xi_i), & \nu \in \mathbb{R}^n \setminus r_0^n \\ 0, & \nu \in r_0^n, \end{cases} \tag{20}$$

and

$$(m+n) g_i(\xi, \nu) = \begin{cases} m\, R^m(\xi)\sin(\psi^m(\xi) - \nu_i), & \xi \in \mathbb{R}^m \setminus r_0^m \\ 0, & \xi \in r_0^m. \end{cases} \tag{21}$$

4.2 Global phase-locking states: the homogeneous case

In this section we assume that $\Omega(\omega, \upsilon) = 0$ and $\Upsilon(\omega, \upsilon) = 0$. Thus we assume that there exists $c \in \mathbb{R}$ such that $\omega_i = \upsilon_j = c$ for $1 \le i \le m$ and $1 \le j \le n$. Under this assumption the system of ODEs (5)-(6) simplifies to

$$\begin{pmatrix} \dot{\theta}(t) \\ \dot{\phi}(t) \end{pmatrix} = c\mathbf{1}_{m+n} + k \begin{pmatrix} f(\theta, \phi) \\ g(\theta, \phi) \end{pmatrix} \tag{22}$$

It is easy to see that the homogeneous system (22) admits a GPLS for all $k > 0$ (as it does, of course, for the trivial case $k = 0$). Indeed, for $k > 0$, we have that $(\xi, \nu) \in \mathbb{R}^m \times \mathbb{R}^n$ is a GPLS if and only if $f(\xi, \nu) = 0$ and $g(\xi, \nu) = 0$. This motivates the following result.

Proposition 2 *Let f and g be defined by (8) and (9) respectively, and let $(\xi, \nu) \in \mathbb{R}^m \times \mathbb{R}^n$. If (ξ, ν) satisfies the following condition:*

$$\sin(\xi_i - \nu_j) \quad = \quad 0, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n. \tag{23}$$

then (ξ, ν) is a solution of the homogeneous system

$$\begin{pmatrix} f(\xi, \nu) \\ g(\xi, \nu) \end{pmatrix} \quad = \quad 0. \tag{24}$$

Moreover, if (ξ, ν) is a solution of (24) and if, in addition, $R^m(\xi)R^n(\nu) \neq 0$, then (23) holds.

Proof: The first part follows trivially from inspection of (8) and (9). To prove the second part, suppose $R^m(\xi) \neq 0$. Then $\psi^m(\xi)$ is well defined and by (9) we have that $\sin(\psi^m(\xi) - \nu_i) = 0$ for all i. This implies that there exists $k \in \mathbb{Z}^n$ such that $\nu_i - \nu_j = (k_i - k_j)\pi$ and hence $\sin(\nu_i - \nu_j) = 0$ for all (i, j). By analogy we have that $\sin(\psi^n(\nu) - \xi_i) = 0$ and $\sin(\xi_i - \xi_j) = 0$ for all (i, j). In addition, by Proposition 1 (property 3) we have that $\sin(\psi^n(\nu) - \nu_i) = \sin(\psi^n(\nu - \nu_i)) = \frac{1}{R^n(\nu)} \frac{1}{n} \sum_{j=1}^n \sin(\nu_j - \nu_i) = 0$ for all i. It follows that

$$\begin{aligned} \sin(\xi_i - \nu_j) \quad &= \quad \sin(\xi_i - \psi^n(\nu) - (\nu_j - \psi^n(\nu)) \\ &= \quad \underbrace{\sin(\xi_i - \psi^n(\nu))}_{=0} \cos(\nu_j - \psi^n(\nu)) - \\ &\qquad \cos(\xi_i - \psi^n(\nu)) \underbrace{\sin(\nu_j - \psi^n(\nu))}_{=0} \quad = \quad 0 \end{aligned}$$

for all (i, j). This concludes the proof ■

Remark 1 *Proposition 2 provides a partial characterization of the set of solutions to the homogeneous equation (24). Condition (23) is a sufficient but not a necessary condition. However, it becomes necessary when the extra condition $R^m(\xi)R^n(\nu) \neq 0$ is imposed, as per the second part of the proposition. The complete set of homogeneous solutions can be found by taking the union of the set of solutions defined by conditions (23) and the set $\{(\xi, \nu) \in \mathbb{R}^m \times \mathbb{R}^n : R^m(\xi)R^n(\nu) = 0\}$. In general, these sets are not disjoint. In fact, it can be shown that these sets are disjoint if and only if both m and n are odd.*

In the remainder of the paper, we shall use the notation $\mathcal{S}_{\text{hom}}$ to denote the set of all global phase-locked states corresponding to the homogeneous system (24). That is,

$$\mathcal{S}_{\text{hom}} \; := \; \{(\xi, \nu) \in \mathbb{R}^m \times \mathbb{R}^n : \begin{pmatrix} f(\xi, \nu) \\ g(\xi, \nu) \end{pmatrix} = 0\}.$$

Figure 2 shows a particular GPLS $(\xi^*, \nu^*) \in \mathcal{S}_{\text{hom}}$ for the case of the complete bipartite graph $K_{5,3}$.

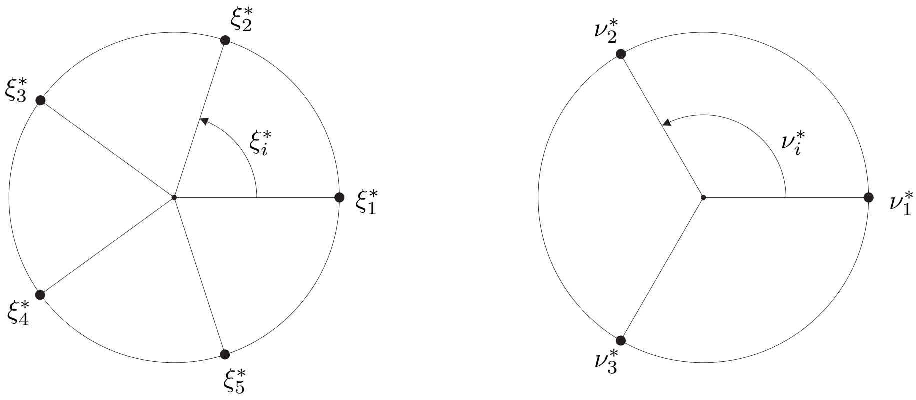

Figure 2: One solution $(\xi^*, \nu^*) \in \mathcal{S}_{\text{hom}}$ of the system of equations (24) for the case of the complete bipartite graph $K_{5,3}$. Note that (ξ^*, ν^*) does not satisfy the conditions of Proposition 2. In this particular example however, we have that $R^m(\xi^*) = 0$ and $R^n(\nu^*) = 0$, and therefore (ξ^*, ν^*) is a GPLS by Remark 1.

4.3 Global phase-locking under strong coupling

Let $\Pi^1 : \mathbb{R}^{m+n} \mapsto \mathbb{R}^m$ and $\Pi^2 : \mathbb{R}^{m+n} \mapsto \mathbb{R}^n$ be given by

$$\Pi^1(x) \;:=\; \begin{pmatrix} x_1 & \cdots & x_m \end{pmatrix}^T \quad ; \quad \Pi^2(x) \;:=\; \begin{pmatrix} x_{m+1} & \cdots & x_{m+n} \end{pmatrix}^T$$

respectively, and define $F : \mathbb{R}^{m+n} \mapsto \mathbb{R}^{m+n}$,

$$F(x) \quad := \quad \begin{pmatrix} f(\Pi^1(x), \Pi^2(x)) \\ g(\Pi^1(x), \Pi^2(x)) \end{pmatrix}. \tag{25}$$

Also, let $\Xi : \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R}^{m+n}$ be given as

$$\Xi(\omega, \upsilon) \;:=\; \begin{pmatrix} \Omega(\omega, \upsilon) \\ \Upsilon(\omega, \upsilon) \end{pmatrix}. \tag{26}$$

Lastly, define the truncation operator $T : \mathbb{R}^{m+n} \mapsto \mathbb{R}^{m+n-1}$,

$$(Tz) \;:=\; z_i, \quad i = 1, \ldots, m+n-1.$$

We now state a technical lemma which will play an important role in the proof of Proposition 3 below. As the proof of this lemma is rather long and technical, we defer it to the Appendix in the interests of clarity.

Lemma 1 *Let $\tilde{F} : \mathbb{R}^{m+n-1} \to \mathbb{R}^{m+n-1}$ be defined by $\tilde{F}(y) = TF(y; 0)$ and suppose $y^* \in \mathbb{R}^{m+n-1}$ satisfies $\tilde{F}(y^*) = 0$ and $R^m(\Pi^1(y^*; 0))R^n(\Pi^2(y^*; 0)) \neq 0$. Then the jacobian $J : \mathbb{R}^{m+n-1} \mapsto \mathbb{R}^{m+n-1}$,*

$$J(y) \;:=\; \frac{\partial \tilde{F}(\zeta)}{\partial \zeta}(y)$$

is nonsingular at $y = y^*$.

We can now state and prove the following proposition, which establishes that the critical coupling is finite for complete bipartite graphs and any given set of finite natural frequencies.

Proposition 3 *Let $(\omega, \upsilon) \in \mathbb{R}^m \times \mathbb{R}^n$ be given and let $\Xi(\omega, \upsilon)$ be defined by (26). Furthermore, let $x^* \in \mathcal{S}_{hom}$ and suppose $R^m(\Pi^1(x^*))R^n(\Pi^2(x^*)) \neq 0$. Then there exists $K > 0$ and an open set $V \subset \mathbb{R}^{m+n-1}$ containing Tx^*, such that for all $k > K$, the system*

$$kF(y_1, \ldots, y_{m+n-1}; x^*_{m+n}) \quad = \quad \Xi(\omega, \upsilon) \tag{27}$$

has a unique solution y' on V.

Proof: Observe that $\sum_{j=1}^{m+n} F_j(\cdot) \equiv 0$ and $\sum_{j=1}^{m+n} \Xi_j(\cdot, \cdot) \equiv 0$. This implies that y is a solution of (27) if and only if $kTF(y; x^*_{m+n}) = T\Xi(\omega, \upsilon)$ or

$$\begin{aligned} kF_1(y_1, \ldots, y_{m+n-1}; x^*_{m+n}) &= \Xi_1(\omega, \upsilon) \\ &\vdots \\ kF_{m+n-1}(y_1, \ldots, y_{m+n-1}; x^*_{m+n}) &= \Xi_{m+n-1}(\omega, \upsilon). \end{aligned} \tag{28}$$

Next, assume without loss of generality that $x^*_{m+n} = 0$ and define $y^* = Tx^*$ and $\tilde{F} : \mathbb{R}^{m+n-1} \mapsto \mathbb{R}^{m+n-1}$, $\tilde{F}(y) := TF(y; 0)$. Then under the hypotheses of the proposition we have $\tilde{F}(y^*) = 0$ and it follows from Lemma 1 that the jacobian $J : \mathbb{R}^{m+n-1} \mapsto \mathbb{R}^{m+n-1}$,

$$J(y) \; := \; \frac{\partial \tilde{F}(\zeta)}{\partial \zeta}(y)$$

is nonsingular at $y = y^*$. Then by the inverse function theorem there exists an open set U, containing the origin, such that on U, the map $\tilde{F}$ has a unique inverse, $\tilde{F}^{-1}$. Define $V := \tilde{F}^{-1}(U)$ and note that V is open by continuity of $\tilde{F}^{-1}$. Now pick $\epsilon > 0$ such that $\mathcal{B}_\epsilon := \{z \in \mathbb{R}^{n+m-1} : \|z\| < \epsilon\} \subset U$ and let $K = \frac{\|\tilde{\Xi}\|}{\epsilon}$. Then it follows that (28) has a unique solution on V for all $k \geq K$. ∎

5 Critical coupling

In the previous section we showed that, for large enough values of k, the heterogeneous system (5)-(6) will exhibit global phase-locked states in (small) open sets containing a global phase-locked state of the associated homogeneous system (24). A particular implication of this result is that the critical coupling k_c is always finite. In the present section we will refine this result and show that the value of the critical coupling can be obtained by solving a particular system of nonlinear equations, the solution of which can be determined very efficiently by means of a bisection algorithm.

Our first result relates the existence of global phase-locked states to solutions of a two-dimensional system of equations, as follows:

Theorem 2 *Let $k > 0$ and let $(\omega, \upsilon) \in \mathbb{R}^m \times \mathbb{R}^n$. Then the system (5)-(6) admits a GPLS if and only if there exists $(x, y) \in [\frac{m+n}{m}\frac{1}{k}\|\Upsilon\|_\infty, 1] \times [\frac{m+n}{n}\frac{1}{k}\|\Omega\|_\infty, 1]$ and $(a, b) \in \{-1, 1\}^m \times \{-1, 1\}^n$ such that*

$$x^2 = \Big(\frac{m+n}{mn}\sum_{j=1}^{m}\frac{\Omega_j}{ky}\Big)^2 + \Big(\frac{1}{m}\sum_{j=1}^{m} a_j\sqrt{1-(\frac{m+n}{n}\frac{\Omega_j}{ky})^2}\Big)^2; \qquad (29)$$

$$y^2 = \Big(\frac{m+n}{mn}\sum_{j=1}^{n}\frac{\Upsilon_j}{kx}\Big)^2 + \Big(\frac{1}{n}\sum_{j=1}^{n} b_j\sqrt{1-(\frac{m+n}{m}\frac{\Upsilon_j}{kx})^2}\Big)^2. \qquad (30)$$

Proof: We distinguish four cases: (a) $\Omega(\omega, \upsilon) = 0$ and $\Upsilon(\omega, \upsilon) = 0$; (b) $\Omega(\omega, \upsilon) = 0$ and $\Upsilon(\omega, \upsilon) \neq 0$; (c) $\Omega(\omega, \upsilon) \neq 0$ and $\Upsilon(\omega, \upsilon) = 0$; (d) $\Omega(\omega, \upsilon) \neq 0$ and $\Upsilon(\omega, \upsilon) \neq 0$. Case (a) corresponds to the homogeneous system (24) that we discussed in the previous section, where we showed that global phase-locked states exist for every $k > 0$. Hence, what we need to show for this case is that for every $k > 0$ there exists $(a, b) \in \{-1, 1\}^m \times \{-1, 1\}^n$ such that (29)-(30) has a solution $(x, y) \in [0, 1] \times [0, 1]$. Under the given hypotheses it suffices to note that the system $\{x^2 = 1, y^2 = 1\}$ has a nonnegative solution that does not depend on k. Case (b) corresponds to the semi-homogeneous system

$$\left.\begin{aligned} 0 &= \tfrac{k}{m+n}\textstyle\sum_{j=1}^{n}\sin(\nu_j - \xi_i) \quad i = 1, \ldots, m; \\ -\Upsilon_i &= \tfrac{k}{m+n}\textstyle\sum_{j=1}^{m}\sin(\xi_j - \nu_i) \quad i = 1, \ldots, n. \end{aligned}\right\} \qquad (31)$$

This system has a solution if and only if $k \geq \frac{m+n}{m}\|\Upsilon\|_\infty$. The necessity is obvious. For sufficiency, note that if $\xi_i = c$ for $1 \leq i \leq m$, we can always find $\nu_1, \ldots, \nu_n$ satisfying the second equation above. But then $\frac{km}{m+n}\sin(\nu_j - c) = \Upsilon_j$ for $1 \leq j \leq n$ and

$$\frac{k}{m+n}\sum_{j=1}^{n}\sin(\nu_j - c) = m(\sum_{j=1}^{n}\Upsilon_j + \sum_{j=1}^{m}\Omega_j) = 0.$$

Hence, what we need to show is that there exists $(a, b) \in \{-1, 1\}^m \times \{-1, 1\}^n$ such that the system (29)-(30) has a solution $(x, y) \in [\frac{m+n}{m}\frac{1}{k}\|\Upsilon\|_\infty, 1] \times [0, 1]$ if and only if $k \geq \frac{m+n}{m}\|\Upsilon\|_\infty$. We note that the condition $x \in [\frac{m+n}{m}\frac{1}{k}\|\Upsilon\|_\infty, 1]$ implies that $k \geq \frac{m+n}{m}\|\Upsilon\|_\infty$, which proves necessity. To prove sufficiency suppose $k \geq \frac{m+n}{m}\|\Upsilon\|_\infty$. Under the given hypotheses it suffices to show that the system

$$\Big\{x^2 = 1, y^2 = \Big(\frac{m+n}{mn}\sum_{j=1}^{n}\frac{\Upsilon_j}{k}\Big)^2 + \Big(\frac{1}{n}\sum_{j=1}^{n}\sqrt{1-(\frac{m+n}{m}\frac{\Upsilon_j}{k})^2}\Big)^2\Big\}$$

has a solution on $[\frac{m+n}{m}\frac{1}{k}\|\Upsilon\|_\infty, 1] \times [0, 1]$. But this amounts to showing that

$$\Big(\frac{m+n}{mn}\sum_{j=1}^{n}\frac{\Upsilon_j}{k}\Big)^2 + \Big(\frac{1}{n}\sum_{j=1}^{n}\sqrt{1-(\frac{m+n}{m}\frac{\Upsilon_j}{k})^2}\Big)^2 \leq 1.$$

But if $k \geq \frac{m+n}{m}\|\Upsilon\|_\infty$, then for each $j = 1, \ldots, n$, we can write $\frac{m+n}{m}\frac{\Upsilon_j}{k} = \cos(x_j)$ for some $x_j \in [-\pi, \pi]$ and thus the above expression reduces to

$$\Big(\frac{1}{n}\sum_{j=1}^{n}\cos(x_j)\Big)^2 + \Big(\frac{1}{n}\sum_{j=1}^{n}\sin(x_j)\Big)^2 = \Big|\frac{1}{n}\sum_{j=1}^{n}e^{ix_j}\Big|^2 \leq 1$$

as required. Case (c) is analogous to case (b). We now consider case (d). Let (ξ, ν) be a GPLS in the sense of Definition 7. The conditions $\Omega(\omega, \upsilon) \neq 0$ and $\Upsilon(\omega, \upsilon) \neq 0$ imply that $R^m(\xi) \neq 0$ and by $R^n(\nu) \neq 0$ respectively. It follows that $\psi^m(\xi)$ and $\psi^n(\nu)$ are well defined and by (20)-(21) we have that

$$\begin{aligned} \sin(\psi^n(\nu) - \xi_i) &= -\frac{(\frac{m+n}{n})\Omega_i}{kR^n(\nu)}, \quad i = 1, \ldots, n; \\ \sin(\psi^m(\xi) - \nu_i) &= -\frac{(\frac{m+n}{m})\Upsilon_i}{kR^m(\xi)}, \quad i = 1, \ldots, m; \end{aligned}$$

Recall that $(R^m(\xi))^2 = (R^m(\xi - \psi^n(\nu)\mathbf{1}_m))^2$ by Proposition 1. It follows that

$$\begin{aligned} (R^m(\xi))^2 &= \Big(\frac{1}{m}\sum_{j=1}^{m}\sin(\xi_j - \psi^n(\nu))\Big)^2 + \Big(\frac{1}{m}\sum_{j=1}^{m}\cos(\xi_j - \psi^n(\nu))\Big)^2; \\ (R^n(\nu))^2 &= \Big(\frac{1}{n}\sum_{j=1}^{n}\sin(\nu_j - \psi^m(\xi))\Big)^2 + \Big(\frac{1}{n}\sum_{j=1}^{n}\cos(\nu_j - \psi^m(\xi))\Big)^2. \end{aligned}$$

Now let $a \in \{-1, 1\}^m$ and $b \in \{-1, 1\}^n$ be respectively defined as

$$\begin{aligned} a_i &:= \begin{cases} +1 & \cos(\xi_i - \psi^n(\nu)) \geq 0 \\ -1 & \cos(\xi_i - \psi^n(\nu)) < 0 \end{cases}, \quad 1 \leq i \leq m; \\ b_i &:= \begin{cases} +1 & \cos(\nu_i - \psi^m(\xi)) \geq 0 \\ -1 & \cos(\nu_i - \psi^m(\xi)) < 0 \end{cases}, \quad 1 \leq i \leq n. \end{aligned}$$

Let $x := R^m(\xi)$, $y := R^n(\nu)$. Then the result follows by substitution. This proves necessity. To prove sufficiency, let $(a', b') \in \{-1, 1\}^m \times \{-1, 1\}^n$ be given and let (x, y) be a solution of (29)-(30) with $(a, b) = (a', b')$. Consider the system of equations

$$\left\{\begin{aligned} \sin(u_i) &= -\frac{(\frac{m+n}{n})\Omega_i}{ky} \\ a_i' \cos(u_i) &\geq 0 \end{aligned}\right. , \quad i = 1, \ldots, m; \tag{32}$$

$$\left\{\begin{aligned} \sin(v_i) &= -\frac{(\frac{m+n}{m})\Upsilon_i}{kx} \\ b_i' \cos(v_i) &\geq 0 \end{aligned}\right. , \quad i = 1, \ldots, n. \tag{33}$$

Under the hypotheses of the proposition, the system (32)-(33) has a unique solution (u', v') on $(-\pi, \pi]^m \times (-\pi, \pi]^n$. We claim that either $\psi^m(u') = -\psi^n(v')$ or $\psi^m(u') - \psi^n(v') = \pi \pmod{2\pi}$. To prove this, recall that $\sum_{j=1}^{m}\Omega_j + \sum_{j=1}^{n}\Upsilon_j =$

0. By (32)-(33) we have that $\sum_{j=1}^n \sin(v_j') = -\frac{ny}{mx}\sum_{j=1}^m \sin(u_j')$. If we can show that $|\sum_{j=1}^n \cos(v_j')| = |\frac{ny}{mx}\sum_{j=1}^m \cos(u_j')|$, this claim will be established. By (29) and (30) we have that

$$\begin{aligned}
(\sum_{j=1}^n \cos(v_j'))^2 &= n^2y^2 - (\sum_{j=1}^n \sin(v_j'))^2 \\
&= \frac{n^2y^2}{m^2x^2}[m^2x^2 - (\sum_{j=1}^m \sin(u_j'))^2] \\
&= (\frac{ny}{mx}\sum_{j=1}^m \cos(u_j'))^2,
\end{aligned}$$

where the last line follows from (29). This proves the claim. Indeed, the above implies that either

$$\sum_{j=1}^n \cos(v_j') = \frac{ny}{mx}\sum_{j=1}^m \cos(u_j'), \tag{34}$$

in which case $\psi^m(u') = -\psi^n(v')$, or

$$\sum_{j=1}^n \cos(v_j') = -\frac{ny}{mx}\sum_{j=1}^m \cos(u_j'), \tag{35}$$

in which case $\psi^m(u') - \psi^n(v') = \pi \pmod{2\pi}$. Suppose (35) holds. Define $(a'', b'') := (a', -b')$ and observe that (x, y) is a solution of (29)-(30) with $(a, b) = (a'', b'')$. Moreover, if (u'', v'') denotes the solution of (32)-(33) with (a', b') replaced with (a'', b''), then we have that $\psi^m(u'') = -\psi^n(v'')$. This implies that we can assume without loss of generality that (34) holds, and hence that $\psi^m(u') = -\psi^n(v')$. Now define

$$\begin{aligned}
\xi_i &:= -u_i', \quad i = 1, \ldots m; \\
\nu_i &:= -v_i' - \psi^m(u'), \quad i = 1, \ldots n;
\end{aligned}$$

Then we have that

$$\begin{aligned}
\sin(\psi^n(\nu) - \xi_i) &= \sin(-\psi^n(v') - \psi^m(u') + u_i) = \sin(u_i), \quad i = 1, \ldots, m; \\
\sin(\psi^m(\xi) - \nu_i) &= \sin(-\psi^m(u') + \psi^m(u') + v_i) = \sin(v_i), \quad i = 1, \ldots, n.
\end{aligned}$$

If we can show that $R^m(\xi) = x$ and $R^n(\xi) = y$ then it follows that (ξ, ν) is a GPLS and the proof is finished. We have that

$$\begin{aligned}
(R^m(\xi))^2 &= (\frac{1}{m}\sum_{j=1}^m \sin(\xi_j - \psi^n(\nu)))^2 + (\frac{1}{m}\sum_{j=1}^m \cos(\xi_j - \psi^n(\nu)))^2; \\
&= \Big(\frac{m+n}{mn}\sum_{j=1}^m \frac{\Omega_j}{ky}\Big)^2 + \Big(\frac{1}{m}\sum_{j=1}^m a_j\sqrt{1 - (\frac{m+n}{n}\frac{\Omega_j}{ky})^2}\Big)^2; \\
&= x^2, \qquad (36)
\end{aligned}$$

where the last step follows from the hypotheses of the proposition. This implies that $R^m(\xi) = x$ and by analogy we have that $R^n(\nu) = y$. This concludes the proof. ∎

Define

$$\begin{aligned} I_v &:= [\big(\tfrac{m+n}{n}\|\Omega\|_\infty\big)^2, \infty) \quad , \quad I_v^\circ := (\big(\tfrac{m+n}{n}\|\Omega\|_\infty\big)^2, \infty) \\ I_w &:= [\big(\tfrac{m+n}{m}\|\Upsilon\|_\infty\big)^2, \infty) \quad , \quad I_w^\circ := (\big(\tfrac{m+n}{m}\|\Upsilon\|_\infty\big)^2, \infty) \end{aligned} \tag{37}$$

and let $p : I_v \times \{-1,1\}^m \mapsto [0,1]$, and $q : I_w \times \{-1,1\}^n \mapsto [0,1]$ be given as

$$p(v,a) := \Big(\frac{m+n}{mn}\sum_{j=1}^m \Omega_j\Big)^2 \frac{1}{v} + \Big(\frac{1}{m}\sum_{j=1}^m a_j\sqrt{1-(\frac{m+n}{n}\Omega_j)^2\frac{1}{v}}\Big)^2; \tag{38}$$

$$q(w,b) := \Big(\frac{m+n}{mn}\sum_{j=1}^n \Upsilon_j\Big)^2 \frac{1}{w} + \Big(\frac{1}{n}\sum_{j=1}^n b_j\sqrt{1-(\frac{m+n}{m}\Upsilon_j)^2\frac{1}{w}}\Big)^2. \tag{39}$$

In terms of the above notation, Theorem 2 states that the system (5)-(6) admits a GPLS if and only if there exist $(v,w) \in I_v \times I_w$ and $(a,b) \in \{-1,1\}^m \times \{-1,1\}^n$ such that $w = k^2 p(v,a)$ and $v = k^2 q(w,b)$. It follows that the critical coupling is the smallest k for which such a solution exists. Before we proceed, let us illustrate the result of Theorem 2 with an example. We consider the case of the complete bipartite graph $G = K_{5,4}$. Thus, let $(m,n) = (5,4)$ and let ω, υ be given as

$$\omega := \begin{pmatrix}1 & 3 & 5 & 7 & 9\end{pmatrix}^T \quad ; \quad \upsilon := \begin{pmatrix}2 & 4 & 6 & 8\end{pmatrix}^T.$$

Then by definition, we have that

$$\Omega(\omega,\upsilon) = \begin{pmatrix}-4 & -2 & 0 & 2 & 4\end{pmatrix}^T \quad ; \quad \Upsilon(\omega,\upsilon) = \begin{pmatrix}-3 & -1 & 1 & 3\end{pmatrix}^T.$$

We note that Theorem 2 implies the weaker result that the system (29)-(30) admits a GPLS if there exists $(v,w) \in I_v \times I_w$ such that $w = k^2 p(v, \mathbf{1}_m)$ and $v = k^2 q(w, \mathbf{1}_n)$. In Figure 3 we have plotted three sets of two graphs for three different values of k. The graphs correspond to the sets of points

$$\Big\{(\frac{v}{k^2}, p(v,\mathbf{1}_m)) \ : \ \frac{v}{k^2} \in [\big(\frac{m+n}{n}\|\Omega\|_\infty\big)^2, 1]\Big\}$$

and

$$\Big\{(q(w,\mathbf{1}_n), \frac{w}{k^2}) \ : \ \frac{w}{k^2} \in [\big(\frac{m+n}{m}\|\Upsilon\|_\infty\big)^2, 1]\Big\}$$

respectively. We observe that for each value of k there is a unique point of intersection. By Theorem 2 such an intersection corresponds to a global phase-locked state.

In Proposition 4 below, we shall show that if, for a given value of k, the system (5)-(6) admits a solution for some $(a,b) \in \{-1,1\}^m \times \{-1,1\}^n$, then it will also admit a solution for $(a,b) = (\mathbf{1}_m, \mathbf{1}_n)$. This suggests that, without loss og

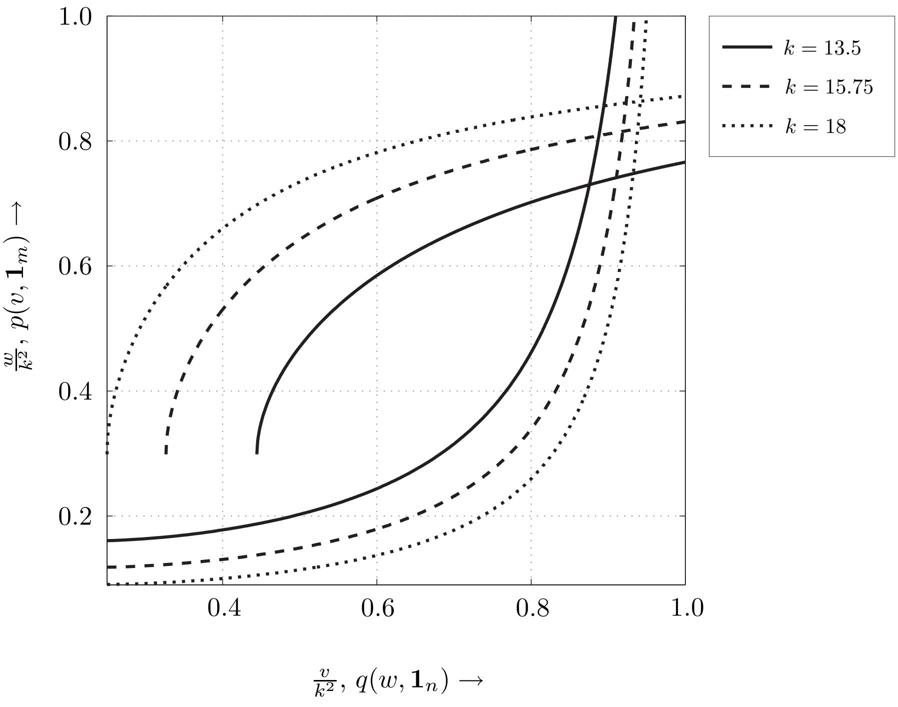

Figure 3: The graphs of $(\frac{v}{k^2}, p(v, \mathbf{1}_m))$ and $(q(w, \mathbf{1}_n), \frac{w}{k^2})$ for different values of the coupling coefficient k. An intersection is a point (v^*, w^*) such that $k^2 p(v^*, \mathbf{1}_m) = w^*$ and $k^2 q(w^*, \mathbf{1}_n) = v^*$. By Theorem 2 such an intersection corresponds to a GPLS of the system (5)-(6).

generality, the critical coupling can be found by determining the smallest value of k for which the system of equations

$$\left.\begin{array}{rcl} w & = & k^2 p(v, \mathbf{1}_m) \\ v & = & k^2 q(w, \mathbf{1}_n) \end{array}\right\} \tag{40}$$

has a solution. To prove this, we will make use of some special properties of the functions p and q, namely: monotonicity (Lemma 2) and concavity (Lemma 3).

Lemma 2 (monotonicity) *Let $(\omega, v) \in \mathbb{R}^m \times \mathbb{R}^n$ be given and let p, q be defined by (38) and (39) respectively. Then:*

(1) *$p(\cdot, \mathbf{1}_m)$ is monotone increasing (on I_v);*

(2) *$p(\cdot, \mathbf{1}_m)$ is strictly monotone iff $\Omega_i(\omega, v) \neq \Omega_j(\omega, v)$ for some (i, j);*

(3) *$p(\cdot, \mathbf{1}_m)$ is either strictly monotone or constant.*

Moreover, the statement remains true if we replace $p(\cdot, \mathbf{1}_m)$ with $q(\cdot, \mathbf{1}_m)$, I_v with I_w, and $\Omega(\omega, v)$ with $\Upsilon(\omega, v)$.

Proof: We prove the result for p. The result for q follows by analogy. Note that $p(\cdot, \mathbf{1}_m)$ is continuous on I_v and differentiable on I_v°. We claim that $\frac{\mathrm{d}p(\xi,\mathbf{1}_m)}{\mathrm{d}\xi}(v) \geq 0$ on I_v°. Note that

$$m^2 \frac{\mathrm{d}p(\xi, \mathbf{1}_m)}{\mathrm{d}\xi}(v) \quad = \quad \frac{1}{v^2}\Big(\underbrace{\Big(\sum_{j=1}^{m} \sqrt{1 - \frac{\tilde{\Omega}_j^2}{v}}\Big)\Big(\sum_{j=1}^{m} \frac{\tilde{\Omega}_j^2}{\sqrt{1-\frac{\tilde{\Omega}_j^2}{v}}}\Big)}_{A} - \Big(\sum_{j=1}^{m} \tilde{\Omega}_j\Big)^2 \Big), \tag{41}$$

for $v \in I_v^\circ$, where $\tilde{\Omega} := \left(\frac{m+n}{n}\right)\Omega$. Using the notation

$$y_i(v) \ := \ \sqrt{1 - \frac{\tilde{\Omega}_i^2}{v}}, \quad 1 \leq i \leq m,$$

we rewrite the term A, as follows:

$$\Big(\sum_{i=1}^{m} y_i(v)\Big)\Big(\sum_{j=1}^{m} \frac{\tilde{\Omega}_j^2}{y_j(v)}\Big) \ = \ \Big(\sum_{i=1}^{m} (\sqrt{y_i(v)})^2\Big)\Big(\sum_{j=1}^{m} (\frac{\tilde{\Omega}_j}{\sqrt{y_j(v)}})^2\Big).$$

By the Cauchy-Schwarz inequality we have that

$$\Big(\sum_{i=1}^{m} (\sqrt{y_i(v)})^2\Big)\Big(\sum_{j=1}^{m} (\frac{\tilde{\Omega}_j}{\sqrt{y_j(v)}})^2\Big) \ \geq \ \Big(\sum_{j=1}^{m} \sqrt{y_j(v)}\frac{\tilde{\Omega}_j}{\sqrt{y_j(v)}}\Big)^2 \ = \ \Big(\sum_{j=1}^{m} \tilde{\Omega}_j\Big)^2$$

for all $v \in I_v^\circ$ where equality holds (for a fixed v) if and only if there exists $\alpha \in \mathbb{R}$ such that $y_i(v) = \alpha\, \Omega_i$ for all i. Inspection shows that this condition is satisfied if and only if $\Omega_i = \Omega_j$ for all (i, j), in which case $p(\cdot, \mathbf{1}_m)$ is a constant function ($p(\cdot, \mathbf{1}_m \equiv 1)$. This concludes the proof.

■

Lemma 3 (concavity) *Let $(\omega, v) \in \mathbb{R}^m \times \mathbb{R}^n$ be given and let p, q be defined by (38) and (39) respectively. Then:*

(1) *$p(\cdot, \mathbf{1}_m)$ is concave (on I_v);*

(2) *$p(\cdot, \mathbf{1}_m)$ is strictly concave iff $\Omega_i(\omega, v) \neq \Omega_j(\omega, v)$ for some (i, j);*

(3) *$p(\cdot, \mathbf{1}_m)$ is either strictly concave or constant.*

Moreover, the statement remains true if we replace $p(\cdot, \mathbf{1}_m)$ with $q(\cdot, \mathbf{1}_m)$, I_v with I_w, and $\Omega(\omega, v)$ with $\Upsilon(\omega, v)$.

Proof: We prove the result for p. The proof for q follows by analogy. Suppose $\Omega_i(\omega, v) = \Omega_j(\omega, v)$ for all (i, j). Then $p(\cdot, \mathbf{1}_m)$ is a constant function by Lemma 2. Hence $p(\cdot, \mathbf{1}_m)$ is concave but not strictly concave. Now suppose $\Omega_i(\omega, v) \neq \Omega_j(\omega, v)$, for some (i, j). We claim that $\frac{\partial^2 p(\zeta, \mathbf{1}_m)}{\partial \zeta^2}(v) < 0$ for all $v \in I_v^\circ$.

It follows easily from (41) that

$$m^2 \frac{\partial p(\xi, \mathbf{1}_m)}{\partial \xi}(v) = \frac{1}{v^2} \sum_{i,j=1, i\neq j}^{m} \Big(\tilde{\Omega}_j^{\,2} \sqrt{\frac{v - \tilde{\Omega}_i^{\,2}}{v - \tilde{\Omega}_j^{\,2}}} - \tilde{\Omega}_i \tilde{\Omega}_j \Big), \tag{42}$$

and hence we may write

$$m^2 \frac{\partial^2 p(\xi, \mathbf{1}_m)}{\partial \xi^2}(v) = -\frac{2}{v} m^2 \frac{\partial p(\xi, \mathbf{1}_m)}{\partial \xi}(v) + \frac{1}{2v^2} B \tag{43}$$

where the term B is given by:

$$\sum_{i,j=1, i\neq j}^{m} \tilde{\Omega}_j^2 \Big((v - \tilde{\Omega}_i^2)^{-1/2} (v - \tilde{\Omega}_j^2)^{-1/2} - (v - \tilde{\Omega}_i^2)^{1/2} (v - \tilde{\Omega}_j^2)^{-3/2} \Big). \tag{44}$$

As $\Omega_i(\omega, v) \neq \Omega_j(\omega, v)$, for some (i, j), it follows from Lemma 2 that the first term on the right hand side of (43) is negative for all $v \in I_v^\circ$. Thus, to complete the proof, it is enough to show that $B \leq 0$ for $v \in I_v^\circ$.

Note first that by rearranging terms we can write

$$\begin{aligned} B &= \sum_{i,j=1, i\neq j}^{m} \tilde{\Omega}_j^2 (v - \tilde{\Omega}_i^2)^{-1/2} (v - \tilde{\Omega}_j^2)^{-1/2} \Big(1 - \frac{v - \tilde{\Omega}_i^2}{v - \tilde{\Omega}_j^2} \Big) \\ &= \sum_{i,j=1, i< j}^{m} (v - \tilde{\Omega}_i^2)^{-1/2} (v - \tilde{\Omega}_j^2)^{-1/2} T_{ij}, \end{aligned} \tag{45}$$

where

$$T_{ij} = \tilde{\Omega}_j^2 \Big(1 - \frac{v - \tilde{\Omega}_i^2}{v - \tilde{\Omega}_j^2} \Big) + \tilde{\Omega}_i^2 \Big(1 - \frac{v - \tilde{\Omega}_j^2}{v - \tilde{\Omega}_i^2} \Big) \tag{46}$$

$$= \tilde{\Omega}_j^2 \Big(\frac{\tilde{\Omega}_i^2 - \tilde{\Omega}_j^2}{v - \tilde{\Omega}_j^2} \Big) + \tilde{\Omega}_i^2 \Big(\frac{\tilde{\Omega}_j^2 - \tilde{\Omega}_i^2}{v - \tilde{\Omega}_i^2} \Big) \tag{47}$$

$$= (\tilde{\Omega}_i^2 - \tilde{\Omega}_j^2) \Big(\frac{\tilde{\Omega}_j^2}{v - \tilde{\Omega}_j^2} - \frac{\tilde{\Omega}_i^2}{v - \tilde{\Omega}_i^2} \Big). \tag{48}$$

It can be readily verified that

$$(\tilde{\Omega}_i^2 - \tilde{\Omega}_j^2) \;\geq\; 0 \;\Leftrightarrow\; \frac{\tilde{\Omega}_j^2}{v - \tilde{\Omega}_j^2} - \frac{\tilde{\Omega}_i^2}{v - \tilde{\Omega}_i^2} \;\leq\; 0$$

and hence $T_{ij} \le 0$ for $1 \le i < j \le m$. But it then follows immediately from (45) that $B \le 0$ for $v \in I_v^\circ$ and hence that $\frac{\partial^2 p(\zeta, \mathbf{1}_m)}{\partial \zeta^2}(v) < 0$ for all $v \in I_v^\circ$ as required.

■

Proposition 4 *Let $k > 0$, and let p and q be given by (38) and (39) respectively. Suppose there exists $(v', w'))$ such that $w' = k^2 p(v', a)$ and $v' = k^2 q(w', b)$ for some $(a, b) \in \{-1, 1\}^m \times \{-1, 1\}^n$. Then there exists $(v'', w''))$ such that $w'' = k^2 p(v'', \mathbf{1}_m)$ and $v'' = k^2 q(w'', \mathbf{1}_n)$.*

Proof: Let $D \subset I_v$ denote the largest (open) subinterval of I_v on which $q(k^2 p(\cdot))$ is defined. Note that if $k^2 p(v^0) \in I_w$ for some $v^0 \in I_v$ then $k^2 p(v) \in I_w$ for all $v \ge v^0$. It follows that

$$D = \begin{cases} \emptyset & \text{if } \{v \in I_v : k^2 p(v) \in I_w\} = \emptyset; \\ [\min\{v \in I_v : k^2 p(v) \in I_w\}, \infty) & \text{otherwise.} \end{cases}$$

Let $s : D \mapsto \mathbb{R}_{\ge 0}$ be given as

$$s(v) := k^2 q(k^2 p(v, \mathbf{1}_m), \mathbf{1}_n).$$

Then there exists $(v, w) \in I_v \times I_w$ such that

$$w = k^2 p(v, \mathbf{1}_m) \quad ; \quad v = k^2 q(w, \mathbf{1}_n) \tag{49}$$

if and only if s has a fixed point on D. Indeed, if v is a fixed point of s, then $(v, w) = (v, k^2 p(v, \mathbf{1}_m))$ is a solution of (49) and vice versa. Inspection of (38)-(39) shows that

$$\begin{array}{rcll} p(v, \mathbf{1}_m) & \ge & p(v, a), \quad \forall a \in \{-1, 1\}^m, & \forall v \in I_v; \\ q(w, \mathbf{1}_n) & \ge & q(w, b), \quad \forall b \in \{-1, 1\}^n, & \forall w \in I_w \end{array} . \tag{50}$$

Let (a', b') be such that $v' = k^2 q(w', b')$ and $w' = k^2 p(v', a')$. As an immediate consequence of (50) we have that

$$s(v') \ge k^2 q(k^2 p(v', \mathbf{1}_m), b'). \tag{51}$$

Moreover, monotonicity of $p(\cdot, \mathbf{1}_m)$ and $q(\cdot, \mathbf{1}_n)$ (see Lemma 2) implies that

$$k^2 q(k^2 p(v', \mathbf{1}_m), b') \ge k^2 q(k^2 p(v', a'), b') = v'. \tag{52}$$

Combining (51) and (52) we see that $s(v') \ge v'$. If $s(v') = v'$, we are done. Suppose therefore that $s(v') > v'$. If we can show that there exists $v^+ \in (v', \infty)$ such that $s(v^+) < v^+$, then by continuity of s there exists $v'' \in [v', v^+)$ such that $s(v'') = v''$. Note that $s(v) \le k^2$ for all $v \ge v'$. Take $v^+ = 2k^2$. It follows that $s(v^+) \le k^2 < v^+$. This concludes the proof. ■

Theorem 3 *Let $(\omega, \upsilon) \in \mathbb{R}^m \times \mathbb{R}^n$ and let p and q be given by (38) and (39) respectively. If $\Omega_i(\omega, \upsilon) \neq \Omega_j(\omega, \upsilon)$ for some (i, j) and $\Upsilon_l(\omega, \upsilon) \neq \Upsilon_m(\omega, \upsilon)$ for some (l, m) then there exists a unique pair $(v, w) \in I_v \times I_w$ such that*

$$\left.\begin{array}{rcl} \frac{\partial q(\zeta, \mathbf{1}_n)}{\partial \zeta}(w) \cdot \frac{\partial p(\zeta, \mathbf{1}_m)}{\partial \zeta}(v) & = & \frac{p(v, \mathbf{1}_m)}{v} \frac{q(w, \mathbf{1}_n)}{w} \\ w \cdot q(w, \mathbf{1}_n) - v \cdot p(v, \mathbf{1}_m) & = & 0 \end{array}\right\}, \tag{53}$$

and we have that

$$k_c = \sqrt{\frac{v}{q(w, \mathbf{1}_n)}} = \sqrt{\frac{w}{p(v, \mathbf{1}_m)}}.$$

In all other cases, k_c is given by

$$k_c = \begin{cases} \frac{m+n}{m} \|\Upsilon\|_\infty, & \Omega_i = \Omega_j \textit{ for all } (i, j); \\ \frac{m+n}{n} \|\Omega\|_\infty, & \Upsilon_l = \Upsilon_m \textit{ for all } (l, m). \end{cases}$$

Proof: First we prove the case $\Omega_i = \Omega_j$ for all (i, j): suppose there exists $c \in \mathbb{R}$ such that $\Omega_i = c$ for all i. Then $c = -\frac{1}{m} \sum_{j=1}^n \Upsilon_j$ by definition of Ω and Υ. We claim that the system

$$\left.\begin{array}{rcll} -c & = & \frac{k}{m+n} \sum_{j=1}^n \sin(\nu_j - \xi_i) & i = 1, \ldots, m; \\ -\Upsilon_i & = & \frac{k}{m+n} \sum_{j=1}^m \sin(\xi_j - \nu_i) & i = 1, \ldots, n. \end{array}\right\} \tag{54}$$

admits a solution (ξ, ν) if and only if $k \geq \frac{m+n}{m} \|\Upsilon\|_\infty$. Necessity follows by inspection. To prove sufficiency, suppose $k \geq \frac{m+n}{m} \|\Upsilon\|_\infty$. Then there exist $\nu' \in \mathbb{R}^n$ such that $\sin(\nu'_i) = \frac{m+n}{km} \Upsilon_i$. Now let $\xi' = 0$. Then by construction $-\Upsilon_i = \frac{k}{m+n} \sum_{j=1}^m \sin(\xi'_j - \nu'_i)$, $1 \leq i \leq n$. Moreover, for $1 \leq i \leq m$, we have that $\frac{k}{m+n} \sum_{j=1}^n \sin(\nu'_j - \xi'_i) = \frac{1}{n} \sum_{j=1}^n \Upsilon = -c$, and it follows that (ξ', ν') is a solution of (54). This proves the claim. The case $\Upsilon_l = \Upsilon_m$ for all (l, m) follows by analogy. Now suppose $\Omega_i(\omega, \upsilon) \neq \Omega_j(\omega, \upsilon)$ for some (i, j) and $\Upsilon_l(\omega, \upsilon) \neq \Upsilon_m(\omega, \upsilon)$ for some (l, m). Let $D := \{v, k) \in I_v \times \mathbb{R}_{\geq 0} : k^2 p(v, \mathbf{1}_m) \in I_w\}$ and let $t : D \mapsto \mathbb{R}_{\geq 0}$ be given as

$$t(v, k) := k^2 q(k^2 p(v, \mathbf{1}_m), \mathbf{1}_n). \tag{55}$$

Then k_c is the smallest k for which $t(\cdot, k)$ is defined on a nonempty interval and has a fixed point on that interval. We claim that the fixed point equation $t(v, k_\mathrm{c}) = v$ has precisely one solution. To prove this, observe first of all that by definition of the critical coupling, it has *at least* one solution. Now suppose by contradiction that there exist $v^{(1)}$,$v^{(2)}$, $v^{(1)} \neq v^{(2)}$, such that $t(v^{(1)}, k_\mathrm{c}) = v^{(1)}$ and $t(v^{(2)}, k_\mathrm{c}) = v^{(2)}$. Without loss of generality, assume that $v^{(2)} > v^{(1)}$. It is easy to verify that the set of points on which $t(\cdot, k_\mathrm{c})$ is defined, is convex (i.e. defines an interval). This implies that $t(v, k_\mathrm{c})$ is defined for all $v \in V := [v^{(1)}, v^{(2)}]$. Under the given hypotheses, t is strictly concave on V by Lemma 3. This implies that there exists $v' \in V^\circ := (v^{(1)}, v^{(2)})$ such that $t(v', k_\mathrm{c}) > v'$ (in fact, $t(v, k_c) > v$ for all $v \in V^\circ$). As t is both strictly monotone and continuous with respect to its second argument, this implies that there exists $k' < k_c$ such that $t(v', k') = v'$, which is a contradiction. We conclude that the fixed point equation $t(v, k_c) = v$ has a unique solution, and we denote this solution by v^*. Let $w^* := k_c^2 p(v^*)$. Then by definition

$$\left.\begin{array}{rcl} (k_c)^2 q(w^*, \mathbf{1}_n) & = & v^* \\[2ex] (k_c)^2 p(v^*, \mathbf{1}_m) & = & w^* \end{array}\right\}. \tag{56}$$

Eqn. (56) implies, firstly, that

$$k_c \;=\; \sqrt{\frac{v^*}{q(w^*, \mathbf{1}_n)}} \;=\; \sqrt{\frac{w^*}{p(v^*, \mathbf{1}_m)}};$$

and secondly, that

$$w^* \cdot q(w^*, \mathbf{1}_n) - v^* \cdot p(v^*, \mathbf{1}_m) \;=\; 0.$$

Next, we will show that

$$\left.\frac{\partial t(\zeta, k_c)}{\partial \zeta}\right|_{\zeta = v^*} \;=\; 1. \tag{57}$$

As a first step towards proving this we will establish the fact that there exists an open subinterval $S \subset I_v$, containing v^*, such that $t(\cdot, k_c)$ is defined on the whole of S. It suffices to show that $v^* > v_{\min} := \min\{v \in I_v : \ (v, k_c) \in D\}$. Suppose by contradiction that $v^* = v_{\min}$. Then by monotonicity, it must be the case that $w^* = (\frac{m+n}{m}\|\Upsilon\|_\infty)^2$. But this implies that there exists $\delta > 0$ such that

$$\frac{\partial t(\zeta, k_c)}{\partial \zeta}(v) \;=\; \left((k_c)^2 \left.\frac{\partial q(\zeta, \mathbf{1}_n)}{\partial \zeta}\right|_{\zeta = k^2 p(v, \mathbf{1}_m)}\right) \cdot \left((k_c)^2 \left.\frac{\partial p(\zeta, \mathbf{1}_m)}{\partial \zeta}\right|_{\zeta = v}\right) \;>\; 1$$

for all $v \in (v^*, v^* + \delta)$. Indeed, by continuity of the partial derivative, and by the fact that p is strictly concave, there exist $\delta > 0$ and $c > 0$, such that $(k_c)^2 \left.\frac{\partial p(\zeta, \mathbf{1}_m)}{\partial \zeta}\right|_{\zeta = v} > \frac{1}{c}$ and $(k_c)^2 \left.\frac{\partial q(\zeta, \mathbf{1}_n)}{\partial \zeta}\right|_{\zeta = k^2 p(v, \mathbf{1}_m)} > c$ for all $v \in (v^*, v^* + \delta)$. But this implies that there exists $v' \in (v^*, v^* + \delta)$ such that $t(v', k_c) > v'$. Noting that $t(v, k_c) < v$ for large enough v, it follows by continuity that there exists $v^{**} > v^*$ such that $t(v^{**}, k_c) = v^{**}$. But this contradicts the uniqueness of the fixed point and we conclude that $v^* > v_{\min}$.

We are now ready to prove equality (57). Suppose that $\frac{\partial t(\zeta, k_c)}{\partial \zeta}(v^*) > 1$. Then by continuity there exists $\delta > 0$ such that $t(v, k_c) > v$ for all $v \in (v^*, v^* + \delta)$. But as we have seen above this leads to a contradiction and we conclude that $\frac{\partial t(\zeta, k_c)}{\partial \zeta}(v^*) \leq 1$. Now suppose that $\frac{\partial t(\zeta, k_c)}{\partial \zeta}(v^*) < 1$. Then by continuity there exists $v' < v^*$ such that $t(v', k_c) > v'$. But this implies that there exists $k' < k_c$ such that $t(v', k') = v'$, which is a contradiction by definition of k_c. We conclude that $\frac{\partial t(\zeta,k_c)}{\partial\zeta}(v^*) = 1$. Straightforward manipulation and substitution of (56) gives us:

$$\begin{aligned}\frac{\partial t(\zeta,k_c)}{\partial\zeta}(v^*) &= \Big((k_c)^2\left.\frac{\partial q(\zeta,\mathbf{1}_n)}{\partial\zeta}\right|_{\zeta=k^2p(v^*,\mathbf{1}_m)}\Big)\cdot\Big((k_c)^2\left.\frac{\partial p(\zeta,\mathbf{1}_m)}{\partial\zeta}\right|_{\zeta=v^*}\Big)\\ &= \Big(\frac{v^*}{q(w^*,\mathbf{1}_n)}\left.\frac{\partial q(\zeta,\mathbf{1}_n)}{\partial\zeta}\right|_{\zeta=w^*}\Big)\cdot\Big(\frac{w^*}{p(v^*,\mathbf{1}_m)}\left.\frac{\partial p(\zeta,\mathbf{1}_m)}{\partial\zeta}\right|_{\zeta=v^*}\Big).\end{aligned}$$

It follows that

$$\frac{\partial q(\zeta,\mathbf{1}_n)}{\partial\zeta}(w^*)\cdot\frac{\partial p(\zeta,\mathbf{1}_m)}{\partial\zeta}(v^*) = \frac{p(v^*,\mathbf{1}_m)}{v^*}\frac{q(w^*,\mathbf{1}_n)}{w^*}. \tag{58}$$

This concludes the proof. ∎

6 Algorithm

Theorem 3 states that the critical coupling of a system of coupled oscillators on a complete bipartite graph can be determined from the unique solution of the system of equations (53). In the present section we will show how this solution may be computed using numerical techniques.

Let $P : I_v^\circ \mapsto \mathbb{R}$ and $Q : I_w^\circ \mapsto \mathbb{R}$ be given as

$$m^2P(v) := \Big(\sum_{j=1}^m\sqrt{1-(\tilde\Omega_j)^2\frac{1}{v}}\Big)\Big(\sum_{j=1}^m\frac{1}{\sqrt{1-(\tilde\Omega_j)^2\frac{1}{v}}}\Big) \tag{59}$$

$$n^2Q(w) := \Big(\sum_{j=1}^n\sqrt{1-(\tilde\Upsilon_j)^2\frac{1}{w}}\Big)\Big(\sum_{j=1}^n\frac{1}{\sqrt{1-(\tilde\Upsilon_j)^2\frac{1}{w}}}\Big) \tag{60}$$

where, as before $\tilde\Omega := \frac{m+n}{n}\Omega$ and $\tilde\Upsilon := \frac{m+n}{m}\Upsilon$. Using the notation

$$z_j(v) := 1-\frac{\left(\frac{m+n}{n}\Omega_j\right)^2}{v},\quad j=1,\ldots,m, \tag{61}$$

we can rewrite $\frac{\partial p(\zeta,\mathbf{1}_m)}{\partial\zeta}(v)$, as follows:

$$\frac{\partial p(\zeta,\mathbf{1}_m)}{\partial\zeta}(v) = -\frac{1}{v}\Big(p(v,\mathbf{1}_m)-\frac{1}{m}\sum_{i=1}^m\sqrt{z_i(v)}\frac{1}{m}\sum_{j=1}^m\frac{1}{\sqrt{z_j(v)}}\Big). \tag{62}$$

An analogous formula also holds for the function $q(\zeta,\mathbf{1}_n)$. Let $(v,w)\in I_v\times I_w$. Then it follows from (62) that (v,w) is a solution of (53) if and only if (v,w) satisfies

$$\Big(\frac{P(v)}{p(v,\mathbf{1}_m)}-1\Big)\Big(\frac{Q(w)}{q(w,\mathbf{1}_n)}-1\Big) = 1. \tag{63}$$

As a first step, we introduce the function $\Gamma : I_v \times I_w \mapsto \{\text{(i)},\text{(ii)},\text{(iii)},\text{(iv)}\}$, which partitions the set $I_v \times I_w$ into 4 disjoint subsets, as follows:

$$\Gamma(v,w) := \begin{cases} \text{(i)} & vp(v,\mathbf{1}_m) - wq(w,\mathbf{1}_n) \leq 0,\ (\frac{P(v)}{p(v,\mathbf{1}_m)}) - 1)(\frac{Q(w)}{q(w,\mathbf{1}_n)}) - 1) \leq 1 \\ \text{(ii)} & vp(v,\mathbf{1}_m) - wq(w,\mathbf{1}_n) > 0,\ (\frac{P(v)}{p(v,\mathbf{1}_m)}) - 1)(\frac{Q(w)}{q(w,\mathbf{1}_n)}) - 1) \leq 1 \\ \text{(iii)} & vp(v,\mathbf{1}_m) - wq(w,\mathbf{1}_n) > 0,\ (\frac{P(v)}{p(v,\mathbf{1}_m)}) - 1)(\frac{Q(w)}{q(w,\mathbf{1}_n)}) - 1) > 1 \\ \text{(iv)} & vp(v,\mathbf{1}_m) - wq(w,\mathbf{1}_n) \leq 0,\ (\frac{P(v)}{p(v,\mathbf{1}_m)}) - 1)(\frac{Q(w)}{q(w,\mathbf{1}_n)}) - 1) > 1 \end{cases} . \tag{64}$$

Note that by definition of Γ, the solution of the system of equations (53) is contained in the set $\{(v,w) : \Gamma(v,w) = \text{(i)}\}$. Figure 4 shows the sets corresponding to the different values of Γ for a particular realization of the complete bipartite graph $K_{24,17}$. We have the following result:

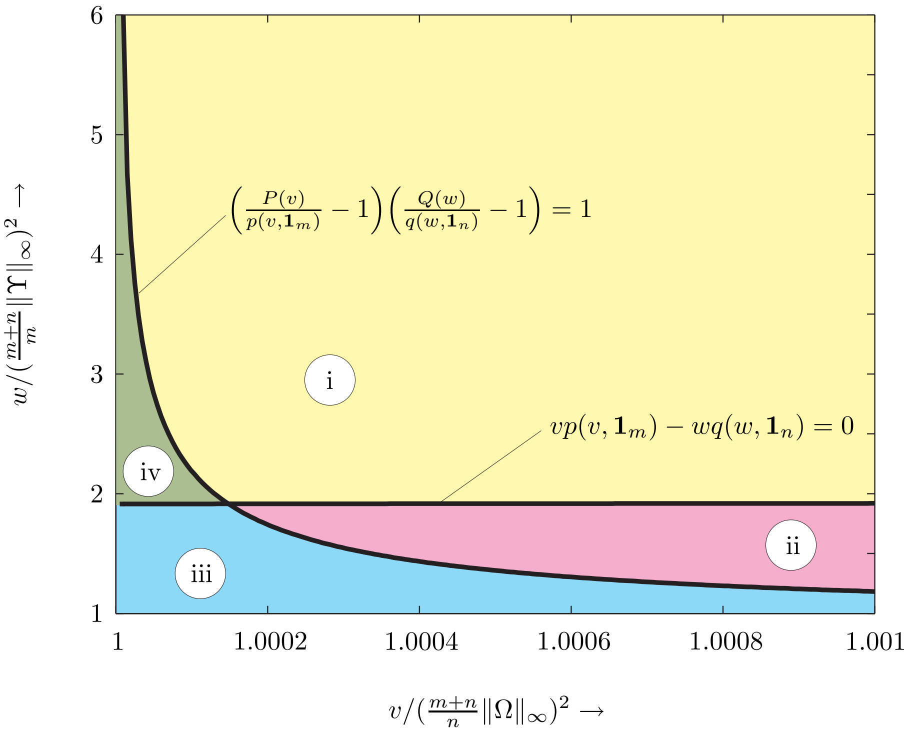

Figure 4: Illustration of the action of the map Γ, given by (64): Γ induces a partition on the set $I_v^\circ \times I_w^\circ$. In the figure, the different subsets are represented by colors, and a lower case roman numeral, e.g. (i) or (ii), is used to denote the value of Γ on that particular subset. The bold lines correspond to the isoclines $\{(v,w) : vp(v,\mathbf{1}_m) - wq(w,\mathbf{1}_n) = 0\}$ and $\{(v,w) : (\frac{P(v)}{p(v,\mathbf{1}_m)} - 1)(\frac{Q(w)}{q(w,\mathbf{1}_n)} - 1) = 1\}$ separating the subsets.

Proposition 5 *Let (v', w') denote the unique solution of the system of equations (53) and let Γ be given by (64). Let $(v, w) \in I_v^\circ \times I_w^\circ$. Then the following implications hold:*

(1) *If $\Gamma(v, w) =$ (i) then $w \geq w'$.*

(2) *If $\Gamma(v, w) =$ (ii) then $v > v'$.*

(3) *If $\Gamma(v, w) =$ (iii) then $w < w'$.*

(4) *If $\Gamma(v, w) =$ (iv) then $v < v'$.*

Proof: As a first step we show that the functions $\frac{P(\cdot)}{p(\cdot, \mathbf{1}_m)} - 1$ and $\frac{Q(\cdot)}{q(\cdot, \mathbf{1}_n)} - 1$ are monotone decreasing and nonnegative on their respective domains. The nonnegativity follows immediately from (62). It is also possible to show using arguments very similar to those used in the proof of Lemma 3 that $\frac{\partial P(\zeta)}{\partial \zeta}(v) =\leq 0$ for all $v \in I_v^\circ$ and that $\frac{\partial Q(\zeta)}{\partial \zeta}(w) \leq 0$ for all $w \in I_w^\circ$. Since p and q are monotone increasing by Lemma 2, it follows that $\frac{P(\cdot)}{p(\cdot, \mathbf{1}_m)}$ and $\frac{Q(\cdot)}{q(\cdot, \mathbf{1}_n)}$ are monotone decreasing. Furthermore we have that $P(v) \geq p(v, \mathbf{1}_m)$ for all $v \in I_v^\circ$, and $Q(w) \geq q(w, \mathbf{1}_n)$ for $w \in I_w^\circ$ (this follows from the proof of Lemma 2, and Eqn. (62) in particular). We conclude that

$$\Big(\frac{P(v)}{p(v, \mathbf{1}_m)} - 1\Big) \geq 0 \ \forall v \in I_v, \text{ and } \Big(\frac{Q(w)}{q(w, \mathbf{1}_n)} - 1\Big) \geq 0 \ \forall w \in I_w.$$

Now let $(v, w) \in I_v^\circ \times I_w^\circ$. It follows that

$$\Big(\frac{P(v)}{p(v, \mathbf{1}_m)} - 1\Big)\Big(\frac{Q(w)}{q(w, \mathbf{1}_n)} - 1\Big) \ > \ 1,$$

if $v < v'$ and $w < w'$. Similarly, we have that

$$v \cdot p(v, \mathbf{1}_m) - w \cdot q(w, \mathbf{1}_n) \ > \ 0.$$

if $v > v'$ and $w < w'$ (by monotonicity of p and q). Now suppose for example that $\Gamma(v, w) =$ (i). Then it follows from the above that ($v \geq v'$ OR $w \geq w'$) AND ($v \leq v'$ OR $w \geq w'$), which implies that $v = v'$ OR $w \geq w'$. Inspection shows that if $v = v'$ and $w < w'$ then necessarily

$$v \cdot p(v, \mathbf{1}_m) - w \cdot q(w, \mathbf{1}_n) \ > \ 0,$$

which contradicts the assumption that $\Gamma(v, w) =$ (i). We conclude that $w \geq w'$, which proves implication (1). The other cases follows similarly. This concludes the proof. ■

Proposition 5 suggests that the solution of the system of equations (53) can be found numerically using a bisection algorithm. The idea is as follows. To initialise the algorithm, we pick $(v^0, w^0) \in I_v^\circ \times I_w^\circ$ such that $v^0 \geq v'$ and

$w^0 \geq w'$ (more about this shortly), and we define $\Delta v := \frac{1}{2}(v^0 - a_v)$ and $\Delta w := \frac{1}{2}(w^0 - a_w)$, where a_v and a_w are given by

$$a_v := \big(\frac{m+n}{n}\|\Omega\|_\infty\big)^2 \; ; \; a_w := \big(\frac{m+n}{m}\|\Upsilon\|_\infty\big)^2. \tag{65}$$

We set $v := v^0 - \Delta_v$ and $w := w^0 - \Delta_w$. Now we invoke Proposition 5 to obtain information about the location of (v, w) relative to (v', w'). There are four possibilities:

(1) $\Gamma(v, w) =$ (i). In this case $w \geq w$ and we update w and Δw according to the update rule $w \mapsto w - \Delta w$, $\Delta w \mapsto \frac{1}{2}\Delta w$;

(2) $\Gamma(v, w) =$ (ii). In this case $v > v'$ and we update v and Δv according to the update rule $v \mapsto v - \Delta v$, $\Delta v \mapsto \frac{1}{2}\Delta v$;

(3) $\Gamma(v, w) =$ (iii). In this case $w < w'$ and we update w and Δw according to the update rule $w \mapsto w + \Delta w$, $\Delta w \mapsto \frac{1}{2}\Delta w$;

(4) $\Gamma(v, w) =$ (iv). In this case $v < v'$ and we update v and Δv according to the update rule $v \mapsto v + \Delta v$, $\Delta v \mapsto \frac{1}{2}\Delta v$.

We repeat until the algorithm terminates (which it does when the error first drops below a certain tolerance level). The error can be defined in different ways. We have defined it as follows:

$$\mathtt{Err}(v, w) \quad := \quad \max\left\{ |vp(v, \mathbf{1}_m) - wq(w, \mathbf{1}_n)|, \Big|\frac{P(v)}{p(v, \mathbf{1}_m)} - 1\Big| \cdot \Big|\frac{Q(w)}{q(w, \mathbf{1}_n)} - 1\Big| \right\}$$

Before we present the actual algorithm, let us discuss how to pick v^0, w^0 such that $v^0 \geq v'$ and $w^0 \geq w'$. One approach is to try and derive analytic upper bounds on v' and w' in terms of $\|\Omega\|_\infty$ and $\|\Upsilon\|_\infty$. We propose a more practical approach, as follows. We pick any pair $(v, w) \in I_v^\circ \times I_w^\circ$. For concreteness, let us say $(v, w) := (2a_v, 2a_w)$. Next we fix $w = 2a_w$ and repeatedly increase v by a factor of 2 until $\Gamma(v, 2a_w) =$ (ii). It is easy to show that this procedure always terminates in a finite number of steps. If we let v^0 denote the value of v after termination of this procedure, then by Proposition 5, we have that $v^0 \geq v'$. Now we fix $v = v^0$ and we repeatedly increase w by a factor of 2 until $\Gamma(v^0, w) =$ (i). Again, this procedure will always terminate in a finite number of steps. Let w^0 denote the value of w after termination of this procedure. Then by Proposition 5 we have that $w^0 \geq w'$. Effectively, if ρ_1 is the smallest nonnegative integer i for which $\Gamma(2^{i+1}a_v, 2a_w) =$ (ii) and ρ_2 is smallest nonnegative integer j for which $\Gamma(2^{\rho_1+1}a_v, 2^{j+1}a_w) =$ (i), then we have that $(v^0, w^0) = (2^{\rho_1+1}a_v, 2^{\rho_2+1}a_w)$. We have the following algorithm:

Algorithm 2

1. $\mathtt{a_v} := (\frac{\mathtt{m+n}}{\mathtt{n}}\|\Omega\|_\infty)^2$;

2. $\mathtt{a_w} := (\frac{\mathtt{m+n}}{\mathtt{m}}\|\Upsilon\|_\infty)^2$;

3. $\mathtt{v} := 2 \cdot \mathtt{a_v}$;

4. $\mathtt{w} := 2 \cdot \mathtt{a_w}$;

5. $\mathtt{AbsTol} := 10^{-6}$;

6. $\mathtt{Err} := 1$;

7. $\mathtt{while}\ \Gamma(\mathtt{v},\mathtt{w}) <> (\mathtt{ii})$

   $\mathtt{v} := 2 \cdot \mathtt{v}$;
   $\mathtt{w} := \mathtt{w}$;

8. end

9. $\mathtt{while}\ \Gamma(\mathtt{v},\mathtt{w}) <> (\mathtt{i})$

   $\mathtt{v} := \mathtt{v}$;
   $\mathtt{w} := 2 \cdot \mathtt{w}$;

10. end

11. $\Delta_\mathtt{v} := \frac{1}{2}(\mathtt{v} - \mathtt{a_v})$;

12. $\Delta_\mathtt{w} := \frac{1}{2}(\mathtt{w} - \mathtt{a_w})$;

13. $\mathtt{while\ Err} > \mathtt{AbsTol}$

    13.1. $\mathtt{switch}\ \Gamma(\mathtt{v},\mathtt{w})$

    13.1.1. case $(\mathtt{i})$

    $$\begin{aligned} \mathtt{w} &:= \mathtt{w} - \Delta_\mathtt{w}; \\ \Delta_\mathtt{w} &:= \tfrac{1}{2}\Delta_\mathtt{w}; \end{aligned}$$

    13.1.2. case $(\mathtt{ii})$

    $$\begin{aligned} \mathtt{v} &:= \mathtt{v} - \Delta_\mathtt{v}; \\ \Delta_\mathtt{v} &:= \tfrac{1}{2}\Delta_\mathtt{v}; \end{aligned}$$

    13.1.3. case $(\mathtt{iii})$

    $$\begin{aligned} \mathtt{w} &:= \mathtt{w} + \Delta_\mathtt{w}; \\ \Delta_\mathtt{w} &:= \tfrac{1}{2}\Delta_\mathtt{w}; \end{aligned}$$

    13.1.4. case $(\mathtt{iv})$

    $$\begin{aligned} \mathtt{v} &:= \mathtt{v} + \Delta_\mathtt{v}; \\ \Delta_\mathtt{v} &:= \tfrac{1}{2}\Delta_\mathtt{v}; \end{aligned}$$

    13.2. end

    13.3. $\mathtt{Err} := \max\left\{|\mathtt{v}\cdot\mathtt{p}(\mathtt{v},\mathbf{1}_\mathtt{m}) - \mathtt{w}\cdot\mathtt{q}(\mathtt{w},\mathbf{1}_\mathtt{n})|, \left|\frac{\mathtt{P(v)}}{\mathtt{p}(\mathtt{v},\mathbf{1}_\mathtt{m})} - 1\right| \cdot \left|\frac{\mathtt{Q(w)}}{\mathtt{q}(\mathtt{w},\mathbf{1}_\mathtt{n})} - 1\right|\right\}$;

14. end

By construction of the algorithm and Proposition 5, v and w exponentially converge to the unique solution (v', w') of the system of equations (53). In the next section we will apply this algorithm in a number of different examples.

7 Numerical Examples

7.1 Example 1

In the first example, we consider the very simple complete bipartite graph $K_{(2,2)}$ depicted in Figure 5, which is also known as the cycle graph C_4. We pick natural

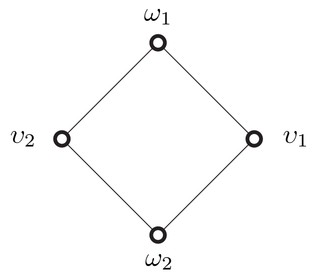

Figure 5: The complete bipartite graph $K_{(2,2)}$. The symbol next to each node represents the intrinsic frequency of the corresponding oscillator.

frequencies $(\omega, \upsilon) \in \mathbb{R}^2 \times \mathbb{R}^2$, as follows: $\omega := (1.0 \;\; 1.3)^T$, $\upsilon := (0.9 \;\; 1.2)^T$. By definition of Ω and Υ we have that

$$\begin{pmatrix} \Omega(\omega, \upsilon) \\ \Upsilon(\omega, \upsilon) \end{pmatrix} := \begin{pmatrix} \omega \\ \upsilon \end{pmatrix} - \frac{1}{4}(\omega_1 + \omega_2 + \upsilon_1 + \upsilon_2) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.1 \\ 0.2 \\ -0.2 \\ 0.1 \end{pmatrix}.$$

Our objective is to determine the smallest $k \geq 0$ for which the system of differential equations

$$\left.\begin{array}{rcl} \dot{\theta}_1(t) & = & \omega_1 + \frac{k}{4}\sin(\phi_1(t) - \theta_1(t)) + \frac{k}{4}\sin(\phi_2(t) - \theta_1(t)) \\ \dot{\theta}_2(t) & = & \omega_2 + \frac{k}{4}\sin(\phi_1(t) - \theta_2(t)) + \frac{k}{4}\sin(\phi_2(t) - \theta_2(t)) \\ \dot{\phi}_1(t) & = & \upsilon_1 + \frac{k}{4}\sin(\theta_1(t) - \phi_1(t)) + \frac{k}{4}\sin(\theta_2(t) - \phi_1(t)) \\ \dot{\phi}_2(t) & = & \upsilon_2 + \frac{k}{4}\sin(\theta_1(t) - \phi_2(t)) + \frac{k}{4}\sin(\theta_2(t) - \phi_2(t)) \end{array}\right\} \quad (66)$$

admits a global phase-locked state. To investigate the phase-locking behaviour of this system, we integrate (66) twice, first with $k = 0.55$, and then with $k = 0.65$. The results are given in Figures 6 and 7. When $k = 0.55$, we observe that the magnitudes of the order parameters do not converge but rather oscillate between the extrema 0 and 1. If a global phase-locked state would exist for this value of k, and if it were stable in the sense that its equivalence class is a stable manifold, then we might expect that at least for some initial conditions the order parameters would converge. This is in fact what we observe in Figure 7. On the

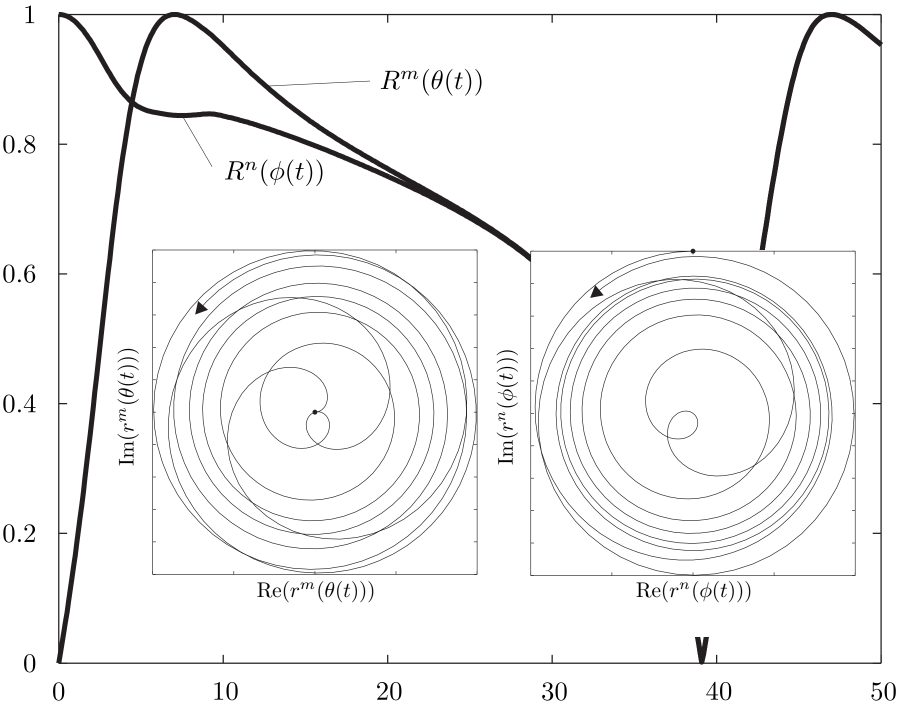

Figure 6: Case $k = 0.55$: no global phase-locked state exists. The graph shows the time evolution of $R^m(\theta)$ and $R^n(\phi)$, where $(\theta(\cdot), \phi(\cdot))$ is the solution to the system (66) subject to the initial condition $(\theta(0), \phi(0)) = ((\frac{1}{2}\pi\ -\frac{1}{2}\pi), (\frac{1}{2}\pi\ \frac{1}{2}\pi))$. Inset: the evolution of the order parameter $r^m(\theta)$ (left panel) and $r^n(\phi)$ (right panel) in the complex plane.

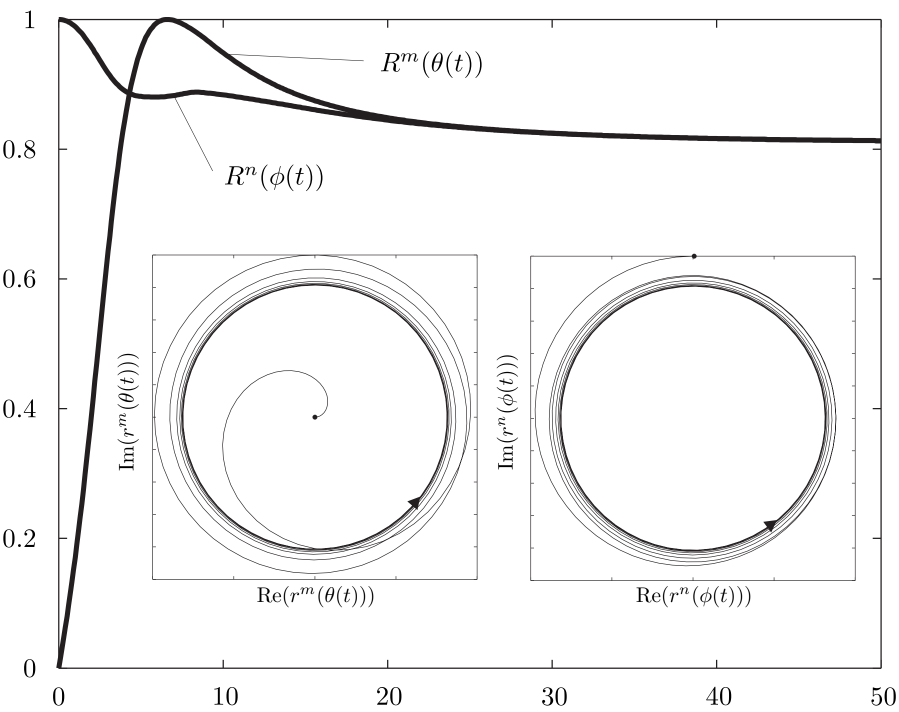

Figure 7: Case $k = 0.65$: the system converges to a global phase-locked state. The graph shows the time evolution of $R^m(\theta)$ and $R^n(\phi)$, where $(\theta(\cdot), \phi(\cdot))$ is the solution to the system (66) subject to the initial condition $(\theta(0), \phi(0)) = ((\frac{1}{2}\pi \;\; -\frac{1}{2}\pi), (\frac{1}{2}\pi \;\; \frac{1}{2}\pi))$. Inset: the evolution of the order parameter $r^m(\theta)$ (left panel) and $r^n(\phi)$ (right panel) in the complex plane.

1	2	3	4	5	6	7	8	9	10	11
0.859	0.750	0.571	0.278	0.179	0.110	0.012	0.053	0.518	0.158	0.040

Table 1: The error term $\mathtt{Err}(\mathtt{v}, \mathtt{w})$ vs. iteration number.

other hand, convergence of the magnitude of the order parameters is a necessary but not a sufficient condition for the existence of a global phase-locked state. In other words, these plots are not conclusive in as far as the (non)existence of global phase-locked states is concerned. To determine whether a GPLS exists in each case we examine the graph of $t(\cdot, k)$ for $k = 0.55$ and $k = 0.65$ to see whether, for one value of k or the other, it has a fixed point on the interval I_v. If it does then we know that a GPLS exists for the corresponding value of k (and in fact for any k greater than or equal to it). And if does not, then we know that no GPLS exists for $k \leq 0.65$. The graphs of $t(\cdot, 0.55)$ and $t(\cdot, 0.65)$ are given in Figure 8. We observe that $t(\cdot, 0.55)$ does not have a fixed point (in the given interval), whereas $t(\cdot, 0.65)$ does. From this we deduce that $0.55 < k_{\mathrm{c}} \leq 0.65$. To compute the exact value of k_{c}, we invoke algorithm 2. It turns out that for this example $k_{\mathrm{c}} = 0.6298$. Table 1 shows the error $\mathtt{Err}(\mathtt{v}, \mathtt{w})$, which was defined in (66), as a function of the number of iterations. We observe that the error decreases rapidly, as expected, though not monotonically. To check whether the given answer is correct we examine the graph of $t(v, 0.6298)$, shown in Figure 9. Inspection shows that for this value of k, the map $t(\cdot, k)$ has exactly one fixed point v^* and $\left.\frac{\partial t(\zeta,k)}{\partial \zeta}\right|_{\zeta = v'} = 1$, as required.

7.2 Example 2

In this example we use Algorithms 1 and 2 to compare the critical coupling of a complete bipartite graph $K_{m,n}$ with the critical coupling of the associated complete graph K_{m+n}. More specifically, we compare the critical coupling of $K_{2i,3i}$ with that of K_{5i}; that of $K_{2i,4i}$ with that of K_{6i}; and that of $K_{3i,4i}$ with that of K_{7i} for selected values of i in the interval $[1, 100]$. The intrinsic frequencies of the respective graphs are picked as follows. We construct the vectors $\alpha \in \mathbb{R}^{300}$ and $\beta \in \mathbb{R}^{400}$ by drawing from a uniform distribution on $[0, 1]$, respectively 300 and 400 times. Then for the set of complete bipartite graphs $\{K_{m,n} :\ (m, n) \in \{(2, 4), (2, 3), (3, 4)\}\}$, we define intrinsic frequencies $\omega \in \mathbb{R}^m$ and $\upsilon \in \mathbb{R}^n$,

$$\begin{aligned} \omega_i &:= \alpha_i, \quad i = 1, \dots m; \\ \upsilon_i &:= \beta_i, \quad i = 1, \dots n, \end{aligned}$$

and for the set of complete graphs $\{K_{m+n} :\ (m, n) \in \{(2, 4), (2, 3), (3, 4)\}\}$, we define $\omega \in \mathbb{R}^{m+n}$,

$$\omega_i := \begin{cases} \alpha_i, & i = 1, \dots, m; \\ \beta_{i-m}, & i = m+1, \dots, m+n. \end{cases}$$

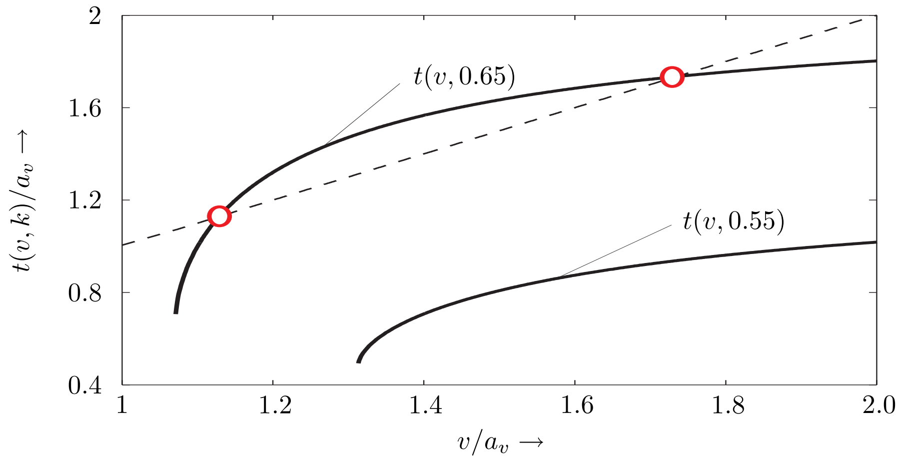

Figure 8: The graphs of $t(v, 0.55)$ and $t(v, 0.65)$ for selected values of v in the interval $[a_v, 2a_v]$. The dashed line corresponds to the identity map $\mathrm{Id} : \mathbb{R} \mapsto \mathbb{R}$. The red circle marking the intersections of $t(v, 0.65)$ with $\mathrm{Id}(v)$ correspond to global phase-locked solutions.

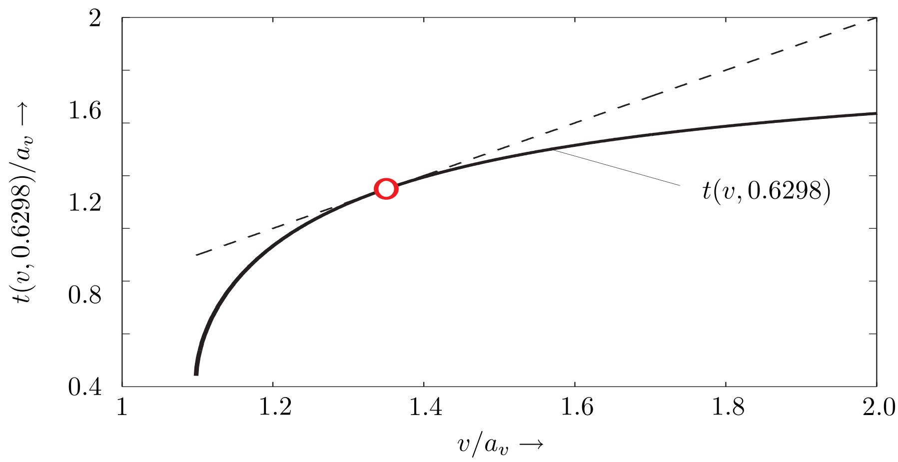

Figure 9: The graph of $t(v, 0.6298)$ for selected values of v in the interval $[a_v, 2a_v]$. The dashed line corresponds to the identity map $\mathrm{Id} : \mathbb{R} \mapsto \mathbb{R}$. The red circle marking the intersection of $t(v, 0.6298)$ with $\mathrm{Id}(v)$ defines a unique fixed point v' of $t(\cdot, 0.6298)$. Note that $\left.\frac{\partial t(\zeta, 0.6298)}{\partial \zeta}\right|_{\zeta = v'} = 1$.

Finally, for each pair of graphs $(K_{m,n}, K_{m+n})$ we compute the critical coupling. The results are depicted in Figure 10. We observe that, in all cases, the critical coupling of the Kuramoto model on the complete graph is strictly smaller than that of its counterpart on a complete bipartite graph. The data also suggest—though we do not have proof to show this—that in each case the value of the critical coupling tends to a limit as the number of oscillators tends to infinity with the limit depending on the ratio of m and n. As regards the first observation, on the critical coupling of a complete graph being smaller than that of its complete bipartite counterpart, we remark that as it is, the comparison is not entirely fair. The reason is that the normalization constant in the bipartite system (5)-(6) scales with the total number of oscillators $(n+m)$, and not with the number of neighbours (n or m), as it does in the case of a complete graph (where the ratio between the two tends to one as $n+m$ tends to infinity). To ensure a fair comparison we consider the modified complete bipartite system given below, where, consistent with the Kuramoto model on a complete graph, the normalization constant is directly proportional to the number of neighbours.

$$\dot{\theta}_i(t) = \omega_i + \frac{k}{n+1}\sum_{j=1}^{n}\sin(\phi_j(t) - \theta_i(t)), \quad i = 1, \ldots, m; \tag{67}$$

$$\dot{\phi}_i(t) = \upsilon_i + \frac{k}{m+1}\sum_{j=1}^{m}\sin(\theta_j(t) - \phi_i(t)), \quad i = 1, \ldots, n. \tag{68}$$

Computing the critical coupling for this system turns out to be equivalent to solving the familiar system of equations

$$\begin{aligned}
\frac{k}{m+n}\sum_{j=1}^{n}\sin(\xi_i - \nu_j) &= -\Omega_i, \quad i = 1, \ldots, m;\\
\frac{k}{m+n}\sum_{j=1}^{m}\sin(\nu_i - \xi_j) &= -\Upsilon_i, \quad i = 1, \ldots, n.
\end{aligned}$$

where, in this case, Ω and Υ are given by

$$\begin{aligned}
\Omega &:= \Big(\frac{n+1}{m+n}\Big)\Big(\omega - \frac{(m+1)\sum_{j=1}^{n}\upsilon_j + (n+1)\sum_{j=1}^{m}\omega_j}{(m+1)n + (n+1)m}\mathbf{1}_m\Big);\\
\Upsilon &:= \Big(\frac{m+1}{m+n}\Big)\Big(\upsilon - \frac{(m+1)\sum_{j=1}^{n}\upsilon_j + (n+1)\sum_{j=1}^{m}\omega_j}{(m+1)n + (n+1)m}\mathbf{1}_n\Big).
\end{aligned}$$

When we compare the critical coupling of this system (Figure 11) with that of the original system (Figure 10) we observe that: (a) the values of the critical coupling of the modified system are closer to that of the corresponding complete system than the values of the original system are, yet the critical coupling of the complete system is still significantly smaller in all cases; (b) as was the case for the original system, the data suggests that the values of the critical coupling of the modified system tend to a limit as the size of the system tends to infinity. Here however, the limit appears to be independent of the ratio of m and n.

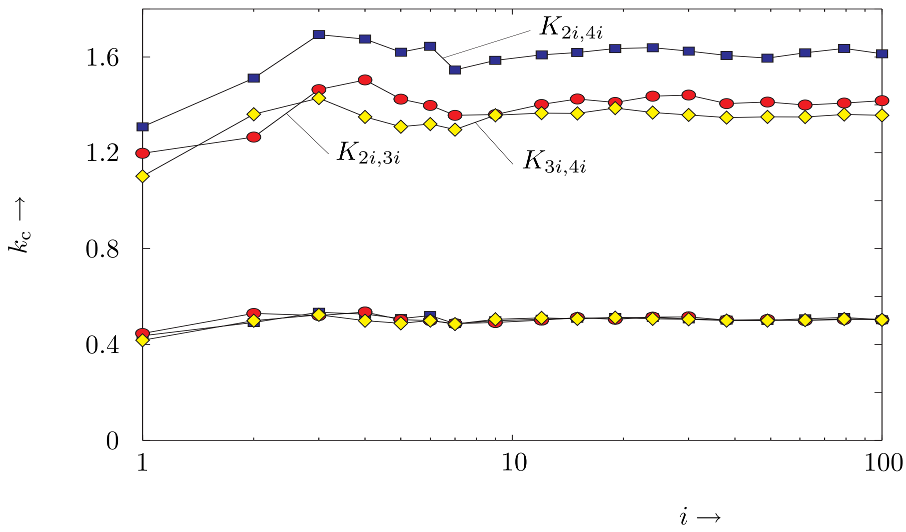

Figure 10: The figure shows the values of the critical coupling of the complete bipartite graphs $K_{2i,4i}$ (blue square), $K_{2i,3i}$ (red circle), and $K_{3i,4i}$ (yellow diamond) along with that of the corresponding complete graphs K_{6i} (blue square), K_{5i} (red circle) and K_{7i} (yellow diamond) for selected values of i.

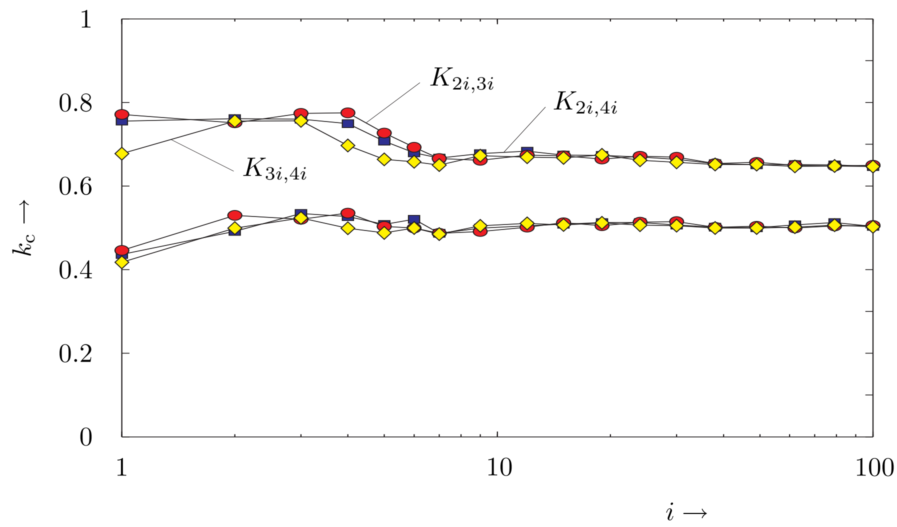

Figure 11: The same as Figure 10, but the data in this figure apply to the modified bipartite system (67)-(68) rather than the original system (5)-(6). Note the difference in scale.

8 Conclusion

In this paper, we have derived new results on the critical coupling of the Kuramoto model for the case of a complete bipartite graph. In particular, we have shown that the value of the critical coupling can be obtained exactly by solving a system of two nonlinear equations that do not depend on the coupling coefficient. We showed that the said system of equations can be solved using efficient numerical techniques and we proposed an algorithm based on the same. Using this algorithm we were able to investigate numerically the relation between the critical coupling of a complete bipartite graph and the critical coupling of the associated complete graph for networks with large numbers of nodes (oscillators). After renormalization of the coupling strength, we found that the critical coupling of the complete graph is significantly smaller than that of the bipartite graph. We also found that the critical coupling of the appropriately renormalized bipartite graph tends to a limit as the number of oscillators tend to infinity. The numerical results suggest that this limit is independent of the ratio of the number of oscillators in each partition, but further analysis is required to show that this is indeed the case.

Appendix A

Proof of Lemma 1

Proof: Let $x^* = (y^*; 0)$. Then as $\sum_{j=1}^{m+n} F_j(\cdot) \equiv 0$, it follows that $F(x^*) = 0$ and under the hypotheses of the lemma, $R^m(\Pi^1(x^*))R^n(\Pi^2(x^*)) \neq 0$. Thus, from Proposition 2 we have that $\sin(x_j^* - 0) = 0$ for $1 \leq j \leq m + n - 1$.

Inspection shows that $J(y^*)$ is of the form

$$J(y^*) = \frac{1}{m+n} \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where A and C are diagonal matrices of size $m \times m$ and $(n-1) \times (n-1)$ respectively, whose entries are given as

$$\begin{aligned} A_{ii} &:= -\sum_{j=1}^{n} \cos(x_{m+j}^* - x_i^*) \\ &= -\sum_{j=1}^{n} \cos(x_{m+j}^*) \cos(x_i^*); \\ C_{ii} &:= -\sum_{j=1}^{m} \cos(x_j^* - x_{m+i}^*) \; = \; -\sum_{j=1}^{m} \cos(x_j^*) \cos(x_{m+i}^*). \end{aligned}$$

The submatrix B is a matrix of dimension $m \times (n-1)$ whose entries satisfy the following relation:

$$\begin{aligned} B_{ij} &:= \cos(x_{m+j}^* - x_i^*). \\ &= \cos(x_{m+j}^*) \cos(x_i^*). \end{aligned}$$

Using the following identities:

$$R^m(\Pi^1(x^*))^2 = (\frac{1}{m}\sum_{j=1}^{m}\cos(x_j^*))^2; \quad (69)$$

$$R^n(\Pi^2(x^*))^2 = (\frac{1}{n}\sum_{j=1}^{n}\cos(x_{m+j}^*))^2, \quad (70)$$

we rewrite A_{ii} and C_{ii} as

$$A_{ii} = -nR^n(\Pi^2(x^*))\cos(x_i^*) \quad ; \quad C_{ii} = -mR^m(\Pi^1(x^*))\cos(x_{m+i}^*).$$

This shows that, under the hypotheses of the proposition all diagonal entries of A and C are non-zero. If we define the non-singular diagonal matrix $D = \mathrm{diag}(\cos(x_1^*),\ldots,\cos(x_{m+n-1}^*))$, then $J(y^*)$ is non-singular if and only if $S = D^{-1}J(y^*)$ is non-singular. We shall show that S is non-singular by contradiction. Suppose there is some non-zero vector $z \in \mathbb{R}^{m+n-1}$ with $Sz = 0$. Then by inspection the components of z must satisfy:

$$az_i + \sum_{j=1}^{n-1} b_j z_{m+j} = 0 \quad \text{for } 1 \le i \le m, \quad (71)$$

$$cz_i + \sum_{j=1}^{m} d_j z_j = 0 \quad \text{for } m \le i \le m+n-1, \quad (72)$$

where $a = -\sum_{j=1}^{n}\cos(x_{m+j}^*)$, $b_j = \cos(x_{m+j}^*)$ $c = -\sum_{j=1}^{m}\cos(x_j^*)$, $d_j = \cos(x_j^*)$. From (69), (70) it follows that a and c are non-zero. Now note the following easily verifiable facts.

(i) If $z_i = 0$ for any i with $1 \le i \le m+n-1$, then $z = 0$. Thus we can assume that each component of z is non-zero.

(ii) There are constants $\kappa_1, \kappa_2 \in \mathbb{R}$ such that $z_i = \kappa_1$ for $1 \le i \le m$ and $z_i = \kappa_2$ for $m \le i \le m+n-1$.

(iii) $a + \sum_{j=1}^{n-1} b_j = -1$.

(iv) $c + \sum_{j=1}^{m} d_j = 0$.

We can assume without loss of generality that $\kappa_1 = 1$. Then it follows from (72) that $c + \kappa_2(\sum_{j=1}^{m} d_j) = 0$. Thus (iv) implies that $\kappa_2 = 1$. But then (71) implies that $a + \sum_{j=1}^{n-1} b_j = 0$ which contradicts (iii) above. Thus, there is no non-zero $z \in \mathbb{R}^{m+n-1}$ satisfying $Sz = 0$ and hence S and $J(y^*)$ are non-singular as claimed. ∎

References

[1] J. Acebrón, L. Bonilla, C. Pérez Vicente, F. Ritort, and R. Spigler. The kuramoto model: a simple paradigm for synchronization phenomena. *Reviews of Modern Physics*, 77:137–185, 2005.

[2] S.J. Aton and E.D. Herzog. Come Together, Right Now: Synchronization of Rhythms in a Mammalian Circadian Clock. *Neuron*, 48(4):531–534, 2005.

[3] M. Bennett, M. Schatz, H. Rockwood, and K. Wiesenfield. Huygen's clocks. *Proc. R. Soc. Lond. A*, 458:563579, 2002.

[4] A. Beuter, L. Glass, M.C. Mackey, and M.S. Titcombe, editors. *Nonlinear Dynamics in Physiology and Medicine*. Springer, 2003.

[5] N. Chopra and MW Spong. On exponential synchronization of Kuramoto oscillators. *IEEE Transactions on Automatic Control (accepted)*, 2008.

[6] F. Cucker and S. Smale. Emergent behavior in flocks. *IEEE Transactions on Automatic Control*, 52(5):852–862, 2007.

[7] F. De Smet and D. Aeyels. Partial entrainment in the finite Kuramoto–Sakaguchi model. *Physica D: Nonlinear Phenomena*, 234(2):81–89, 2007.

[8] R. Diestel. *Graph Theory*. Springer-Verlag, 2000.

[9] S. Eubank et al. Modelling disease outbreaks in realistic urban social networks. *Nature*, 429:180–184, 2004.

[10] Ergun G. Human sexual contact network as a bipartite graph. *Physica A*, 308:483–488, 15 May 2002.

[11] L. Glass. Synchronization and rhythmic processes in physiology. *Nature Review*, 410:277–284, 2001.

[12] M. Golubitsky and I. Stewart. *The Symmetry Perspective: from Equilibrium to Chaos in Phase Space and Physical Space*. Birkhaüser Verlag, 2000.

[13] M. Golubitsky and Ian Stewart. Nonlinear Dynamics of Networks: the Groupoid Formalism. *Bulletin of the American Mathematical Society*, 43(3):305–364, 2006.

[14] C. Hammond, H. Bergman, and P. Brown. Pathological synchronization in parkinsons disease: networks, models and treatments. *Trends in Neurosciences*, 30(7):357–364, 2007.

[15] W. Huber, V. Carey, L. Long, S. Falcon, and R. Gentleman. Graphs in molecular biology. *BMC Bioinformatics*, 8 (Suppl. 6)(S8):1–14, 2007.

[16] A. Jadbabaie, J. Lin, and AS Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *Automatic Control, IEEE Transactions on*, 48(6):988–1001, 2003.

[17] A. Jadbabaie, N. Motee, and M. Barahona. On the stability of the kuramoto model of coupled nonlinear oscillators. In *Proceedings of the American Control Conference, Boston, USA*, pages 4296 – 4301, 2004.

[18] Y. Kuramoto. Self-entrainment of a population of coupled nonlinear oscillators. In H. Araki, editor, *International Symposium on Mathematical Problems in Theoretical Physics*, volume 39 of *Lecture Notes in Physics*. Springer, 1975.

[19] Y. Kuramoto. *Chemical Oscillations, Waves and Turbulence*. Springer, 1984.

[20] Y. Maistrenko, O. Popovych, and P. Tass. *Desynchronization and Chaos in the Kuramoto Model*, volume 671 of *Lecture Notes in Physics*. 2005.

[21] Y. Maistrenko, V. Popovych, O. Burylko, and P. Tass. Mechanism of Desynchronization in the Finite-Dimensional Kuramoto Model. *Physical Review Letters*, 93(8), 2004.

[22] Y. Maistrenko, V. Popovych, and P. Tass. Chaotic Attractor in the Kuramoto Model. *International Journal of Bifurcation and Chaos*, 15(11):3457–3466, 2005.

[23] L. Melloni, C. Molina, M. Pena, D. Torres, W. Singer, and E. Rodriguez. Synchronization of Neural Activity across Cortical Areas Correlates with Conscious Perception. *Journal of Neuroscience*, 27(11):2858, 2007.

[24] DC Michaels, EP Matyas, and J Jalife. Mechanisms of sinoatrial pacemaker synchronization: a new hypothesis. *Circ Res*, 61(5):704–714, 1987.

[25] R. Mirollo and S. Strogatz. The spectrum of the locked state for the kuramoto model of coupled oscillators. *Physica D*, 205:249266, 2005.

[26] R. Mirollo and S. Strogatz. The Spectrum of the Partially Locked State for the Kuramoto Model. *Journal of Nonlinear Science*, 7(4):309–347, 2007.

[27] J.D. Murray. *Mathematical Biology*. Springer, 3rd edition, 2002.

[28] R. Olfati-Saber and RM Murray. Consensus problems in networks of agents with switching topology and time-delays. *Automatic Control, IEEE Transactions on*, 49(9):1520–1533, 2004.

[29] G. Osipov, J. Kurths, and C. Zhou. *Synchronization in Oscillatory Networks*. Springer Series in Synergetics. Springer-Verlag, 2007.

[30] Derek Paley, Naomi Ehrich Leonard, Rodolphe Sepulchre, Daniel Grünbaum, and Julia K. Parrish. Oscillator models and collective motion: Spatial patterns in the dynamics of engineered and biological networks. *IEEE Control Systems Magazine*, 27(4):89–105, 2007.

[31] A. Pikovsky, M. Rosenblum, and J. Kurths. *Synchronization: a universal concept in nonlinear sciences*. Cambridge University Press, 2003.

[32] V. Popovych, Y. Maistrenko, and P. Tass. Phase chaos in coupled oscillators. *Physical Review E*, 71(065201), 2005.

[33] J. Restrepo, E. Ott, and B. Hunt. Onset of synchronization in large networks of coupled oscillators. *Physical Review E*, 71(3), 2005.

[34] Juan G. Restrepo, Edward Ott, and Brian R. Hunt. Emergence of coherence in complex networks of heterogeneous dynamical systems. *Physical Review Letters*, 96(25):254103, 2006.

[35] Juan G. Restrepo, Edward Ott, and Brian R. Hunt. Synchronization in large directed networks of coupled phase oscillators. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 16(1):015107, 2006.

[36] J. Rogge and D. Aeyels. Existence of partial entrainment and stability of phase locking behavior of coupled oscillators. *Progress of Theoretical Physics*, 112:921–942, 2004.

[37] J. Rogge and D. Aeyels. Stability of phase locking in a ring of unidirectionally coupled oscillators. *Journal of Physics A*, 37:11135–11148, 2004.

[38] Luca Scardovi, Alain Sarlette, and Rodolphe Sepulchre. Synchronization and balancing on the n-torus. *Systems and Control Letters*, 56(5):335–341, 2007.

[39] R. Sepulchre. Oscillators as systems and synchrony as a design principle. In L. Menini, L. Zaccarian, and C. T. Abdallah, editors, *Current trends in nonlinear systems and control*, Lecture Notes in Control and Information Sciences. Birkhauser, 2006.

[40] R. Sepulchre, D. Paley, and N.E. Leonard. Group coordination and cooperative control of steered particles in the plane. In K. Y. Pettersen, J. T. Gravdahl, and H. Nijmeijer, editors, *Group Coordination and Cooperative control*, Lecture Notes in Control and Information Sciences, pages 217–232. Spinger-Verlag, London, 2006.

[41] Rodolphe Sepulchre, Derek Paley, and Naomi Ehrich Leonard. Stabilization of planar collective motion with all-to-all communication. *IEEE Transactions on Automatic Control*, 52(5):811–824, May 2007.

[42] Rodolphe Sepulchre, Derek Paley, and Naomi Ehrich Leonard. Stabilization of planar collective motion with limited communication. *IEEE Transactions on Automatic Control*, 53(3):706–719, 2008.

[43] Jackie Shen. Cucker–smale flocking under hierarchical leadership. *SIAM Journal on Applied Mathematics*, 68(3):694–719, 2007.

[44] S. Strogatz. From kuramoto to crawford: exploring the onset of synchronization in populations of coupled oscillators. *Physica D*, 143:1–20, 2000.

[45] S. Strogatz and R. Mirollo. Stability of incoherence in a population of coupled oscillators. *Journal of Statistical Physics*, 63(3-4):613–635, 1991.

[46] P.A. Tass. Desynchronization of brain rhythms with soft phase-resetting techniques. *Biological Cybernetics*, 87(2):102–115, 2002.

[47] P.A. Tass. A model of desynchronizing deep brain stimulation with a demand-controlled coordinated reset of neural subpopulations. *Biological Cybernetics*, 89(2):81–88, 2003.

[48] T. To, M. Henson, E. Herzog, and F. Doyle. A molecular model for intercellular synchronization in the mammalian circadian clock. *Biophys. J.*, 92(11):3792–3803, 2007.

[49] F. Varela, J.-P. Lachaux, E. Rodriguez, and J. Martinerie. The brainweb: phase synchronization and large-scale integration. *Nature Reviews Neuroscience*, 2:229–239, 2001.

[50] M. Verwoerd and O. Mason. Global phase-locking in finite populations of phase-coupled oscillators. *SIAM Journal on Applied Dynamical Systems*, 7(1):134–160, 2008.

[51] K. Wiesenfeld, P. Colet, and S.H. Strogatz. Synchronization Transitions in a Disordered Josephson Series Array. *Physical Review Letters*, 76(3):404–407, 1996.

[52] K. Wiesenfeld, P. Colet, and S.H. Strogatz. Frequency locking in Josephson arrays: Connection with the Kuramoto model. *Physical Review E*, 57(2):1563–1569, 1998.

[53] S. Yamaguchi, H. Isejima, T. Matsuo, R. Okura, K. Yagita, M. Kobayashi, and H. Okamura. Synchronization of cellular clocks in the suprachiasmatic nucleus. *Science*, 302:1408–1412, 2003.