MANIFOLDS OF LOW COHOMOGENEITY
AND POSITIVE RICCI CURVATURE

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Abstract: *We classify compact asystatic G-manifolds with fixed point singular orbits in cohomogeneity ≤ 3 up to equivariant diffeomorphism. From this we derive existence results for invariant metrics of positive Ricci curvature on such objects. We also develop non-existence results for invariant metrics of positive Ricci curvature in cohomogeneity four.*

§1 Introduction

The most studied families of (Riemannian) manifolds are almost certainly those which display lots of symmetry. The homogeneous spaces (equipped with homogeneous metrics) are the most symmetric family of all. These are manifolds admitting a smooth (isometric) Lie group action which is transitive. Put another way, a homogeneous space is a manifold admitting a Lie group action for which the space of orbits consists of a single point. The topology and geometry of these spaces, is, for the most part well-understood.

The next most symmetric family of manifolds are those which admit a smooth action from a compact Lie group for which the space of orbits is one dimensional. These are the so-called cohomogeneity one manifolds. Such manifolds have a simple topological description. The space of orbits is either a cicle or an interval. In the first case, the manifold is just a homogeneous space bundle over the circle, and all orbits are principal orbits. In the second case, there are two non-principal orbits corresponding to the ends of the interval. Topologically, the manifold is a union of two disc bundles, for which the non-principal orbits form the zero-section. The boundary of each disc bundle (indeed every distance sphere, given an invariant metric) is a principal orbit, and therefore a homogeneous space. The entire manifold can be described by a group diagram, involving the main group, the principal isotropy and the two non-principal isotropy subgroups (see **[GZ1]**).

The geometry of cohomogeneity one manifolds, especially those for which the space of orbits is an interval, has been studied intensively in recent times. The reason that these objects form such a good family to study is that on the one hand, they have a simple topoplogical description, as discussed above. On the other hand, however, they form a large and rich class containing many interesting and important examples.

Of particular note is the role that cohomogeneity one manifolds continue to play in the search for new examples of manifolds with good curvature characteristics. If one

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considers invariant metrics, then symmetry reduces the problem of describing and analysing such metrics to one which has a reasonable chance of being tractable. For example, new families of manifolds with non-negative sectional curvature, including many exotic spheres in dimension seven, have been been discovered as a result of this approach [**GZ1**]. The cohomogeneity one condition in the context of positive sectional curvature has attracted particular attention due to the work of Grove, Ziller, Wilking, Verdiani and others. (See for example [**GWZ**], [**GVWZ**], [**V1**], [**V2**].) In an exciting new development, Grove, Verdiani and Ziller [**GVZ**] and independently Dearricott [**D**] have just announced the existence of a new cohomogeneity-one manifold with positive sectional curvature. Together with the recent announcement of a positive sectional curvature metric on the Gromoll-Meyer sphere by Petersen and Wilhelm [**PW**], these are the first new examples of manifolds admitting positive sectional curvature metrics for a number of years.

Given the successes achieved by the study of cohomogeneity one manifolds, it is natural to ask about manifolds of cohomogeneity two, or of other low cohomogeneities. It seems reasonable to expect that new interesting geometric phenomena should arise in these contexts also. However, as in the cohomogeneity one case, it seems prudent to first understand the topological consequences of admitting a Lie group action with low cohomogeneity. In cohomogeneity two, for example, the space of orbits will be a surface, but in general this surface will have boundaries and/or singularities, and understanding the various possibilities is a challenging problem. Furthermore, as the cohomogeneity increases, the possible manifold structures increase in complexity quite dramatically.

The main motivation behind this paper was to better understand the structure of manifolds of cohomogeneity two and three, and to derive some geometric consequences. This is most likely a difficult and long term proposal, however the results in this paper can be viewed as a first attempt to address this issue.

To try and make the situation tractable, we impose two extra conditions on the group action besides the cohomogeneity restriction. These conditions are satisfied, for example, whenever the principal orbits are isotropy irreducible, [**PS**].

The actions of low cohomogeneity we will study in this paper will be assumed to be *asystatic*:

Definition [AA]. *Let G be a compact, connected Lie group, and K a closed subgroup.*

(i) The homogeneous space G/K is called *asystatic* if the isotropy representation of K has no fixed vector.
(ii) A manifold X on which G acts smoothly is called asystatic if the principal orbits are asystatic.

For more details about the asystatic condition and its related geometry, see §2 and [**AA**]. The motivation behind adopting this condition comes from [**AA**], where it is shown that asystatic manifolds form a rich class of spaces with interesting geometric properties. Note also the similarity between the asystatic condition and the (Riemannian) notion of a *polar action*. This is explored in Lemma 3 below.

In general, orbit types fall into three categories: principal orbits, exceptional orbits (that is, non-principal orbits with the same dimension as principal orbits), and singular orbits (that is orbits of lower dimension). Our second assumption about the group actions on our manifolds concerns the singular orbits. Understanding the singular orbits is a crucial issue if one aims to understand the structure of manifolds with low cohomogeneity. With this is mind, we demand that singular orbits take only the simplest possible form: we assume that the singular orbits are precisely the fixed points of the action.

Throughout this paper, G will be a compact connected Lie group acting smoothly on the compact connected manifold X^n. By K we will denote a principal isotropy group so that the principal orbits are $F = G/K$. The main topological results established in this paper are as follows.

Theorem A. *Let X^n be a compact asystatic G-manifold of cohomogeneity 2 with finite fundamental group. Suppose that the singular orbits (if any) are precisely the fixed points. Then X is equivariantly diffeomorphic to one of the following:*

1. *$S^n \subset \mathbb{R}^2 \times \mathbb{R}^{n-1}$ where G acts transitively on the sphere of the second factor,*
2. *$\mathbb{R}\mathrm{P}^n = (\mathbb{R}^2 \times \mathbb{R}^{n-1})/\mathbb{R}^\times$ where G acts transitively on the sphere of the second factor,*
3. *$(S^2 \times G/K)/\Gamma$ or $(\mathbb{R}\mathrm{P}^2 \times G/K)/\Gamma$ where K is a principal isotropy group and $\Gamma \subset N_G K/K$ is any subgroup of the Weyl group acting on S^2 or $\mathbb{R}\mathrm{P}^2$ from the right.*

Theorem B. *Let X^n be a compact asystatic G-manifold of cohomogeneity 3 with finite fundamental group. Suppose that the singular orbits (if any) are precisely the fixed points. Then X is equivariantly diffeomorphic to a quotient $\tilde{X}/\Gamma$ of one of the following by a free action of a finite group $\Gamma \subset O(4) \times N_G K/K$:*

1. *$\tilde{X} = S^n \subset \mathbb{R}^3 \times \mathbb{R}^{n-2}$ where G acts transitively on the sphere S^{n-3} in the second factor,*
2. *$\tilde{X} = \#_k S^2 \times S^{n-2}$, $k > 1$, where the G-action is given by the diffeomorphism of Lemma 12,*
3. *$\tilde{X} = S^3 \times G/K$.*

Note that compact cohomogeneity-1-manifolds with fixed point singular orbits can only be $S^n \subset \mathbb{R} \times \mathbb{R}^n$, and $\mathbb{R}\mathrm{P}^n = S^n/\pm$ with some G action transitive on the sphere in $\mathbb{R}^n$.

With a little further work, it is possible to derive some geometric consequences of Theorem A. To put these in context, let us recall that a compact homogeneous space always admits an invariant metric of non-negative sectional curvature, and admits an invariant metric of positive Ricci curvature if and only if the fundamental group is finite. In the case of cohomogeneity one, the analogous statement for non-negative sectional curvature is not true (see [**GVWZ**]). On the other hand, the corresponding statement about positive Ricci curvature is still valid:

Theorem [Grove, Ziller [GZ2]]. *A compact cohomogeneity one manifold admits an invariant metric with positive Ricci curvature if and only if its fundamental group is finite.*

Grove and Ziller ask what is the largest cohomogeneity for which the conclusion of the above theorem is true. As a corollary of Theorem A we obtain the analogue of the result of Grove and Ziller for the type of G-actions considered here:

Theorem C. *Suppose that X^n is a compact asystatic G-manifold of cohomogeneity two or three, for which the singular orbits (if any) are precisely the fixed points. Then X admits a G-invariant metric of positive Ricci curvataure if and only if the fundamental group $\pi_1(X)$ is finite.*

The extra orbit conditions which we impose might actually be superfluous from the positive Ricci curvature point of view. Thus it might be true generally that a compact G-manifold of cohomogeneity two or three admits an invariant metric of positive Ricci curvature if and only if the fundamental group is finite.

The proofs of Theorems B and C hinge on the fact that an action of a finite group on S^3 is conjugate to an isometric one. For free actions this is a consequence of Perelman's Ellipization Theorem, [**P1,P2,P3**] and for non-free orientation preserving actions it follows from the Orbifold Geometrization Theorem in [**BLP**] and [**CHK**]. For cyclic groups it is a consequence of Theorem 2.2 of [**K**] together with the fact that the Smith Conjecture has been shown to be true (see for example [**MB**]). Without any of these assumptions the full conjugacy result was recently proved in [**DL**].

In [**GZ2**] it is remarked that the cohomogeneity one result above cannot possibly be extended to cohomogeneity four: for example if G is the trivial group acting on a $K3$-surface (a simply-connected 4-manifold which fails to admit even a positive scalar curvature metric). However, this raises the question of whether simply-connected manifolds of dimension greater than four admitting a cohomogeneity four action necessarily admit invaraint metrics of positive Ricci curvature. We show that again, the answer is no:

Theorem D. *Suppose that G/K is a compact isotropy irreducible homogeneous space of dimension p, and let $X = \Sigma^n \times G/K$ for some compact manifold Σ^n. View X as a G-manifold with the obvious G-action and assume X admits a G-invariant metric of positive Ricci curvature. If*

$$n = 2 \quad \text{or} \quad n \geq 3 \text{ and } p \leq \frac{4n-4}{n-2} \,,$$

then Σ admits a metric of positive scalar curvature.

This immediately gives:

Theorem E. *There are compact, simply-connected manifolds in dimensions 5 through 10 admitting an asystatic fixed-point-free cohomogeneity four action, which admit no invariant metric with positive Ricci curvature.*

Example. *If $p \leq 6$ and Σ^4 is a simply-connected spin manifold with non vanishing signature (such as a K3 surface), then $X = \Sigma^4 \times S^p$ does not admit a metric of positive Ricci curvature invariant under a group (such as $SO(p+1)$) which acts transitively and isotropy irreducibly on S^p.*

Notice that we cannot rule out the possibility that examples such as the above admit *some* Ricci positive metric, only that there is no such metric invariant under the given action. It is an open question whether there is necessarily a Ricci positive metric on a (compact) product manifold where one factor admits a Ricci positive metric and the other factor is simply-connected and does not admit such a metric.

This paper is laid out as follows. In §2 we present the basic results about asystatic manifolds which we will need in later sections. In §3 we prove Theorems A and C in cohomogeneity 2. In §4 we prove Theorems B and C in cohomogeneity 3 and in §5 we establish Theorem D.

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§2 Asystatic manifolds

Let G be be a compact connected Lie group acting smoothly on a compact connected manifold X. An orbit Gq, $q \in X$, is principal if the isotropy group $G_q = \{g \in G \mid gq = q\}$ is minimal among all isotropy groups of the G-action on X. The regular points, i.e. those lying in a principal orbit, form an open dense subset of X. We fix such a regular point $p \in X$ and let $K = G_p$ be its isotropy group. The fixed point set $X^K \subset X$ of K is a closed submanifold of X. We denote by X_0^K the connected component of X^K containing p. The Lie algebras of G, H, K will be denoted by $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ respectively.

We will often rely on the following

Lemma 1. *Each G-orbit meets X_0^K, and the principal orbits meet X_0^K transversally. Moreover if $X^G \neq \emptyset$ then $X_0^K = X^K$, i.e. X^K is connected.*

Proof: The first statement follows from [**AA**], Lemma 1.5. For the second statement, we observe that each component of X^K must intersect every orbit. But each point in X^G is itself an orbit. Therefore all components of X^K must have all points of X^G in common, hence there can only be one connected component. ∎

The following lemma is self-evident:

Lemma 2. *For a G-manifold X and a regular point $p \in X$ as above the following are equivalent.*

(i) The intersection $X_0^K \cap Gp$ is discrete.

(ii) There is no nonzero vector in the isotropy representation $T_pG_p \cong \mathfrak{g}/\mathfrak{k}$ fixed by $K = G_p$.

The G-manifold X is asystatic if these conditions are satisfied.

Lemma 3. *If X is an asystatic G-manifold with G a compact Lie group, then the following statements hold.*

(i) With respect to any G-invariant Riemannian metric on X the submanifold X_0^K is totally geodesic. Also the G-action is polar with section X_0^K, i.e. X_0^K meets each G orbit and the intersection is perpendicular.

(ii) If $x \in X_0^K$ lies in a principal orbit and $g \in G$ with $gx \in X_0^K$, then $gX_0^K = X_0^K$.

(iii) $N_GK = \{g \in G \mid gX^K = X^K\}$.

(iv) The map

$$\Phi\colon X_0^K \times G/K \to X \quad , \quad (s, gK) \mapsto gs \tag{4}$$

is G-equivariant and surjective. Over the set X^0 of regular or exceptional points, Φ is a covering.

(v) *The group of deck transformations of this covering is a subgroup W_0 of the Weyl group $W = N_G K/K$. It is finite and acts on X_0^K and freely on G/K. Explicitly, the action is given by*

$$w(s, gK) = (qs, gq^{-1}K)$$

where $w \in W$ can be represented as qK for some $q \in N_G K$.

Proof: We offer a justification for the covering property of (4), refering the reader otherwise to **[AA]**. Recall that X^0 is the set of points whose orbits have maximal dimension. If $q \in X^0$ we may assume that $K \subset H = G_q$ is of finite index. Then $\mathfrak{h} = \mathfrak{k}$ and the differential of Φ,

$$d_{(q,eK)}\Phi\colon T_q X_0^K \oplus T_e(G/K) = T_q X_0^K \oplus T_e(G/H) = T_q X_0^K \oplus T_q Gq \to T_q X$$

is an isomorphism. By G equivariance Φ is a local diffeomorphism over all of X^0. ∎

The isotropy group of G at $q \in X$ is a closed subgroup conjugate to some group H with $K \subseteq H \subseteq G$. If the principal orbits are isotropy irreducible, i.e. K acts irreducibly on $T_x Gx \cong \mathfrak{g}/\mathfrak{k}$, then there are three possibilities:
(i) q is a fixed point, $H = G$;
(ii) q is exceptional, $K \subset H \subset G$;
(iii) q is regular, $H = K$.
Furthermore a G-invariant metric g^F on $F = G/K$ is always Einstein, i.e. the Ricci tensor $\mathrm{ric}^F = \lambda g^F$ with a positive constant λ. For most of the following we will only need that there are no more than these three types of orbits and that F admits an invariant metric of positive Ricci curvature.

For a submanifold $L \subset X$ we will denote by $X \backslash\!\backslash L$ the manifold with boundary obtained by closing the interior of $X \setminus L$ with the sphere bundle of the normal bundle $\nu(L, X)$ of L in X. Thus $X \backslash\!\backslash L$ is a manifold with

$$\partial(X \backslash\!\backslash L) = S\nu(L, X) \text{ and } \mathrm{int}(X \backslash\!\backslash L) = X \setminus L \ .$$

If X is a manifold with boundary and F is a manifold we will denote by $X \times_\partial F$ the space obtained by collapsing F over the boundary. Thus

$$X \times_\partial F = X \times F/_\sim \cong X \times F \cup_\partial \partial X \times CF$$

where CF denotes the cone over F and we identify $(\xi, f) \sim (\xi', f') \in X \times F$ if $\xi = \xi' \in \partial X$. If X is a manifold *without* boundary, we set $X \times_\partial F = X \times F$.

Lemma 5. *Let X be an asystatic G-manifold and assume that the G-orbits are either fixed points, exceptional or principal orbits. Then we have two possibilities:*
(i) There are no fixed points. In this case (4) is a covering.
(ii) There are fixed points. In this case $G/K = S^p$ is a sphere, $X^G \subset X_0^K = X^K$ is a totally geodesic hypersurface and

$$(X^K \backslash\!\backslash X^G)_0 \times_\partial S^p \to X \tag{6}$$

is a 2-fold covering or a diffeomorphism, depending on whether X^G separates X^K or not. Here $(X^K \backslash\backslash X^G)_0$ denotes the connected component containing p. The Weyl group has order $|W_0| = |W| = 2$.

Proof: If $X^G \neq \emptyset$ then G acts on $S\nu(X^G, X)$ and has only principal or exceptional orbits there. We have coverings

$$S\nu(X^G, X^K) \times G/K \to S\nu(X^G, X) \text{ and } S\nu_q(X^G, X^K) \times G/K \to S\nu_q(X^G, X)$$

which is only possible if $S\nu_q(X^G, X^K) \cong S^0$, $G/K \cong S\nu_q(X^G, X) \cong S^p$. In particular $\dim X^G = \dim X^K - 1$. Since (4) is a covering away from the fixed points, we have a covering $X^K \backslash\backslash X^G \times G/K \to X \backslash\backslash X^G$. This induces the covering (6). ■

For the proofs of Theorems A and B we will need to replace the section $(X^K \backslash\backslash X^G)_0$ by a simply connected one by passing to a suitable covering.

Lemma 7. *Compact asystatic G-manifolds X with only principal, exceptional and fixed point orbits admit a G-equivariant covering*

$$\Sigma \times_\partial F \longrightarrow X \tag{8}$$

where Σ is a simply connected compact manifold which has non-empty boundary precisely when X has G-fixed points, and $F = G/K$ is a connected compact homogeneous space. This action of the group Γ of deck transformations of (8) on $\Sigma \times_\partial F$ is the product of an action of Γ on Σ and an action on F.

Proof: We have already seen that we have a covering $(X^K \backslash\backslash X^G)_0 \times_\partial F \to X$ whose group of deck transformations is W_0. Let $\Sigma = \widetilde{(X^K \backslash\backslash X^G)}_0$ be the universal covering of $(X^K \backslash\backslash X^G)_0$. Clearly, $\partial\Sigma \neq \emptyset$ if and only if $X^G \neq \emptyset$. The coverings

$$\Sigma \times_\partial F \longrightarrow (X^K \backslash\backslash X^G)_0 \times_\partial F \longrightarrow X$$

are G equivariant and therefore the Γ-action commutes with that of G.

In order to see the product property, we write this action as

$$\gamma(s, f) = (\gamma_\Sigma(s, f),\ \gamma_F(s, f)),\ \gamma \in \Gamma,\ s \in \Sigma,\ f \in F\ .$$

By G equivariance, the first component $\gamma_\Sigma(s, f) = \gamma_\Sigma(s)$ is independent of f and defines an action of Γ on Σ. Also, $\gamma_F(s, f) \in F^K$ if $f \in F^K$, which is discrete because the G-action on F is asystatic. Since Σ is connected, $\gamma_F(s, f) = \gamma_F(f)$, $f \in F^K$, cannot depend on s, and again by G-equivariance, this holds for all $f \in F$. We therefore have actions of Γ on Σ and F separately. ■

On the fibre we may always assume that the Γ-action preserves a Ricci positive metric. We show

Lemma 9. *Let $F = G/K$ be a normal Riemannian homogeneous space with an action of a finite group Γ which commutes with the action of G. Then Γ acts isometrially. In particular we have a $\Gamma \times G$-invariant metric of positive Ricci curvature on F provided $\pi_1(F)$ is finite. If $F = S^p$ we can take the round metric.*

Proof: The group Γ acts on F via the Weyl group. The homomophism $\Gamma \to W = N_G(K)/K$ is given as follows: since the Γ-action commutes with that of G, we have $\gamma hK = h(\gamma K) = h(gK)$ for some $g = g(\gamma) \in G$ and all $h \in G$. Also $\Gamma F^K = F^K$ and therefore $g(\gamma) \in N_G(K)$. Thus the Γ-action stems from the right action of W on G which preserves a bi-invariant metric on G. For the last claim of the Lemma, we have from **[B]** that a compact normal homogeneous space with finite fundamental group has positive Ricci curvature. It is well known, (see 7.13 of **[Be]**), that an effective compact connected transitive group of diffeomorphisms of S^p is conjugate to a subgroup of $\mathrm{SO}(p+1)$. The standard metric is then normal with respect to such a group and therefore invariant under Γ. ∎

§3 Cohomogeneity Two

In this section we prove Theorem A and the cohomogeneity-2-part of Theorem C. Therefore let X^n be a compact asystatic G-manifold of cohomogeneity two, for which the singular orbits (if any) are precisely the fixed points. Recall from (8) that in either case we have a covering

$$\Sigma \times_\partial F \longrightarrow X$$

where Σ is a simply connected surface, hence $\Sigma = S^2$ or D^2.

In the case $X^G = \emptyset$ we have $\Sigma \times_\partial F = \Sigma \times F$, the only possibilities for $(X^K \setminus\setminus X^G)_0 = \Sigma/\Gamma$ are S^2, $\mathbb{R}\mathrm{P}^2$. Thus $X = S^2 \times F/\Gamma$ or $X = \mathbb{R}\mathrm{P}^2 \times F/\Gamma$ with Γ acting G-equivariantly on F, hence $\Gamma \subset N_G K/K$.

If $X^G \neq \emptyset$ then $\Sigma = D^2 = (X^K \setminus\setminus X^G)_0$ has no quotients. We must have $X = D^2 \times_\partial S^p \cong S^{p+2}$ or $X = (D^2 \times_\partial S^p)/\tau \cong \mathbb{R}\mathrm{P}^{p+2}$ where τ is the involution given by $\tau(s,x) = (-s,-x)$ for $s \in D^2$ and $x \in S^p \subset \mathbb{R}^{p+1}$. (As we are assuming that G is connected, this latter situation can only arise when the map $S^p \to S^p$ given by $x \mapsto -x$ is orientation preserving, that is, when p is odd.)

In order to have any metric of positive Ricci curvature we must have $\pi_1(X)$ finite by Myers' Theorem. Conversely, if $\pi_1(X)$ is finite then so is $\pi_1(G/K)$. By Lemma 9, F has a Γ-invariant metric of positive Ricci curvature and thus the manifolds listed above admit a G-invariant metric of positive Ricci curvature ∎

§4 Cohomogeneity three

We now turn our attention to the case of a cohomogeneity three action. In order to list the possible manifolds X we again rely on the covering (8),

$$\Sigma \times_\partial F \longrightarrow X \ ,$$

where Σ is now a simply connected 3-manifold. In the case that $X^G = \partial\Sigma = \emptyset$, we have $\Sigma = S^3$ by Perelman's resolution of the Poincaré conjecture ([**P1,P2,P3**]), and Γ acts on S^3. From [**DL**] we have that this action must be orthogonal (up to conjugacy by a diffeomorphism of S^3). Thus X is a quotient of $S^3 \times G/K$ by a subgroup $\Gamma \subset O(4) \times N_G K/K$.

If G has fixed points then $X^G = \partial\Sigma$ is a nonempty surface. From the long exact sequence of the pair $(\Sigma, \partial\Sigma)$ and Poincaré duality we have

$$0 = H^1(\Sigma; \mathbb{Z}/2) \cong H_2(\Sigma, \partial\Sigma; \mathbb{Z}/2) \to H_1(\partial\Sigma; \mathbb{Z}/2) \to H_1(\Sigma; \mathbb{Z}/2) = 0 \ , \tag{10}$$

and from this we infer that $H_1(\partial\Sigma; \mathbb{Z}/2) = 0$ and therefore $\partial\Sigma$ is a disjoint union of spheres $S^2_1 \cup \cdots \cup S^2_r$. Glueing Σ with discs D^3_j along the boundary we get a closed simply connected 3-manifold

$$\Sigma \cup_{S^2_1 \cup \cdots \cup S^2_r} (D^3_1 \cup \cdots \cup D^3_r)$$

which is diffeomorphic to S^3, again by [**P1,P2,P3**]. The Γ-action extends in the obvious way to again give an action on S^3. As before we may assume that this action is isometric with respect to the standard (Ricci positive) metric on S^3 and $\Gamma \subset O(4) \times N_G K/K$.

The principal orbits $F = S^{n-3}$ are spheres (Lemma 5) and we have a covering

$$\left(S^3 \setminus \coprod_{i=0}^{k} D^3_i\right) \times_\partial S^p \cong \left(S^3 - \coprod_{i=0}^{k} D^3_i\right) \times S^{n-3} \cup_{\mathrm{id}} \coprod_{i=0}^{k} S^2_i \times D^{n-2} \longrightarrow X \ . \tag{11}$$

By the following well-known topological (surgery) result the left hand side of (11) is a connected sum. See for example [**W**] for a proof.

Lemma 12. *We have a diffeomorphism*

$$\sharp_{i=1}^{k} S^m \times S^{p+1} \cong (S^{m+1} - \coprod_{i=0}^{k} D^{m+1}_i) \times S^p \cup_{id} \coprod_{i=0}^{k} S^m_i \times D^{p+1} \tag{13}$$

where the left hand side is read as S^{m+p+1} when $k = 0$.

This completes the proof of Theorem B.

For the proof of the second part of Theorem C in the case $X^G = \emptyset$ we can use the product of the standard metric with a normal one on F as before because of Lemma 9. If $X^G \neq \emptyset$ we use the Ricci positive metric on the left hand side of (11) provided by a construction in [**SY**]:

Lemma 14. *If $m, p+1 \geq 2$, we can choose a product of round metrics on $S^{m+1} \times S^p$ and a warped product of round metrics on each $S^m_i \times D^{p+1}$ of the form $dr^2 + f^2(r) ds^2_m + h^2(r) ds^2_p$ for suitable scaling functions $h(r)$, $f(r)$, $(r \geq 0)$, such that the resulting metric on the right hand side of (13) is both smooth and Ricci positive.*

Proof: See Lemma 1 of [**SY**]. ■

In particular this Lemma applied to the left hand side of (11) yields a metric invariant under the actions of $\mathrm{SO}(p+1)$ and Γ. Therefore the metric given in Lemma 14 descends to a metric on X. This finishes the proof of Theorem C. ■

§5 The proof of Theorem D

We have a product manifold $X = \Sigma^n \times G/K$ with the obvious G-action, and where G/K is an isotropy irreducible, compact homogeneous space of dimension p, with $\pi_1(G/K)$ finite. The finiteness of $\pi_1(G/K)$ means that G/K admits an invariant Ricci positive metric. Moreover, as G/K is isotropy irreducible, this (Einstein) metric is unique up to scaling. We fix such a metric g and let $\lambda > 0$ be its Einstein constant.

Lemma 15. *A G-invariant metric on $\Sigma \times G/K$ is a warped product $\sigma + f^2 g$ where σ is a metric on Σ, and $f\colon \Sigma \to \mathbb{R}^+$ a positive function on Σ.*

Proof: The initial step is to show that Σ is orthogonal to Gx at all points $x \in \Sigma$. To see this, consider the isotropy represesentation of K at x on

$$T_x X = T_x \Sigma \oplus T_x Gx \ .$$

As K-modules, $T_x Gx$ is irreducible and non trivial and $T_x\Sigma$ is trivial. The scalar product defines a K-equivariant homomoprhism $\alpha\colon T_x Gx \to (T_x\Sigma)^*$, assigning to $v \in T_x Gx$ the linear form $\alpha(v)$ on $T_x\Sigma$ with $\alpha(v)s = \langle v \mid s\rangle$. But by Schur's Lemma α must vanish.

By G-invariance the translates $\Sigma \times gK$ carry the same metric σ for all $g \in G$, and the metrics on the orbits G/K are determined up to scaling. It follows that the metric is of the form $\sigma + f^2 g$ where $f(p)^2 g$ is the metric of the orbit Gx. ■

Let us assume there exists a G-invariant metric of positive Ricci curvature $\Sigma \times G/K$. By Lemma 15, such a metric must necessarily be a warped product

$$\sigma + f^2 g \text{ for some function } f\colon \Sigma \to \mathbb{R}^+ \ .$$

By §9J of **[Be]** the warping function f must satisfy

$$\begin{aligned} \lambda + f\Delta f - (p-1)|df|^2 &> 0 \text{ and} \\ \mathrm{Ric}(\sigma)(v,v) - p\frac{\nabla df(v,v)}{f} &> 0 \text{ for all } v \in T\Sigma \end{aligned}$$

in order to have positive Ricci curvature on X. Taking the trace of the second expression gives

$$\mathrm{scal}(\sigma) + p\frac{\Delta f}{f} > 0 \ .$$

For this function f, set $h = f^r$ for some $r \in \mathbb{R}^+$ to be chosen below, and consider the metric $h^2\sigma$ on Σ. This will have positive scalar curvature if

$$\mathrm{scal}(\sigma) + 2(n-1)\frac{\Delta h}{h} \ > \ (n-4)(n-1)\frac{|dh|^2}{h^2} \ . \tag{16}$$

Inserting

$$dh = rf^{r-1}df \quad \text{and} \quad \Delta h = rf^{r-1}\Delta f - r(r-1)f^{r-2}|df|^2$$

in (16) yields

$$\mathrm{scal}(\sigma) + 2(n-1)r\frac{\Delta f}{f} \; > \; \big(2(n-1)r(r-1) + r^2(n-4)(n-1)\big)\,\frac{|df|^2}{f^2}\ . \tag{17}$$

With $r = p/(2n-2)$ the left hand side of (17) is positive. The right hand side of (17) becomes nonpositive if

$$2(n-1)r(r-1) + r^2(n-4)(n-1) = p\left(\frac{n-2}{4n-4}p - 1\right) \leq 0\ ,$$

which is the assumption of Theorem D. ■

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