Rabbit Mathematics

Anthony G. O'Farrell

1. Fibonacci numbers. Leonardo of Pisa, also known as Fibonacci, introduced the Arabic numerals 0, 1, 2, 3, ... to Europeans, in his book, Liber abaci (1202). His most famous invention is the sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

of Fibonacci numbers. You can arrive at this by imagining what would happen if you started with one pair of newborn rabbits, and assume that

(1) the time from birth to maturity equals the gestation period, and

(2) each mature pair produced a new pair at the end of each period. Ignoring all further possible complications, the initial pair mature in one period, then produce another pair at the end of the next period, giving two pairs. At the end of the next period you have a new pair from the original pair, making 3 in all. At the end of the next period you have 2 new pairs, making 5, and so on.

If we denote by Y_n the number of pairs at the end of the n-th period, we see that

$$Y_o = Y_1 = 1 \tag{1.1}$$

$$Y_{n+1} = Y_n + Y_{n-1}. \tag{1.2}$$

The equation (1.2) is an example of what is called a recursion relation, or, sometimes, a difference equation. The equation (1.1) specifies the initial conditions. If we vary (1.1) without changing (1.2), we arrive at other sequences that behave in the same kind of way as the Fibonacci numbers. For instance, if you take

$$Y_o = 1,$$

$$Y_1 = 3,$$

you get the sequence

1, 3, 4, 7, 11, 18, 29, 47, ...

of Lucas numbers. In terms of rabbits, this is what happens if you start with one mature and two immature pairs.

Those of you with computers may enjoy writing a program which takes Y_o and Y_1 and produces Y_2, Y_3, Y_4, and so on. If you do this, you will notice that the sequence usually grows extremely rapidly, and it is impractical to get Y_n for large n. Indeed, even for quite modest n, like 500, the numbers usually overflow.

If you get the computer to calculate the ratio Y_n/Y_{n-1}, it is remarkable that this invariably approaches a limit of about 1.6.

I will have more to say about calculating Y_n, and about this ratio, further on.

2. Difference equations Let a, b, and c be any constant (real) numbers, with $a \neq 0$, and consider the equation

$$a\,Y_{n+1} + b\,Y_n + c\,Y_{n-1} = 0. \tag{2.1}$$

This is the general type of a "second-order constant-coefficient homogeneous difference equation". It reduces to the Fibonacci example (1.2) when $a = 1$, $b = c = -1$.

(Some of you are familiar with the properties of second-order constant-coefficient ordinary differential equations

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = 0.$$

You will notice that there is a complete formal analogy between what I am about to show you and the theory of such differential equations.)

Let V denote the set of sequences $\{Y_n\}(n = 0, 1, 2, 3, \ldots)$ which satisfy (2.1).

If Y_n^1 and Y_n^2 are two sequences belonging to V, then their sum $Y_n^1 + Y_n^2$ belongs to V. For,

$$a\, Y_{n+1}^1 + b\, Y_n^1 + c\, Y_{n-1}^1 = 0,$$

$$a\, Y_{n+1}^2 + b\, Y_n^2 + c\, Y_{n-1}^2 = 0,$$

so, adding,

$$a(Y_{n+1}^1 + Y_{n+1}^2) + b(Y_n^1 + Y_n^2) + c(Y_{n-1}^1 + Y_{n-1}^2) = 0.$$

Further, if Y_n belongs to V and α is any real number, then αY_n belongs to V. For, on multiplying across in

$$a\, Y_{n+1} + b\, Y_n + c\, Y_{n-1} = 0$$

by α, we obtain

$$a(\alpha Y_{n+1}) + b(\alpha Y_n) + c\,(\alpha Y_{n-1}) = 0.$$

Thus V is a vector space over the real numbers.

Let Y_n^1, Y_n^2, Y_n^3 be three sequences belonging to V. Then we can choose real numbers α, β, γ, such that

$$\left.\begin{aligned} \alpha Y_o^1 + \beta Y_o^2 + \gamma Y_o^3 &= 0 \\ \alpha Y_1^1 + \beta Y_1^2 + \gamma Y_1^3 &= 0 \end{aligned}\right\} \qquad (2.2)$$

(This can be done because (2.2) just says that the vector (α, β, γ) is perpendicular to the two vectors (Y_o^1, Y_o^2, Y_o^3) and (Y_1^1, Y_1^2, Y_1^3)). It then follows from (2.1), and induction, that

$$\alpha Y_n^1 + \beta Y_n^2 + \gamma Y_n^3 = 0$$

for $n = 2, 3, 4, \ldots$.

Thus any three sequences belonging to V are linearly related, so the dimension of V is at most 2.

On the other hand, the sequences

$$Y_o^1 = 0,\quad Y_1^1 = 1,\quad Y_2^1 = -{}^b/_a,\quad Y_3^1 = -{}^c/_a + {}^{b^2}/_{a^2},\ \ldots$$

$$Y_o^2 = 1,\quad Y_1^2 = 0,\quad Y_2^2 = -{}^c/_a,\quad Y_3^2 = {}^{bc}/_{a^2},\ldots$$

provide 2 linearly independent solutions of (2.1). so the dimension of V is exactly 2.

This means that, whenever we find two independent solutions Y_n^1 and Y_n^2 for (2.1), the general solution is

$$Y_n = \alpha Y_n^1 + \beta Y_n^2$$

for constant α and β.

3. Explicit solutions. Suppose we look for a solution to (2.1) in the form

$$Y_n = \lambda^n$$

for some constant λ. Substituting, we obtain

$$a\,\lambda^{n+1} + b\lambda^n + c\lambda^{n-1} = 0. \tag{3.1}$$

Assuming $\lambda \neq 0$, we get

$$a\lambda^2 + b\lambda + c = 0, \tag{3.2}$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.3}$$

$$= \lambda_1,\ \lambda_2,\ \text{say.}$$

Turning it around, we see that whenever λ satisfies the quadratic (3.2), $Y_n = \lambda^n$ is a solution of (2.1). So as long as $\lambda_1 \neq \lambda_2$, we get the general solution of (2.1) in the form

$$Y_n = \alpha\lambda_1^n + \beta\lambda_2^n.$$

4. Fibonacci again. Let us go back and apply this to the Fibonacci sequence. The difference equation was

$$Y_{n+1} - Y_n - Y_{n-1} = 0,$$

so the quadratic is

$$\lambda^2 - \lambda - 1 = 0.$$

This has the roots

$$\lambda_1, \lambda_2 = \frac{1 \pm \sqrt{5}}{2}.$$

Notice that $\lambda_1\lambda_2 = -1$, i.e.

$$\frac{1 + \sqrt{5}}{2} = -\left(\frac{1 - \sqrt{5}}{2}\right)^{-1}.$$

According to what we learned just now, the general solution of (1.2) is

$$Y_n = \alpha\left(\frac{1 + \sqrt{5}}{2}\right)^n + \beta\left(\frac{1 - \sqrt{5}}{2}\right)^n \qquad (4.1)$$

for constant α and β. Putting in the initial conditions $Y_o = Y_1 = 1$ for the Fibonacci numbers, we get

$$1 = \alpha + \beta,$$

$$1 = \alpha\left(\frac{1 + \sqrt{5}}{2}\right) + \beta\left(\frac{1 - \sqrt{5}}{2}\right),$$

$$\alpha = \frac{1 + \sqrt{5}}{2\sqrt{5}}, \qquad \beta = \frac{\sqrt{5} - 1}{2\sqrt{5}},$$

and we arrive at

$$Y_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1 + \sqrt{5}}{2}\right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{n+1}\right\}. \qquad (4.2)$$

This is a startling formula for whole numbers! For instance, it says that

$$233 = \frac{1}{\sqrt{5}}\left\{\left(\frac{1 + \sqrt{5}}{2}\right)^{13} - \left(\frac{1 - \sqrt{5}}{2}\right)^{13}\right\}.$$

Of course, what happens is that when you multiply out the right-hand side, the $\sqrt{5}$'s all drop out and the 2's under the line all cancel.

You might like to try your hand at finding the corresponding formula for the Lucas numbers.

The number $\frac{1 + \sqrt{5}}{2}$ is about 1.618, so its powers grow rapidly. For instance,

$$\left(\frac{1 + 5}{2}\right)^{100} \doteqdot 7.9 \times 10^{20}.$$

By the same token, its reciprocal $\frac{\sqrt{5} - 1}{2}$ has rapidly-decaying powers. Thus in formula (4.2), the second term on the right is rapidly dwarfed by the first, and we get

$$Y_n \doteqdot \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^{n+1} \tag{4.3}$$

to a large number of significant figures. For instance, already by n = 20, (4.3) gives Y_n correct to the nearest integer.

Those of you who have written the program suggested above, can add the calculation of (4.3), and compare.

Applying the same reasoning to (4.1), we see that unless $\alpha = 0$, we have

$$Y_n = \alpha \left(\frac{1 + 5}{2}\right)^n$$

for large n. This explains how I could predict that the ratio of successive terms would "always" tend to about 1.6. I was gambling that you would not hit upon something like

$$Y_o = 1, \quad Y_1 = \frac{1 - \sqrt{5}}{2} = \quad - 0.618034$$

as initial values; and even if you did, I was relying on roundoff error to make you drift off the exact solution

$$Y_n = \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

that would result. Notice, however that this solution is quite different from the typical ones. It alternates in sign, and instead of

$$\frac{Y_{n+1}}{Y_n} \doteq \frac{1 + \sqrt{5}}{2} ,$$

we have

$$\frac{Y_{n+1}}{Y_n} = \frac{-2}{1 + \sqrt{5}} .$$

Notice also the superiority of mathematics over brute force. Your original computer program is completely outclassed by formula (4.3), when it comes to calculating large Fibonacci numbers.

5. The golden ratio. The ratio $\frac{1 + \sqrt{5}}{2} = 1.618 \ldots$ is known as the golden ratio. We found it lurking in the Fibonacci numbers, but in fact it crops up all over the place. Indeed, it crops up in crops! It first arose long ago when someone considered

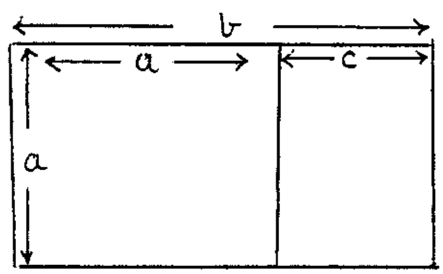

figure 1

the following problem: Make a rectangle, a by b, so that removing the square on the short side a, leaves a similar rectangle a by c. In other words, we want (figure 1)

$$\frac{a}{b} = \frac{c}{a} .$$

Since c = b - a, this becomes

$$\frac{a}{b} = \frac{b}{a} - 1 .$$

Letting $\lambda = b/a$, this gives

$$\lambda^2 - \lambda - 1 = 0,$$

so λ is the golden ratio.

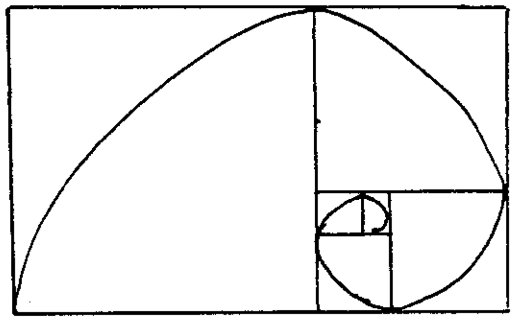

figure 2

The wonderful thing about this is that if you have a rectangle with this ratio (a golden rectangle), you can continue the process of removing squares indefinitely, obtained an infinite sequence of diminishing golden rectanges, vanishing to a point (figure 2). There is a curve (the logarithmic spiral) which fits neatly into this picture. This curve has been observed in numerous things that grow, such as sunflower heads, snail shells, the pattern of leaf buds on a stem, and animal horns.

The golden ratio is supposed to be particularly pleasing to the eye. It occurs in a number of places in Leonardo da Vinci's notion of the perfectly-proportioned human body. For a long time it was the preferred ratio for (rectangular) paintings.

6. Exchange reserves. The general theory of constant-coefficient difference equations has many uses besides the Fibonacci example. Just as surds like $\sqrt{5}$ can slip into a problem about whole numbers, so complex numbers can slip into problems about real numbers. Here is an example, completely different from the Fibonacci sequence, which illustrates these points. It concerns exchange reserves.

Imagine, if you will, a country with exchange reserves. Let R_n stand for the level of the reserves at the start of time period n. Let M_n be the payment for imports in period n, and X_n be the income from exports. Then

$$R_{n+1} = R_n + X_n - M_n$$

Assume that there is a simple relationship between imports and reserves:

$$M_n = a + b\ R_{n-1}.$$

This just means that imports have a base level a, and then vary in proportion to reserves in the previous time period. In other words, if we find ourselves flush with cash, we splash out, otherwise, we rein in. The n-1 allows for the lag between orders and invoices. Now we have

$$R_{n+1} - R_n + b\ R_{n-1} = X_n - a. \tag{6.1}$$

This is a second-order, constant-coefficient difference equation. It doesn't quite fit the pattern of (2.1), becuase it is not homogeneous, i.e. the right-hand side is not zero. This is not a major problem, however. If $R_n = P_n$ is any particular solution to (6.1), then the general solution is

$$R_n = Y_n + P_n, \tag{6.2}$$

where Y_n is the general solution to the associated homogeneous equation

$$Y_{n+1} - Y_n + b\ Y_{n-1} = 0. \tag{6.3}$$

The quadratic associated to (6.3) is

$$\lambda^2 - \lambda + b = 0,$$

and the roots are

$$\lambda_1, \lambda_2 = \frac{1 \pm \sqrt{1 - 4b}}{2}.$$

Now these are only real if $b \leq \frac{1}{4}$. This is not observed in actual practice. In practice b is greater than 1. This means that if we find ourselves with reserves of fifty million, we increase expenditure by more than fifty million. For such b, letting

$$\nu = \sqrt{4b - 1}$$

we have

$$\lambda_1, \lambda_2 = \tfrac{1}{2}(1 \pm \nu i) = r(\cos\theta + i\ \sin\theta),$$

where

$$r = \tfrac{1}{2}\sqrt{1 + \nu^2} = \sqrt{b}, \quad \theta = \arctan 2\nu.$$

The general solution of (6.3) is

$$Y = \alpha\lambda_1^n + \beta\lambda_2^n$$

for constant α and β. Using de Moivre's theorem, this becomes

$$Y_n = r^n \{(\alpha + \beta) \cos n\theta + i(\alpha - \beta) \sin n\theta\}.$$

This can be written in the form

$$Y_n = \gamma b^{\frac{n}{2}} \cos (n\theta + \delta) \tag{6.4}$$

for constants γ and δ. Thus the general Y_n is oscillating, with exponentially growing amplitude, $b^n/_2$.

To get the full solution (6.2), we need some particular solution $R_n = P_n$ for equation (6.1). This can only be done in closed form if we make some assumption about the explicit form of exports X_n. For instance, let us assume that

$$X_n = cg^n$$

i.e. that exports grow at an exponential rate, increasing by a factor g in each period. The equation (6.1) then becomes

$$R_{n+1} - R_n + b\, R_{n-1} = cg^n - a. \tag{6.5}$$

Let us try for a particular solution of the form

$$R_n = A\, g^n + B.$$

We get

$$Ag^{n+1} - Ag^n + bAg^{n-1} + bB = cg^n - a,$$

$$A\{g - 1 + \frac{b}{g}\}\, g^n + bB = cg^n - a,$$

so a solution is obtained if

$$A\{g - 1 + \frac{b}{g}\} = c, \quad bB = -a,$$

or, in other words, if

$$A = \frac{cg}{g^2-g+b}, \quad B = -\frac{a}{b} .$$

This particular solution is

$$R_n = \frac{cg^{n+1}}{g^2-g+b} - \frac{a}{b} .$$

Putting it all together, we get the general solution for (6.5) in the form

$$R_n = \gamma b^{\frac{n}{2}} \cos(n\theta + \delta) + \frac{cg^{n+1}}{g^2-g+b} - \frac{a}{b} .$$

The behaviour of this is influenced to some degree by the initial conditions (which determine γ and δ), and to a lesser extent (in the short term) by the ratio $^a/_b$ of fixed discretionary imports, but the main thing that determines the long-term behaviour is the ratio of $b^{\frac{1}{2}}$ and g, or equivalently, of b and g^2. If b exceeds g^2, the reserves eventually begin to oscillate wildly between ever-increasing bounds.
If g^2 exceeds b, then the reserves eventually settle to a growth rate g, equal to the rate of growth in exports. Thus this (extremely over simplified) model dictates that discretionary spending must be held to a level below g^2R, where g is the rate of growth in exports and R is the level of reserves.

For example, taking the period equal to a year, if exports grow at 4%, then g = 1.04, so discretionary spending should be held to 1.08 times reserves.

7. Acknowledgments. I first learned about difference equations from my favourite teacher, Professor Philip G. Gormley, God rest him, who once used n-th order constant-coefficient linear difference equations to weed first honours. I also recall hearing a nice talk on the subject by Professor Ernie Schlesinger, in or about 1969. The bit about exchange holdings I learned from my old classmate Sean Mooney. The present article is based on a talk I gave to the Science Society at Maynooth, in March 1983.