Capacities in Function Theory

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1. Introduction.

Ever since the famous thesis of Frostman, capacities have been important in many areas of function theory. In this talk I shall be concerned only with one–variable function theory on arbitrary open subsets of the complex plane, **C**. It is important to stress that the open sets need not be connected. I will discuss the use of analytic capacities in connection with problems of removable singularities, holomorphic approximation, and boundary smoothness.

A brief reference to the applications is in order. The connections between analytic (and harmonic) functions and the physics of perfect fluids, electrostatics, magnetostatics, classical gravitation and heat are well–known. Much of what I shall say about analytic functions (solutions of the d–bar equation) applies also to solutions of other elliptic equations, and so there are other applications, for instance to elasticity (connected to the bi-Laplacian and the d-bar-squared operator). It should also be noted that students of capacitary problems were among the first to discover and examine many of the weird–looking sets formerly regarded as pathological by most people, but now known as fractals and accepted as natural objects of study for many applications.

Here is a brief outline of the talk. Let U be an open subset of **C**, and $F \subset \mathcal{D}'(\mathbf{C},\mathbf{C})$ be a topological vector space of distributions (generalised functions) on **C**. We define

$$AF(U) = \{f \in F : \bar\partial f = 0 \text{ on } U\}.$$

In other words, $AF(U)$ consists of those distributions $f \in F$ that are analytic on U. Typically, the space F is supposed to be characterised by some real–variable property, such as boundedness, continuity, etc. One expects that the functions $f \in AF(U)$ are in some respects nicer than typical elements of F. To the space F, we shall associate an analytic capacity

$$\bar\partial - F - \text{cap}: \quad 2^{\mathbf{C}} \to [0, +\infty].$$

The definition of $\bar\partial$–F–cap will be given presently, but for now it is enough to note that it is a monotone nonnegative set function. The main idea of analytic capacity theory is that it is possible to recover many *local* properties of the spaces $AF(U)$ from a knowledge of the $\bar\partial$–F capacity. Consider the following problems.

Removability:

A compact $K \subset \mathbf{C}$ is said to be $\bar{\partial}$–F–null if

$$AF(U \sim K) = AF(U), \qquad \forall \text{ open } U \subset \mathbf{C}.$$

The removability question is to characterise the $\bar{\partial}$–F–null sets in some explicit way.

Approximation:

For which compact sets $X \subset \mathbf{C}$ is

$$\bigcup \{AF(U) : X \subset U, U \text{ open}\}$$

dense in $AF(U)$ in the topology of F?

Boundary smoothness:

For which open $U \subset \mathbf{C}$, $a \in \text{bdy}U$, and nonnegative integers k, is the map

$$f \mapsto f^{(k)}(Q)$$

reasonably well–defined on $AF(U)$?

2. Example: lipα.

To appreciate these problems, it is useful to look at an example. Consider the case $F =$lipα.

Let $0 < \alpha \leq 1$.

Definition. $f \in \text{Lip}\alpha$ means $f : \mathbf{C} \to \mathbf{C}$ and

$$|f(z) - f(w)| \leq \kappa |z - w|^\alpha \qquad \forall z, w \in \mathbf{C}$$

for some constant κ, independent of z and w.

Lipα becomes a Banach space when given the norm

$$\|f\|_{\text{Lip}\alpha} = |f(0)| + \text{ least } \kappa.$$

The subspace of Lipα consisting of those f such that

$$\lim_{|z-w|\downarrow 0} \frac{|f(z) - f(w)|}{|z - w|^\alpha} = 0$$

and

$$\lim_{|z-w|\uparrow \infty} \frac{|f(z) - f(w)|}{|z - w|^\alpha} = 0$$

is a closed subspace, the closure of the space $\mathcal{D}$ of test functions. We denote it lipα.

The $\bar\partial$–Lipα–cap turned out to be equivalent to $(1+\alpha)$–*dimensional Hausdorff content*, $M^{1+\alpha}$ (Dolzhenko, 1964). The Hausdorff content corresponding to a non-negative increasing function h on $[0, \infty)$ is defined by

$$M_h(E) = \inf_{\mathcal{S}} \sum_{B \in \mathcal{S}} h(\text{diam} B), \qquad \forall E \subset \mathbf{C},$$

where $\mathcal{S}$ runs over all countable coverings of E. The $\bar\partial$–lipα–cap is equivalent to $(1 + \alpha)$–dimensional *lower* Hausdorff content, $M_*^{1+\alpha}$. This content is defined by

$$M_*^\beta(E) = \sup \left\{ M_h(E) : h(r) \leq r^\beta, r^{-\beta} h(r) \to 0 \text{ as } r \downarrow 0 \right\}.$$

The solutions to the three problems for lipα are as follows (Similar results hold for Lipα, but the statements are less straightforward, because in that case reference must be made to the weak–star topology on Lipα).

Theorem 1 (Dolzhenko). *Let $K \subset \mathbf{C}$ be compact. Then K is $\bar{\partial}$–lipα–null if and only if*

$$M_*^{1+\alpha}(K) = 0.$$

Theorem 2. *Let X be a compact subset of $\mathbf{C}$. Then*

$$\bigcup \{Alip\alpha(V) : V \text{ is a neighbourhood of } X\}$$

is dense in $Alip\alpha(\text{int}X)$ if and only if

$$M_*^{1+\alpha}(D \sim X) \geq \text{const} \cdot M_*^{1+\alpha}(D \sim \text{int}X), \qquad \forall \text{ discs } D \subset \mathbf{C}.$$

In fact, in this statement, one may replace the $M_*^{1+\alpha}(D \sim X)$ on the left–hand side by $M^{1+\alpha}(D \sim X)$. The statement as given serves to emphasise the fact that the problem may be solved using only the capacity $M_*^{1+\alpha}$.

To understand the statement intuitively, observe that the approximation problem involves the approximation of functions whose singularities lie in $\mathbf{C} \sim \text{int}X$ by functions whose singularities lie in the smaller set $\mathbf{C} \sim X$. It is a question of "pushing" singularities off bdy$X$ into the complement of X. The capacity $\bar{\partial}$–lipα–cap ($\sim M_*^{1+\alpha}$) measures, in fact, the *capacity to carry singularities* of lipα analytic functions. Also, the problem is local. So it is reasonable that the solution should involve the comparison of the capacity of $D \sim X$ and $D \sim \text{int}X$ for all disks D. Naturally, it is not important that disks are used; squares, or arbitrary open sets would do just as well.

The first theorem of this type for analytic functions was for continuous functions and uniform approximation, and was proved by Vitushkin. Earlier, Keldysh proved the corresponding result for harmonic functions and uniform approximation.

Lemma 1. *Let $U \subset \mathbf{C}$ be open and $a \in \mathbf{C}$. Then the set of all $f \in Alip\alpha(U)$ that are analytic on a neighbourhood of a (depending on f) is a dense subset of $Alip\alpha(U)$.*

The uniform approximation version of this lemma was first discovered by Arens.

This lemma enables us to state more precisely the boundary smoothness question for $\bar{\partial}$–lipα. It is: to characterise those U, a, and $k \in \mathbf{N}$ such that the functional $f \mapsto f^{(k)}(a)$, defined on

$$\{f \in \text{Alip}\alpha(U) : f \text{ is analytic near } a\}$$

extends continuously to all of Alip$\alpha(U)$. When such a continuous extension (necessarily unique) exists, we say that Alip$\alpha(U)$ admits a *k–th order continuous point derivation* at a.

Only points a that lie on the boundary of U are of interest, because Alip$\alpha(U)$ admits continuous point derivations of all orders at points of U, and of no order $k \geq 1$ at points outside clos$U$.

Theorem 3. *Let $U \subset \mathbf{C}$ be open, $a \in \mathbf{C}$, and $k \in \mathbf{N}$. Then Alip$\alpha(U)$ admits a k–th order continuous point derivation at a if and only if*

$$\sum_{n=1}^{\infty} 2^{(k+1)n} M_*^{1+\alpha}(A_n \sim U) < +\infty.$$

Here A_n denotes the annulus

$$A_n = \left\{ z \in \mathbf{C} : \frac{1}{2^{n+1}} < |z-a| \leq \frac{1}{2^n} \right\}.$$

3. Symmetric Concrete Spaces.

A Symmetric Concrete Space (SCS) on $\mathbf{R}^d$ is a complete LCTVS, F, such that

1. $\mathcal{D} \hookrightarrow F \hookrightarrow \mathcal{D}'$,

2. $f \mapsto \bar{f}$ maps $F \to F$ continuously,

3.
$$\left\{ \begin{array}{c} \mathcal{D} \times F \to F \\ (\phi, f) \mapsto \phi f \end{array} \right\}$$

makes F a topological $\mathcal{D}$–module.

4. for each $T \in$Aff,
$$c_T : \left\{ \begin{array}{c} F \to F \\ f \mapsto f \circ T \end{array} \right\} \text{ is continuous}$$

and $T \mapsto c_T$ maps compact subsets of Aff to equicontinuous subsets of End(F).

If an SCS is normable, we call it a symmetric concrete Banach space (SCBS).

An SCS is called *small* if $\mathcal{D}$ is sequentially dense in it.

As examples, L^p, C^k, Lipα, lipα, BMO, VMO, Sobolev spaces, Besov spaces, Bloch space, Zygmund class (ZC) and Zygmund smooth class (ZS) are SCS. The space L^p is small if $p < +\infty$. Other small spaces are C^k, lipα, VMO, and ZS $_{\text{loc}}$.

I now give a brief account of basic SCS theory.

Let F be an SCS. To a closed subset $X \subset \mathbf{R}^d$, we associate spaces $F(X)$ — germs on X, F_X — elements of F that are supported on X, and

$$F_{\text{loc}} = \mathcal{E} \cdot F \qquad ; \qquad F_{\text{cs}} = \mathcal{D} \cdot F.$$

The spaces F_{loc} and F_{cs} inherit natural topologies which make them into SCS's.

The most useful equivalence relation on SCS is *local equivalence*, defined by

$$F_1 \overset{\text{loc}}{=} F_2 \Leftrightarrow F_{1\text{loc}} = F_{2\text{loc}}.$$

The notion of *local inclusion*, defined by

$$F_1 \overset{\text{loc}}{\hookrightarrow} F_2 \Leftrightarrow F_{1\text{loc}} \subset F_{2\text{loc}}$$

gives a partial order on the local equivalence classes. It turns out that for SCS, $F_1 \subset F_2$ is equivalent to $F_1 \hookrightarrow F_2$, so that

$$F_1 \overset{\text{loc}}{\hookrightarrow} F_2 \Leftrightarrow F_{1\text{loc}} \hookrightarrow F_{2\text{loc}}.$$

The usefulness of this notion is illustrated by the L^p spaces. One never has $\mathrm{L}^p \hookrightarrow \mathrm{L}^q$ if $p \neq q$, but $\mathrm{L}^p \overset{\text{loc}}{\hookrightarrow} \mathrm{L}^q$ if and only if $p \geq q$, so $\overset{\text{loc}}{\hookrightarrow}$ gives a linear order on the L^p spaces. In general, $\overset{\text{loc}}{\hookrightarrow}$ is not a total order on the SCS. For instance, in two dimensions, the space $C = C^0$ of continuous functions and the Sobolev space $W^{1,2}$ are unrelated by $\overset{\text{loc}}{\hookrightarrow}$

The following "F_∞ construction" is useful:

The space F_∞ associated to an SCS, F, is the set of all those $f \in F_{\text{loc}}$ such that

$$f(\cdot + a)|\mathbf{B}(0,1) \to 0$$

in $F(\mathbf{B}(0,1))$–topology as $a \to \infty$. This construction produces a new SCS, locally–equivalent to the original. It may be larger or smaller than the original. For instance, C^0_∞ is the space, often denoted C_0, of continuous functions that tend to zero at infinity, and is smaller than C^0, whereas L^2_∞ is the space of measurable functions that have

$$\int_{\mathbf{B}(a,1)} |f|^2 dx \to 0$$

as $a \to \infty$, and is larger than L^2. The space F_∞ has some canonical properties. For instance, if there is an SCBS in the local equivalence class of F, then F_∞ is normable.

For any SCS, F, there is a canonical map $i : F^* \to \mathcal{D}'$, the adjoint of the inclusion map $\mathcal{D} \hookrightarrow F$. This map is injective if and only if $\mathcal{D}$ is dense in F, and then $i(F^*)$ is an SCS, a canonical SCS realisation of F^*. Similarly, if F is a dual space, then there is a canonical SCS realisation of the predual F_* if and only if $\mathcal{D}$ is weak–star dense in F; in that case, the canonical predual is just the canonical dual of (F, weak–star).

There is a duality between restriction spaces and support spaces, given by

$$F(X)^* = (F^*)_X, \qquad (F_X)^* = F^*(X),$$

for all $F \in$SCS and all closed $X \subset \mathbf{R}^d$.

If F_1 and F_2 are SCS, then so are the intersection $F_1 \cap F_2$ and the sum $F_1 + F_2$ (— the inner vector space sum in $\mathcal{D}'$), when endowed with the obvious topologies. The only point that requires a little care in the proof is the completeness of these spaces. A corollary is the fact, quoted above, that inclusion inplies continuous inclusion, for SCS.

I conclude this brief summary of the basic properties of SCS with an important observation about the action of convolution. The convolution $f * g$ makes sense when f is a distribution having compact support and g is any distribution. Thus we may consider it for $f \in F_{\mathrm{cs}}$ and $g \in \mathrm{L}^1{}_{\mathrm{loc}}$.

Theorem 4. *Suppose (a) F is a small SCS,* **or** *(b) F is the SCS dual of a small SCS. Then the map*

$$(f, g) \mapsto f * g$$

maps

$$F_{\mathrm{cs}} \times L^1{}_{\mathrm{loc}} \to F_{\mathrm{loc}}$$

continuously.

The conditions given on F may be relaxed, but those given cover all spaces of interest to me.

To indicate the proof, one may begin by remarking that the case (b) may be reduced to case (a). The difficult thing is to demonstrate that convolution maps

$$F_{\mathrm{cs}} \times \mathrm{L}^1{}_{\mathrm{loc}} \to F_{\mathrm{loc}};$$

once this is known, continuity is not a problem. Thus part (b) is obtained by applying part (a) to the SCS (F, weak–star).

To prove the result on hypothesis (a), fix $f \in F_{\text{cs}}$ and $g \in \mathrm{L}^1{}_{\text{loc}}$. Then $f * g$ is defined, as an element of $\mathcal{D}'$, by the usual formula:

$$\langle \psi, f * g \rangle = \langle \langle \psi(x + y), f_x \rangle, g_y \rangle, \quad \forall \psi \in \mathcal{D}. \qquad (*).$$

The first step is to extend the domain of $f * g$ from $\mathcal{D}$ to $(F_{\text{loc}})^*$. It is straightforward to check that (*) makes sense whenever we replace ψ by an element of $(F_{\text{loc}})^* = (F^*)_{\text{cs}}$, and defines an element $f * g$ of $(F_{\text{loc}})^{**}$.

The next step is to apply the Banach–Grothendieck Theorem to show that in fact $f * g$ lies in the image of F_{loc} under the canonical injection of F_{loc} into $(F_{\text{loc}})^{**}$. According to that theorem, this amounts to showing that $f * g$ is uniformly continuous on the polar of each neighbourhood of zero in F_{loc}. Because of smallness, this reduces to checking a simple sequential statement. □

One might remark that the intuition behind this result is quite simple. Convolving a distribution with a locally–integrable function is a process of taking limits of averages over translates of reflections (in 0) of the distribution. SCS are nicely preserved under translation and reflection, and they are complete, so that it is reasonable to suppose that they will be essentially preserved by convolutions.

Figure 1 presents a picture of part of the class of SCS. The arrows indicate the local inclusion relations.

4. Analytic Capacities.

Now we define the analytic capacities associated to an SCS on the plane. For simplicity, we restrict to the case where the SCS F is locally–equivalent to an SCBS. In that case, F_∞ may be normed. We assume this done, and (as is always possible) that the norm on F_∞ is translation–invariant.

Let K be a compact subset of $\mathbf{C}$. Let $\phi \in \mathcal{D}$ be a test function such that $\phi = 1$ on a neighbourhood of K. We define the analytic–F– capacity

$$\bar{\partial} - F - \text{cap}(K) = \sup \left\{ \frac{1}{\pi} |\langle \bar{\partial}\phi, f \rangle| : f \in \text{Ball } \mathrm{A}F_\infty(\mathbf{C} \sim K) \right\}.$$

This capacity is a nonnegative, monotone set function, which carries information about the space F and the analytic functions. The quantity

$$\frac{1}{\pi} \langle \bar{\partial}\phi, f \rangle$$

represents the coefficient a_1 in the Laurent expansion

$$f = a_0 + \frac{a_1}{z} + \cdots$$

of f about ∞.

Given two set functions α and β on a family of subsets of $\mathbf{C}$, we say that they are locally equivalent if for each compact set $X \subset \mathbf{C}$ there is a constant $\kappa > 0$ such that

$$\frac{1}{\kappa}\alpha(K) \leq \beta(K) \leq \kappa\alpha(K)$$

whenever $K \subset X$.

For application to the problems we have mentioned above, the precise values of the capacities are not important: all that matters is the local equivalence class of the appropriate capacity. Many of the interesting analytic capacities have been explicitly identified up to local equivalence. Here is a brief summary of the main results:

The L^∞ capacity (*the* analytic capacity, in ordinary parlance) was the first to be introduced (Ahlfors, 1947), and has at least four distinct descriptions.

The C analytic capacity generates the same outer capacity as the L^∞ analytic capacity, so both are as well (or as little) understood on sets with fat interior. On sets with no interior, the two functions are in general quite different, and not much is known about the C capacity. It is worth mentioning the remarkable result of Browder–Wermer–Carleson–Garnett– Bishop–Jones that arcs with no tangents have positive $\bar{\partial}$–C–capacity.

The L^2 analytic capacity is locally–equivalent to logarithmic capacity, which is very well understood (Hedberg). It may be computed as a kernel capacity, a Chebyshev constant, a transfinite diameter, a condenser capacity associated to the Sobolev space $W^{1,2}$ (Dirichlet space).

For $2 < p < +\infty$, the L^p analytic capacity is equivalent to a condenser capacity, an extremal length, and an iterated potential capacity. (Hedberg, Havin, Mazya, Ziemer)

The C^1 analytic capacity of a compact set is equivalent to the area of its interior. (Nguyen, Hrushchev).

The Lipα capacity $(0 < \alpha < 1)$ is equivalent to the Haudorff content $M^{1+\alpha}$, and the lipα capacity is equivalent to the corresponding lower content $M_*^{1+\alpha}$. (Dolzhenko, Gonchar, Mergelyan, Garnett, author).

The Lip1 capacity is equivalent to area (Nguyen, Hrushchev). The space lip1 is not an SCS (too small).

The BMO capacity is equivalent to M^1, and the VMO capacity is equivalent to M_*^1 (Kaufman).

For $p > 1$, the $W^{1,p}$ capacity is equivalent to $|\mathrm{area}|^{1/q}$, where q is the index conjugate to p.

The C^k capacity of a compact K is equivalent to

$$\int_K \operatorname{dist}(z, \mathbf{C} \sim K)^{k-1} dxdy.$$

A few interesting analytic capacities remain to be constructively identified, such as the $W^{1,1}$, $W^{2,p}$, ZC and ZS capacities. Other interesting questions concern the local affine invariance and quasisubadditivity of various capacities. Both questions remain open for the Ahlfors capacity $\bar\partial$–L^∞–cap. Quasisubadditivity is not a necessary property of analytic capacities in general (— for instance, it fails for the lip(3/2) capacity), but no example is known of an SCS analytic capacity that is not locally affine–invariant.

There is, naturally, no problem of removable singularities for spaces like C^3: the sets of removable singularities are exactly the sets with no interior. The capacity associated to such a space is however of value is examining boundary smoothness. For instance, at which points on the boundary do we have 4–th order bounded point derivations on $\mathrm{AC}^3(U)$? This question requires quantitative estimates on the thinness of the complement of U at boundary points.

5. Operators.

Now we consider operators that are associated to analytic function theory.

Definition. Let S be an operator defined on a set of distributions, with values in the set of distributions.

We say that a topological vector space $F \subset \mathcal{D}'$ is S–invariant if F lies in the domain of S and S maps F into F, continuously (with respect to the topology of F). This is standard terminology, but we wish to introduce some more, useful when F is an SCS.

We say that F is locally S–invariant if $S : F \to F_{\mathrm{loc}}$, continuously.

We say that F is co–locally S–invariant if $S : F_{\mathrm{cs}} \to F$, continuously.

We say that F is bi–locally S–invariant if $S : F_{\mathrm{cs}} \to F_{\mathrm{loc}}$, continuously.

The important integral operators for function theory on general plane open sets are:

1) The **Cauchy transform**:

$$C : f \mapsto \frac{-1}{\pi z} * f.$$

This inverts the $\bar\partial$ operator:

$$\bar\partial C f = f = C \bar\partial f, \qquad \forall f \in \mathcal{E}'.$$

2) The **Beurling transform**:

$$B : f \mapsto \frac{\mathrm{PV}}{\pi z^2} * f.$$

This has

$$\frac{\partial B f}{\partial \bar{z}} = \frac{\partial f}{\partial z} \qquad \forall f \in \mathcal{E}'.$$

3) The **Vitushkin localisation operator**:

$$f \mapsto T_\phi f = \frac{-1}{\pi z} * \left(\phi \frac{\partial f}{\partial \bar{z}} \right),$$

defined for all $f \in \mathcal{D}'$ and all $\phi \in \mathcal{D}$. This satisfies

$$\bar{\partial} T_\phi f = \phi \bar{\partial} f,$$

and so may be used to localise singularities.

We make the convention that T–invariant means T_ϕ–invariant, for each $\phi \in \mathcal{D}$.

Theorem 5. *Let F be a small SCS or the SCS dual of a small SCS. Then F is locally T–invariant, and F_∞ is T–invariant.*

Proof. The function $-1/\pi z$ belongs to ${\mathrm{L}^1}_{\mathrm{loc}}$, so Theorem 4 shows that F is bilocally C–invariant:

$$C : F_{\mathrm{cs}} \to F_{\mathrm{loc}}.$$

Using Leibnitz' rule,

$$T_\phi f = \phi \cdot f + C \left(\frac{\partial \phi}{\partial \bar{z}} f \right),$$

so the local T–invariance of F follows easily, using the $\mathcal{D}$–module property of F.

Thus F_{loc} is T–invariant.

Given this, it is easy to see that F itself is T–invariant if and only if it contains all those functions $f \in F_{\mathrm{loc}}$ that are analytic near ∞ and vanish at ∞. Since F_∞ has this property, and $F_{\infty\,\mathrm{loc}} = F_{\mathrm{loc}}$, we get the last assertion of the Theorem. □

It is noteworthy that previous proofs of the T–invariance of various special SCS, such as C, L^p $(p > 2)$, Lipα, and BMO, have involved substantial spadework. This theorem uncovers the essential pattern in these results. Of course, not all the SCS properties are needed for this result: full affine local invariance may be relaxed to local invariance under translations and reflection in a point. The theorem also throws up important new observations, such as the availability of a T–invariant SCBS that is locally–equivalent to L^p, when $p \leq 2$.

To illustrate the use of the ideas introduced here, we give a simple application. This is a result which originated with Ahlfors in the case $F = \mathrm{L}^\infty$, and it solves the removability problem, modulo constructive identification of the $\bar\partial$–F–cap null sets (a big modulo).

Corollary. *Let F be a small SCS, or the dual of a small SCS. Let $K \subset \mathbf{C}$ be compact. Then the following are equivalent:*
(1) K is $\bar\partial$–F–null;
(2) $AF_\infty(\mathbf{C} \sim K) = \{0\}$.
(3) $\bar\partial$–F–cap$(K) = 0$.

Proof. It is easy to see that the family of $\bar\partial$–F–null compacts depends only on F_{loc}. So we may take $F = F_\infty$, without loss in generality.

(1) $\Rightarrow$ (2) is easy: Suppose that K is $\bar\partial$–F–null. Let $f \in \mathrm{A}F(\mathbf{C} \sim K)$. Then $F \in AF(\mathbf{C})$. So f is entire. Since $f \in F_\infty$, we deduce $f(a + \cdot)|\mathbf{B}(0,1) \to 0$ in the topology of $\mathcal{D}'(\mathbf{B}(0,1)$ as $a \to \infty$. Using a smeared Cauchy integral formula, this is readily seen to yield $f(a) \to 0$ as $a \to \infty$, so Liouville's theorem gives $f = 0$ identically.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (2): Suppose that $\bar\partial$–F–cap$(K) = 0$. Suppose f were a nonzero element of $AF(\mathbf{C} \sim K)$. Choose $n \in \mathbf{N}$ such that

$$f(z) = \frac{a_n}{z^n} + \cdots, \text{ near}\infty,$$

with $a_n \neq 0$. Using the $\mathcal{D}$–module property, and the fact that $F = F_\infty$, one sees that $z^{n-1}f$ also belongs to $AF(\mathbf{C} \sim K)$, hence the $\bar\partial$–F–cap of K is nonzero.

(2) $\Rightarrow$ (1): Suppose that $\mathrm{A}F(\mathbf{C} \sim K) = \{0\}$. Let $f \in \mathrm{A}F(U \sim K)$ for some open set U. We wish to show that $f \in AF(U)$.

It is enough to show that f is holomorphic on each open disc D such that $\mathrm{clos}(D) \subset U$. Fix such a disc D.

Take $\phi \in \mathcal{D}$ with $\phi = 1$ near $K \cap \mathrm{clos}D$ and $\mathrm{spt}\phi \subset U$, and form $T_\phi f$. Since F is T–invariant, and $T_\phi f$ is analytic off $\mathrm{spt}\phi$, we have $T_\phi f \in AF(\mathbf{C} \sim K)$, hence $T_\phi f = 0$. So

$$\bar\partial f = \bar\partial(f - T_\phi f) = (1 - \phi)\bar\partial f,$$

hence f is holomorphic on $D \sim K$ and on a neighbourhood of $K \cap \mathrm{clos}(D)$, and hence on all of D. ◻

Work in progress offers the prospect of attaining a similarly general solution to the other two main problems posed above. In particular, the author and J. Verdera are working on implementing the Vitushkin "coefficient matching" technique, without using algebra structure or B–invariance.

6. The one–reduction.

"Most" SCS are bi–locally B–invariant, and these are easier to work with. We illustrate this by describing the most important technique that depends on B–invariance: 1–reduction.

Before starting, we note that the "most" excludes some very, very interesting spaces.

There is a straightforward way to "differentiate" a SCS, F. You just form

$$\left\{ \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} : f_1, f_2 \in F \right\},$$

and give it the naturally–induced topology. This space does not necessarily contain $\mathcal{D}$, so in fact we define the derivative of F by

$$DF = \mathcal{D} + \left\{ \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} : f_1, f_2 \in F \right\}.$$

Similarly, we define the integral of F by

$$\int F = \left\{ f \in \mathcal{D}' : \frac{\partial f}{\partial x} \in F \text{ and } \frac{\partial f}{\partial y} \in F \right\}.$$

Again, there is a natural topology to use.

Theorem 6. *If F is a bi–locally B–invariant SCS, then DF and $\int F$ are also SCS, and*

$$\int DF \overset{\text{loc}}{=} F \overset{\text{loc}}{=} D \int F.$$

This is the Fundamental Theorem of the Integral Calculus, for function spaces. Part of the proof is the formula

$$\bar{\partial} F \overset{\text{loc}}{=} DF,$$

where $\bar{\partial} F$ stands for the collection of $\bar{\partial} f$, where f ranges over F. Now for the SCS of Theorem 5, to say that K is $\bar{\partial}$–F–null is the same as saying that no function $g \in \bar{\partial} F$ has support in K. This yields the 1–reduction for $\bar{\partial}$:

Corollary. *Let F be a bi–locally B–invariant SCS, such that F or F_* is a small SCS. Let $K \subset \mathbf{C}$ be compact. Then K is $\bar{\partial}$–F–null if and only if K supports no nonzero element of DF.*

This reduces the nullity problem for $\bar{\partial}$–F to a problem about supports, a *real–variable* problem. In many cases, one finds that the real–variable problem has already been studied and elucidated.

As an example, consider the case $F = W^{2,p}$. The interesting range of p is $[1, 2)$. For $1 < p < 2$, the Corollary applies, and $DW^{2,p} \overset{\text{loc}}{=} W^{1,p}$, so the $\bar{\partial}$–$W^{2,p}$–null sets are the compact sets that cannot support nonzero $W^{1,p}$ functions. These have been intensively studied (Havin, Hedberg, Bagby), because they are the sets K such that each function $f \in \mathrm{L}^q$ is the L^q limit of functions holomorphic on a neighbourhood of K.

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