Superalgebras, the Brauer-Wall Group and the Super Frobenius-Schur indicator

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Abstract

In this thesis, we will study the theory of superalgebras, which are algebras with a C_2-grading. One of our main aims is to show that many concepts and theorems in Algebra Theory have their counterparts in Superalgebra Theory. For example, we will state and prove the superalgebra counterparts of Schur's Lemma, Maschke's Theorem, and Wedderburn's theorem.

In Algebra Theory, each field F has a group called the Brauer Group of F (denoted as $Br(F)$), which is a group of equivalence classes of central simple F-algebras. We will be showing that there is a superalgebra equivalent, namely the Brauer-Wall group of F (denoted as $BW(F)$), which is a group of equivalence classes of super central simple F-superalgebras.

Additionally, we will be studying group superalgebras, super representations, and super characters. In the study of ordinary group algebras, the Frobenius-Schur indicator meaningfully associates an irreducible $\mathbb{C}$-character of a finite group G with a division algebra over $\mathbb{R}$. In this thesis, we will introduce the Super Frobenius-Schur indicator, which associates a super irreducible $\mathbb{C}$-super character with a super division algebra over $\mathbb{R}$. We will also give the full decomposition of group superalgebras over $\mathbb{R}$ and $\mathbb{C}$.

Finally, we will discuss Clifford Algebras, another family of examples of superalgebras.

Contents

Introduction

This thesis aims to study finite dimensional superalgebras, with a particular focus on super semisimple superalgebras and group superalgebras over $\mathbb{R}$ and $\mathbb{C}$.

In section 1, we will recall elementary concepts in Algebra theory. We will define objects such as algebras, ideals, modules, algebra homomorphisms, module homomorphisms, and division algebras. We will also define what it means for a module to be irreducible, and what it means for a module to be completely reducible. From there, we will state three important theorems: Schur's Lemma, Maschke's Theorem, and Wedderburn's Theorem.

We will then study central simple algebras. Given a field F, we can define an equivalence relation on the set of central simple F-algebras, and from there we introduce an important group called the Brauer Group of F (denoted by $Br(F)$). We will not focus much on the proofs of important theorems in this section. We will instead refer the reader to I.M. Isaacs' *Character Theory of Finite Groups* [1] and W. Scharlau's *Quadratic and Hermitian Forms* [2].

In section 2, we will introduce the concept of a superalgebra. Many concepts from Algebra Theory have their equivalent concepts in Superalgebra Theory. A term in Superalgebra Theory will often have the prefix "super-" to distinguish it from its ordinary Algebra Theory counterpart. We will also prove the super versions of Schur's Lemma, Maschke's Theorem, and Wedderburn's Theorem. We will base a lot of the proofs of statements in Superalgebra Theory on the proofs of their Algebra Theory equivalents.

Afterwards, we will define super central simple F-superalgebras, and introduce the super version of the Brauer group called the Brauer-Wall group of F (denoted by $BW(F)$). To examine the group structure of $BW(F)$, we will follow the theory from C.T.C. Wall's *Graded Brauer Groups* [5]. We will explore properties of super central simple superalgebras in detail, focusing on the case where $\mathrm{char}(F) \neq 2$. In this situation, some super central simple superalgebras are also central simple in the ordinary algebra sense, and some are not. The subset $P(F) \subset BW(F)$ containing classes of ordinary central simple algebras forms a subgroup. One of the major results in this section is that, for any field F such that $\mathrm{char}(F) \neq 2$, $BW(F)/P(F) \cong C_2$, and $P(F)/Br(F) \cong F^\times/F^{\times 2}$. We will then show that $BW(\mathbb{C}) \cong C_2$ and $BW(\mathbb{R}) \cong C_8$.

We will then build our way towards studying group superalgebras over $\mathbb{R}$ and $\mathbb{C}$. To prepare us for that, we will go through some preliminary definitions and results in Representation and Character Theory of finite groups in section 3. One important definition is the Frobenius-Schur (FS) indicator of an irreducible $\mathbb{C}$-character. For any finite group G, we will then describe the two-sided ideal decomposition of $\mathbb{R}G$ and $\mathbb{C}G$. Each minimal two-sided ideal of $\mathbb{C}G$ corresponds to an irreducible $\mathbb{C}$-representation. A minimal two-sided ideal of $\mathbb{R}G$ is a matrix algebra over either $\mathbb{R}$, $\mathbb{H}$ or $\mathbb{C}$. These cases respectively correspond to either a single $\mathbb{C}$-character with FS indicator 1, a single $\mathbb{C}$-character with FS indicator -1, or a pair of $\mathbb{C}$-characters with FS indicator 0.

We will then specialise to the case where G has a subgroup N such that $|G : N| = 2$. We will use Clifford Theory to study the relationship between characters of G, and characters of N. This will allow us to describe super representations and super characters of G over N. We will define two new indicators: the Gow indicator (an indicator for irreducible characters of N), and the Super Frobenius-Schur (SFS) indicator. The SFS indicator of a super irreducible super character can take 8 possible non-zero values. Each possible non-zero value corresponds to an element of $BW(\mathbb{R})$.

This will bring us to the discussion of group superalgebras over $\mathbb{R}$ and $\mathbb{C}$, our main topic of section 4. We will see how the decomposition of $\mathbb{C}G$ will help us describe the decomposition of the group superalgebra $\mathbb{C}[G, N]$. We will then explore how the decomposition of $\mathbb{C}[G, N]$, $\mathbb{R}G$ and $\mathbb{R}N$ will help us describe the decomposition of $\mathbb{R}[G, N]$. Afterwards will be able to describe super irreducible $\mathbb{C}[G, N]$- and $\mathbb{R}[G, N]$-supermodules and how they come about from $\mathbb{C}G$- and $\mathbb{R}G$-modules.

In section 5, we will briefly go through another family of examples of superalgebras, namely Clifford Algebras. We will investigate how we can construct Clifford Algebras over $\mathbb{R}$ and $\mathbb{C}$, and how we can relate them to an element of $BW(\mathbb{R})$ and $BW(\mathbb{C})$.

The diagram below lists the key concepts in Algebra Theory that have counterparts in Superalgebra Theory. The down arrows tell us when one topic leads to another topic.

$$\begin{tikzcd}
\text{Algebras} \arrow[r] \arrow[d] & \text{Superalgebras} \arrow[d] \\
\text{Central simple algebras} \arrow[r] \arrow[d] & \text{Super central simple superalgebras} \arrow[d] \\
\text{Brauer group} \arrow[r] \arrow[d] & \text{Brauer-Wall group} \arrow[d] \\
\text{Frobenius-Schur indicator} \arrow[r] \arrow[d] & \text{Super Frobenius-Schur indicator} \arrow[d] \\
\text{Group Algebras} \arrow[r] & \text{Group Superalgebras}
\end{tikzcd}$$

Throughout the thesis, we will omit the proofs of some theorems and lemmas. When this happens, a reference will be provided at the end of the theorem/lemma, allowing the reader to locate the proof.

1 Algebras

1.1 Algebra Theory preliminaries

$\star$ Throughout the thesis, F will refer to an arbitrary field.
In this section, we will list some important definitions and theorems from the theory of Algebra. Much of the material in this section comes from [1, Chapter 1] and [2, Chapter 8].

An ***F-algebra*** A is an F-vector space with an additional ring structure. In an F-algebra, the addition operation in the vector space and the ring structure coincide, and the scalar multiplication in the vector space and multiplication in the ring are related in the following way: for all $c \in F$ and $x, y \in A$,

$$(cx)y = c(xy) = x(cy).$$

$\star$ Throughout the thesis, we will assume that any algebra is a ring with a multiplicative identity. Throughout this section, A will refer to an arbitrary F-algebra with multiplicative identity denoted as 1_A.

An F-algebra A is said to be ***finite-dimensional*** if it is finite-dimensional as an F-vector space. We define the dimension of A over F by the vector space dimension of A over F.

Example 1.1 Let us list examples of F-algebras, and mention their dimensions over F.

- The field F itself is an F-algebra, and $\dim_F(F) = 1$.

- Let K be a finite field extension of F. Then K is an F-algebra, and $\dim_F(K) = [K : F]$.

- The set $M_n(F)$ of $n \times n$ matrices over F is an F-algebra, and $\dim_F(M_n(F)) = n^2$.

- Given an F-vector space V, then $\mathrm{End}_F(V)$, the set of endomorphisms (or F-linear transformations) of V is an F-algebra. Multiplication in $\mathrm{End}_F(V)$ is given by composition of maps, and given $c \in F$ and $\alpha \in \mathrm{End}_F(V)$, $c\alpha$ is defined by $c\alpha(v) = \alpha(cv)$ for any $v \in V$. If V is finite-dimensional over F, then $\dim_F(\mathrm{End}_F(V)) = (\dim_F(V))^2$.

- Given a finite group G, the ***group algebra*** of G over F, denoted FG, is an F-algebra with a basis $\{e_g : g \in G\}$ whose elements are indexed by the elements of G. So FG can be expressed as

$$FG = \left\{ \sum_{g \in G} c_g e_g : c_g \in F \right\}.$$

  In this algebra, the multiplication of the basis elements $\{e_g : g \in G\}$ is inherited from the group multiplication in G. So if $g, h \in G$, then, in FG, we have $e_g e_h = e_{gh}$. We extend this linearly to all of FG to define multiplication in FG. We note that e_{1_G} is the multiplicative identity of FG. Furthermore, $\dim_F(FG) = |G|$.

- The ring $F[x]$ of polynomials in x with coefficients in F is an F-algebra. We note that this is an infinite-dimensional F-algebra.

Since we have defined F-algebras to be rings, we can talk about the notion of left and right ideals of algebras.

A ***left ideal*** $I \subset A$ of an F-algebra A is a subspace of A that is closed under left multiplication by A. In other words, for any $a \in A$ and $x \in I$, $ax \in I$. Right ideals are defined correspondingly. We call I a ***two-sided ideal*** of A if it is both a left and a right ideal of A.

Since we assumed that any F-algebra A has an identity 1_A, the condition that an ideal is a subspace of A would come automatically from the ring theoretic definition of an ideal. Indeed, if I is a left ideal of A seen as a ring, then for any $c \in F$ and $x \in I$, $cx = c(1_A x) = (c1_A)x \in I$.

Definition 1.2

- The ***centre*** of A is the set $Z(A) := \{z \in A : za = az \ \forall a \in A\}$.
- Given a subset $B \subset A$, the ***centralizer*** of B in A is the set $Z_A(B) := \{a \in A : ab = ba \ \forall b \in B\}$.

A ***subalgebra*** $B \subset A$ of an F-algebra A is a subspace of A that is itself an F-algebra, with addition and multiplication inherited from A.
While addition and multiplication in B are inherited from A, the identity 1_B of B does not necessarily coincide with the identity 1_A of A. We say that a subalgebra $B \subset A$ is called ***unital*** if 1_B and 1_A coincide.

Example 1.3 Given an F-algebra A, let us mention examples of subalgebras of A.

- The set $F \cdot 1_A = \{c1_A : c \in F\}$ is a subalgebra of A. In this case, the identities of $F \cdot 1_A$ and A coincide. Also, we can identify F with $F \cdot 1_A$.
- The centre $Z(A)$ of A is a subalgebra. Additionally, for any subset $B \subset A$, its centralizer $Z_A(B)$ is a subalgebra of A.

Definition 1.4 Given an F-algebra A, we say that A is ***simple*** if it has no non-trivial proper two-sided ideals.

Definition 1.5 An algebra is called a ***division algebra*** if every non-zero element has a multiplicative inverse.

Since algebras are both vector spaces and rings, we would like homomorphisms of algebras to be both vector space and ring homomorphisms.

Definition 1.6 Let A and B be F-algebras. Then an F***-algebra homomorphism*** is a map $\varphi : A \to B$ from A to B that satisfies the following conditions:

1. $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$,
2. $\varphi(1_A) = 1_B$,
3. φ is an F-linear transformation.

As expected, a map $\varphi : A \to B$ from A to B is called an F***-algebra isomorphism*** if it is a bijective algebra homomorphism.

Definition 1.7 Let A be an F-algebra. Then an A***-module*** V is a finite-dimensional vector space that is equipped with a left A-action map

$$A \times V \to V, \qquad (a, v) \mapsto a \cdot v = av$$

that satisfies the following properties for any $x, y \in A$, $v, w \in V$ and $c \in F$:

1. $x(v+w) = xv + xw$,
2. $(x+y)v = xv + yv$,
3. $(xy)v = x(yv)$,
4. $x(cv) = c(xv) = (cx)v$,
5. $1_A v = v$.

The properties the left A-action needs to satisfy can be equivalently stated as follows:

1. For any $x \in A$, $v \in V$, the map $V \to V$, $v \mapsto xv$ is an endomorphism of V,
2. The map defined by $A \to \mathrm{End}_F(V)$, $x \mapsto (v \mapsto xv)$ for any $x \in A, v \in V$ is an F-algebra homomorphism.

We define right A-modules in a corresponding way.

$\star$ Throughout the thesis, the term A-*module* will refer to a left A-module unless specified otherwise.

Definition 1.8 Let A be an F-algebra. The ***regular*** A***-module***, denoted as A°, is A viewed as an A-module under left multiplication.

Definition 1.9 Let V be an A-module. Then a subspace $W \subset V$ is called an A***-submodule*** of V if W is closed under left multiplication by A. In other words, for any $a \in A$ and $w \in W$, $aw \in W$.

Definition 1.10 Let V and W be two A-modules. Then an A***-module homomorphism*** is a linear transformation $\varphi : V \to W$ from V to W that satisfies $x\varphi(v) = \varphi(xv)$ for any $v \in V$ and $x \in A$.

The set of all A-module homomorphisms from V to W is denoted as $\mathrm{Hom}_A(V,W)$. By convention, $\mathrm{Hom}_A(V,V)$ will be denoted as $\mathrm{End}_A(V)$.

Given two A-modules V,W and an A-module homomorphism $\varphi \in \mathrm{Hom}_A(V,W)$, we define the ***kernel*** of φ as $\ker\varphi := \{v \in V : \varphi(v) = 0_W\}$. The ***image*** of φ is given by $\mathrm{im}\varphi := \{w \in W : \exists v \in V \text{ such that } \varphi(v) = w\}$. $\ker\varphi$ and $\mathrm{im}\varphi$ are submodules of V and W respectively.

Given two A-modules V,W, an A-module homomorphism $\varphi \in \mathrm{Hom}_A(V,W)$ is called an A***-module isomorphism*** if it is bijective, and we would say that V and W are ***isomorphic*** as A-modules. An A-module isomorphism $\varphi \in \mathrm{Hom}_A(V,W)$ satisfies the conditions $\ker\varphi = 0_V$ and $\mathrm{im}\varphi = W$.

Definition 1.11 Let V be a non-zero A-module. Then V is said to be ***irreducible*** if the only submodules of V are 0 and V.

We now state an important lemma involving irreducible A-modules:

Lemma 1.12 (Schur's Lemma) *If V and W are irreducible A-modules, then every non-zero A-module homomorphism in $Hom_A(V,W)$ has an inverse in $Hom_A(W,V)$. [1, Lemma 1.5]*

A consequence of Schur's Lemma is that if F is an algebraically closed field, A is an F-algebra, and V is an irreducible A-module, then $\mathrm{End}_A(V) = F \cdot 1$, the set of scalar multiplications on V.

Definition 1.13 Let V be an A-module. Then V is said to be ***completely reducible*** if for every submodule $W \subseteq V$, there exists a submodule $U \subseteq V$ such that $V = W \oplus U$.

Definition 1.14 An F-algebra A is said to be ***semisimple*** if the regular module A° is completely reducible.

Theorem 1.15 (Maschke's Theorem) *Let G be a finite group and suppose F is a field whose characteristic does not divide the order of G. Then every FG-module is completely reducible. [1, Theorem 1.9]*

Theorem 1.16 *Let V be an A-module. Then the following are equivalent:*

1. *V is completely reducible.*
2. *V is a sum of irreducible submodules.*
3. *V is a direct sum of irreducible submodules.*

[1, Theorem 1.10 + Lemma 1.11]

Let A be an F-algebra with multiplication $*_A$. The ***opposite algebra*** of A, denoted as A^{op}, is the F-algebra that is A as a vector space, but with multiplication $*_{A^{\mathrm{op}}}$ defined by $a *_{A^{\mathrm{op}}} b = b *_A a$ for any $a, b \in A$.

Definition 1.17 Let V be a completely reducible A-module and let M be an irreducible A-module. We define the M***-homogeneous part*** of V as the sum of all the submodules of V that are isomorphic to M. We denote this as $V(M)$.

Lemma 1.18 *Let $V = \bigoplus W_i$ be a direct sum of a finite number of irreducible A-modules W_i. Let M be any irreducible A-module. Then the following holds:*

1. *$V(M)$ is an $\mathrm{End}_A(V)$-submodule of V.*
2. *$V(M) = \sum \{W_i : W_i \cong M\}$.*
3. *The number of W_i that are isomorphic to M is an invariant of V, and is independent of the given direct sum decomposition.*

[1, Lemma 1.13]

Given a semisimple algebra A and an irreducible A-module M, it can be shown that $A^\circ(M)$ is a two-sided ideal of A. We will denote $A^\circ(M)$, regarded as an ideal of A, as $A(M)$.

We let $\mathfrak{M}(A)$ be a set of irreducible A-modules such that every irreducible A-module is isomorphic to exactly one element of $\mathfrak{M}(A)$.

Theorem 1.19 (Wedderburn's Theorem) *Let A be a semisimple algebra and let M be an irreducible A-module. Then the following holds:*

1. *$A(M)$ is a minimal two-sided ideal of A.*
2. *If W is an irreducible A-module, then W is annihilated by $A(M)$ unless $W \cong M$.*
3. *$\mathfrak{M}(A)$ is a finite set.*

[1, Theorem 1.15]

Corollary 1.20 *If A is a simple F-algebra, then any two irreducible A-modules are isomorphic, and $|\mathfrak{M}(A)| = 1$.*

We note that for any irreducible A-module M, $A(M)$ is a simple subalgebra of A that is not necessarily unital. The algebra A can then be decomposed in the following way as a direct sum of simple subalgebras:

$$A = \bigoplus_{M \in \mathfrak{M}(A)} A(M).$$

Lemma 1.21 *Let A be a finite-dimensional simple F-algebra. Then for some positive integer n, $A \cong M_n(D)$ for some division algebra D whose centre contains F. [2, Chapter 8, Theorem 1.5 + Corollary 1.6]*

In combination with Wedderburn's Theorem, this allows us to say that any semisimple F-algebra A is a direct product of simple algebras of the form $M_n(D)$ where D is a division algebra over F.

Definition 1.22 (Tensor Product of Algebras) Let A, B be F-algebras. Then the F-algebra tensor product $A \otimes B$ is $A \otimes_F B$ as an F-vector space with the following multiplication: for any $a_1, a_2 \in A$, $b_1, b_2 \in B$, $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.

It is an exercise to show that multiplication in $A \otimes B$ is associative, that $A \otimes B \cong B \otimes A$, and that $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ for any F-algebras A, B, C.

Recall that the centre of an F-algebra A, $Z(A)$, is defined as the set $\{z \in A : za = az \ \forall a \in A\}$. In this discussion, let A be a simple F-algebra. Then $A \cong M_n(D)$ for some positive integer n and some division algebra D whose centre contains F. From this, we can say that $Z(A) \cong Z(D)$, and for any division algebra D, $Z(D)$ is a field. Note that $Z(D)$ is the largest field over which we can view A as an algebra. We now state the following definition.

Definition 1.23 A is said to be ***central over*** F if $Z(A) = F$.

So if a simple F-algebra A is central, then F is the largest field over which we can view A as an algebra. If A is not central, then there are finite field extensions of F over which we can view A as an algebra.

Now let $A = \bigoplus_i A_i$ be a semisimple F-algebra such that each A_i is simple. Suppose $A_i = M_{n_i}(D_i)$. Then $Z(A) = \bigoplus_i Z(A_i) = \bigoplus_i Z(D_i)$. So the centre of a semisimple algebra is a direct product of fields. Clearly, if a semisimple algebra A is central, then A is simple.

Definition 1.24 An F-algebra is said to be ***central simple*** if it is central over F, and simple.

Theorem 1.25 *If A is a central simple F-algebra and B is a simple F-algebra, then $A \otimes B$ is simple. [2, Chapter 8, Theorem 3.2 (ii)]*

Theorem 1.26 *Let A and B be two F-algebras, and let A', B' be unital subalgebras of A and B respectively. Then $Z_{A \otimes B}(A' \otimes B') = Z_A(A') \otimes Z_B(B')$. [2, Chapter 8, Theorem 3.2 (i)]*

Note that this implies in particular that $Z(A \otimes B) = Z(A) \otimes Z(B)$.

Corollary 1.27 *If A and B are both central simple F-algebras, then $A \otimes B$ is central simple.*

Theorem 1.28 (Skolem-Noether theorem) *Let A be a central simple F-algebra, and let B be a simple F-algebra. Let $\sigma : B \to A$ and $\tau : B \to A$ each be unital F-algebra homomorphisms. Then there exists an inner automorphism ϕ of A such that $\tau = \phi\sigma$, and we obtain the commutative diagram.*

$$\begin{tikzcd} B \arrow[r, "\sigma"] \arrow[dr, "\tau"'] & A \arrow[d, "\phi"] \\ & A \end{tikzcd}$$

[2, Chapter 8, Theorem 4.2]

In particular, this implies that any F-algebra automorphism of a central simple F-algebra is an inner automorphism.

Theorem 1.29 (Centralizer Theorem) *Let A be a central simple F-algebra, and let B be a simple unital subalgebra of A. Then the following holds:*

1. *$Z_A(B)$ is a simple unital subalgebra of A.*
2. *$\dim_F(A) = \dim_F(B) \cdot \dim_F(Z_A(B))$.*
3. *If B is central, then $A \cong B \otimes_F Z_A(B)$.*

[2, Chapter 8, Theorem 4.5]

Corollary 1.30 *Let A be a central simple F-algebra, and let B be a simple unital subalgebra of A. Then $Z_A(Z_A(B)) = B$. [2, Chapter 8, Corollary 4.8]*

Lemma 1.31 *If A is a central simple F-algebra, then $A \otimes A^{\mathrm{op}} \cong M_n(F)$ where $n = \dim_F(A)$. [2, Chapter 8, Theorem 3.4]*

Corollary 1.32 *Let D be a division algebra that is central over F. Then $D \otimes D^{\mathrm{op}} \cong M_n(F)$ where $n = \dim_F(D)$.*

Lemma 1.33 *Let $A = M_n(D)$ be a central simple F-algebra, where D is a central division algebra over F. Then $A^{\mathrm{op}} \cong M_n(D^{\mathrm{op}})$.*

Proof. We can express A as $A \cong M_n(F) \otimes D$, and $M_n(D^{\mathrm{op}}) \cong M_n(F) \otimes D^{\mathrm{op}}$. For any $i, j \leq n$, we define $E_{i,j}$ to be the matrix in $M_n(F)$ with a 1 in the $(i,j)^{th}$ entry and 0 everywhere else. The required isomorphism $\varphi : (M_n(F) \otimes D)^{\mathrm{op}} \to M_n(F) \otimes D^{\mathrm{op}}$ is defined as: $\varphi(E_{i,j} \otimes x) = E_{j,i} \otimes x$ for any $x \in D$. ∎

The tensor product of central simple algebras somewhat exhibits the behaviour of a group operation. Let us see how:

- Firstly, as mentioned before, as an operation on F-algebras, the algebra tensor product is associative. In general, for any F-algebras A, B, C, $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$. This of course still holds for central simple algebras.

- Let A and B be two central simple F-algebras. Then, by Corollary 1.27, $A = M_{n_1}(D_1)$, $B = M_{n_2}(D_2)$ and $A \otimes B = M_{n_3}(D_3)$ for some positive integers n_i and for some division algebras D_i such that $Z(D_i) = F$ for all i. So the algebra tensor product is closed in the set of central simple F-algebras.

- For any central simple algebra $M_n(D)$, $M_n(D) \otimes M_m(F) \cong M_m(F) \otimes M_n(D) = M_{nm}(D)$. In a sense, "tensor multiplying" a central simple algebra by $M_n(F)$ does not change the associated (central) division algebra. So the matrix algebras of the form $M_n(F)$ act like identity elements under the tensor product operation.

- Since $D \otimes D^{\text{op}} = M_d(F)$, $M_n(D) \otimes M_m(D^{\text{op}}) \cong M_m(D^{\text{op}}) \otimes M_n(D) \cong M_{nm}(F) \otimes (D \otimes D^{\text{op}}) \cong M_{nm}(F) \otimes M_d(F) \cong M_{nmd}(F)$. So given a central simple algebra $M_n(D)$, central simple algebras of the form $M_m(D^{\text{op}})$ act like inverses under algebra tensor product.

We remark that the isomorphism classes of central simple F-algebras is a monoid under the operation of algebra tensor product. We now formalize the group structure of central simple F-algebras in a way introduced by Richard Brauer.

1.2 The Brauer Group of F

Let us first describe the set of elements in the Brauer group of F. For any central simple F-algebras A, B, define $A \sim B$ if $A = M_n(D_A)$, $B = M_m(D_B)$, and $D_A \cong D_B$. One can check that $\sim$ defines an equivalence relation on all central simple algebras over F.

Let F be a field. We denote the set of equivalence classes of central simple F-algebras as $Br(F)$. Then $[A]$ will denote the set of central simple F-algebras in the same equivalence class as A. We define a binary operation on $Br(F)$ in the following way:

For any two equivalence classes $[A], [B]$, set $[A] \cdot [B] := [A \otimes B]$. Let us verify that this operation is well defined. Let $A = M_n(D_A)$ and $B = M_m(D_B)$ for some central division algebras D_A, D_B. For any $A' \sim A$, $A' = M_{n'}(D_A)$ and for any $B' \sim B$, $B' = M_{m'}(D_B)$.

Our task is to show that $[A' \otimes B'] = [A \otimes B]$. Suppose $D_A \otimes D_B = M_r(D)$ where D is a central division algebra. Then $A \otimes B \cong M_{nm}(F) \otimes (D_A \otimes D_B) \cong M_{nm}(F) \otimes (M_r(F) \otimes D) \cong M_{nmr}(D)$. On the other hand, $A' \otimes B' \cong M_{n'm'}(F) \otimes (D_A \otimes D_B) \cong M_{n'm'}(F) \otimes (M_r(F) \otimes D) \cong M_{n'm'r}(D)$. This has the same central division algebra part as $A \otimes B$, and so $A \otimes B \sim A' \otimes B'$, hence $[A' \otimes B'] = [A \otimes B]$ and the binary operation on the set of equivalence classes is well defined.

Since the tensor product is associative on central simple F-algebras, the binary operation $\cdot$ is associative. Note that $[A] \cdot [F] = [A]$. So the equivalence class $[F]$ acts as the identity element under the operation. Also if $A \cong M_n(D)$, then $[A] \cdot [D^{\text{op}}] = [D] \cdot [D^{\text{op}}] = [D^{\text{op}}] \cdot [D] = [F]$. So the inverse of $[D]$ is $[D^{\text{op}}]$. We can now define the group $Br(F)$.

Definition 1.34 The ***Brauer group of*** F is the set $Br(F)$ of equivalence classes of central simple F-algebras equipped with the group operation defined by $[A] \cdot [B] = [A \otimes B]$.

As $\otimes$ is commutative, we note that $Br(F)$ is abelian.

It must be emphasised that the Brauer group is different for different fields. Note that given a field F, the order of $Br(F)$ is equal to the number of distinct central division algebras over F.

Corollary 1.35 *If a field F is algebraically closed, then $Br(F) = 1$, the trivial group.*

When it comes to finite fields, we note that by Wedderburn's little theorem, every finite division algebra is a field. Hence, if F is a finite field, then the only finite-dimensional central division algebra over F is F itself. Hence, if F is a finite field, then $Br(F) = 1$.

By Frobenius, there are only two finite-dimensional central division algebras over $\mathbb{R}$. They are $\mathbb{R}$ and $\mathbb{H}$, the algebra of quaternions. Hence $Br(\mathbb{R}) \cong C_2$. The only other finite-dimensional division algebra over $\mathbb{R}$ is $\mathbb{C}$. It is clear that $\mathbb{C}$ is not central over $\mathbb{R}$.

2 Superalgebras

2.1 Superalgebra Theory preliminaries

In this section, we will introduce the concept of a superalgebra, which is a specific type of algebra. Many of the definitions and theorems we encountered in Algebra Theory will have their equivalents in Superalgebra Theory. As such, section 2 will follow a similar structure to section 1. This time, we will provide full proofs of the "super" theorems, such as the Super Schur's Lemma, Super Maschke's Theorem, and the Super Wedderburn's Theorem. Many of these proofs are based on the proofs of the theorems from ordinary Algebra Theory. We will eventually build up to super central simple superalgebras and the Brauer-Wall group. Many of the definitions mentioned in this section comes from [3, Chapters 5 + 6].

$\star$ Again, F will refer to an arbitrary field, and A will refer to an arbitrary F-algebra. Suppose G is a finite group. A G***-grading*** on A is a direct sum decomposition $A = \bigoplus_{g \in G} A_g$ of A indexed by the elements of G that satisfies the following conditions:

- $F \cdot 1_A \subset A_{1_G}$,
- For any $g, h \in G$, $A_g \cdot A_h \subset A_{gh}$.

A is a G***-graded*** F***-algebra*** if it is equipped with a G-grading $A = \bigoplus_{g \in G} A_g$. The subspace A_g is called the g***-component*** of A.

Definition 2.1 An F***-superalgebra*** is a C_2-graded F-algebra.

Given an F-superalgebra with a C_2-grading, by convention we will denote the component of A indexed by the trivial element of C_2 as A_0, and we will denote the component indexed by the non-trivial element of C_2 as A_1. A_0 is called the 0***-component*** of A, while A_1 is its 1***-component***.

Let $A = A_0 \oplus A_1$ be an F-superalgebra. Then the set $A_0 \cup A_1$ is called the set of ***homogeneous*** elements of A. Homogeneous elements in A_0 and A_1 are called ***even*** and ***odd*** elements respectively. An F-superalgebra A is ***purely even*** if $A_1 = \{0\}$. Given $i \in \{0, 1\}$, a homogeneous element x in A_i is said to have ***degree*** $d(x) = i$.

Remark 2.2
- An F-algebra A can be given different gradings. We will mention this fact in the list of examples that follow this remark.
- From the definition of a superalgebra, we know that A_0 is an ordinary unital subalgebra of A.
- Like with F-algebras, the dimension of an F-superalgebra A is given by its vector space dimension over F.

Example 2.3 Given a field F, let us list examples of F-superalgebras.

- The field F itself can be regarded as a purely even F-superalgebra.
- Let $r, s \geq 0$. Then the matrix superalgebra over F, $M_{(r,s)}(F)$, is $M_{r+s}(F)$ as an F-algebra with the following grading: $\big(M_{(r,s)}(F)\big)_0$ is the set of matrices of the form

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$$

where A and B are $r \times r$ and $s \times s$ matrices over F respectively, while $\left(M_{(r,s)}(F)\right)_1$ is the set of matrices of the form

$$\left(\begin{array}{c|c} 0 & C \\ \hline D & 0 \end{array}\right)$$

where C and D are $r \times s$ and $s \times r$ matrices over F respectively. From this, we can see that for any $n \geq 2$, $M_n(F)$ can be given different gradings. For example, the F-superalgebras $M_{(2,2)}(F)$ and $M_{(1,3)}(F)$ are both $M_4(F)$ as algebras, but have different gradings.

- We can define the notion of an F-super vector space and its super endomorphism ring over F. Let V be a finite-dimensional F-vector space. Let r, s be two non-negative integers such that $r + s = \dim_F(V)$. Then the ***F-super vector space*** $V^{(r,s)}$ is V as an F-vector space with a direct sum decomposition $V = V_0 \oplus V_1$ where $\dim_F(V_0) = r$ and $\dim_F(V_1) = s$. The ***super endomorphism ring*** $\mathrm{End}^{\mathfrak{s}}_F\left(V^{(r,s)}\right)$ of $V^{(r,s)}$ is $\mathrm{End}_F(V)$ as an F-algebra with the following grading:
  - $\left(\mathrm{End}^{\mathfrak{s}}_F(V^{(r,s)})\right)_0 = \{\varphi \in \mathrm{End}_F(V) : \varphi(v_0) \in V_0 \text{ and } \varphi(v_1) \in V_1\ \forall v_0 \in V_0, v_1 \in V_1\}$, and
  - $\left(\mathrm{End}^{\mathfrak{s}}_F(V^{(r,s)})\right)_1 = \{\varphi \in \mathrm{End}_F(V) : \varphi(v_0) \in V_1 \text{ and } \varphi(v_1) \in V_0\ \forall v_0 \in V_0, v_1 \in V_1\}$.
- Let G be a finite group with a normal subgroup $N \triangleleft G$ of index 2. Then the ***group superalgebra*** $F[G, N]$ is FG as an F-algebra with the following grading: $(F[G,N])_0 = FN$, while $(F[G,N])_1 = F[G\backslash N]$.
- The ring $F[x]$ of polynomials in x with coefficients in F can be equipped with the following grading: $F[x]_0$ is the set of all polynomials of the form $\sum c_n x^{2n}$, while $F[x]_1$ is the set of all polynomials of the form $\sum c_n x^{2n+1}$. With this grading, $F[x]$ is an infinite-dimensional F-superalgebra.

If A, B are F-superalgebras, then $A \times B$ is an F-superalgebra with 0-component $A_0 \times B_0$ and 1-component $A_1 \times B_1$. In general, a finite direct product of F-superalgebras $\prod A_i$ is itself an F-superalgebra with 0-component $\prod (A_i)_0$ and 1-component $\prod (A_i)_1$.

$\star$ In this section, we set A to be an F-superalgebra.

Definition 2.4 A ***left superideal*** $I \subset A$ of A is a subspace of A with a decomposition $I = I_0 \oplus I_1$ that satisfies the following properties:

- $I_0 \subset A_0$ and $I_1 \subset A_1$.
- For any $i, j \in \{0, 1\}$, $A_i \cdot I_j \subset I_{(i+j)\mathrm{mod}2}$.

Right superideals are defined correspondingly. A ***two-sided superideal*** of A is a left superideal that is also a right superideal.

We note that superideals of a superalgebra are ordinary ideals of the underlying algebra.

A ***subsuperalgebra*** $B \subset A$ is a subspace of A that is also an F-superalgebra, and is equipped with a grading $B = B_0 \oplus B_1$ that satisfies $B_0 \subset A_0$ and $B_1 \subset A_1$. A subsuperalgebra $B \subset A$ is called ***unital*** if $1_B = 1_A$.

Let us mention examples of subsuperalgebras of A.

- $F \cdot 1_A$ is a purely even subsuperalgebra of A.
- $Z(A)$ is a subsuperalgebra of A with grading $Z(A)_0 = Z(A) \cap A_0$ and $Z(A)_1 = Z(A) \cap A_1$.

Definition 2.5 The ***super centre*** of A, denoted by $Z^{\mathfrak{s}}(A)$, is the subsuperalgebra of A with the following grading:

- $(Z^{\mathfrak{s}}(A))_0 = \{z_0 \in A_0 : z_0 a_0 = a_0 z_0 \text{ and } z_0 a_1 = a_1 z_0 \text{ for any } a_0 \in A_0, a_1 \in A_1\}$ and
- $(Z^{\mathfrak{s}}(A))_1 = \{z_1 \in A_1 : z_1 a_0 = a_0 z_1 \text{ and } z_1 a_1 = -a_1 z_1 \text{ for any } a_0 \in A_0, a_1 \in A_1\}$.

Let $B \subset A$ be a subsuperalgebra of A. Then the ***super centralizer*** of B in A, denoted by $Z^{\mathfrak{s}}_A(B)$, is the subsuperalgebra of A with the following grading:

- $(Z^{\mathfrak{s}}_A(B))_0 = \{z_0 \in A_0 : z_0 b_0 = b_0 z_0 \text{ and } z_0 b_1 = b_1 z_0 \text{ for any } b_0 \in B_0, b_1 \in B_1\}$ and
- $(Z^{\mathfrak{s}}_A(B))_1 = \{z_1 \in A_1 : z_1 b_0 = b_0 z_1 \text{ and } z_1 b_1 = -b_1 z_1 \text{ for any } b_0 \in B_0, b_1 \in B_1\}$.

Definition 2.6 A is ***super simple*** if it has no non-trivial proper two-sided superideals.

An F-superalgebra is called a ***super division algebra*** if every non-zero homogeneous element has a multiplicative inverse.

Remark 2.7
- Let $D = D_0 \oplus D_1$ be a super division algebra, and let $x \in D_i$ be a non-zero homogeneous element of degree i. Then it is straightforward to show that x^{-1} is also a homogeneous element of degree i.
- Given a super division algebra $D = D_0 \oplus D_1$, D_0 itself is an ungraded division algebra. If $D_1 \neq 0$, then for any non-zero $v \in D_1$, $D_1 = D_0 v$.

Definition 2.8 Let A and B be F-superalgebras. Then an ***F-superalgebra homomorphism*** is a linear map $\varphi : A \to B$ which satisfies:

1. $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$,
2. $\varphi(1_A) = 1_B$,
3. $\varphi(A_i) \subset B_i$ for any $i \in \{0, 1\}$.

As expected, an ***F-superalgebra isomorphism*** is a bijective superalgebra homomorphism.

Definition 2.9 A ***left A-supermodule*** V is a finite-dimensional vector space that is equipped with a direct sum decomposition $V = V_0 \oplus V_1$ (called its *grading*) and a left A-action map

$$A \times V \to V, \qquad (a, v) \to a \cdot v = av$$

that satisfies the same properties as a left A-action map for an ordinary A-module, and additionally satisfies $A_i \cdot V_j \subset V_{(i+j)\bmod 2}$ for any $i, j \in \{0, 1\}$.

We define right A-supermodules in a corresponding way. Throughout the thesis, the term A-supermodule will refer to a left A-supermodule unless specified otherwise.

Definition 2.10 Let A be an F-superalgebra. The ***regular A-supermodule***, denoted as A°, is A viewed as an A-supermodule under left multiplication.

Let A be an F-superalgebra and let V be an A-supermodule. Then a subspace $W \subset V$ is called an ***A*-*subsupermodule*** of V if W is closed under left multiplication by A, and is equipped with a direct sum decomposition $W = W_0 \oplus W_1$ that satisfies $W_0 \subset V_0$, $W_1 \subset V_1$ and $A_i \cdot W_j \subset W_{(i+j)\mathrm{mod}2}$ for any $i, j \in \{0, 1\}$.

Definition 2.11 Let V and W be two A-supermodules, and let $i \in \{0, 1\}$. Then a map $\varphi : V \to W$ from V to W is called an ***A*-*supermodule homomorphism of degree*** i if it is a linear transformation that satisfies:

- $\varphi(V_j) \subset W_{_{(j+i)\mathrm{mod}2}}$ for any $j \in \{0, 1\}$.
- $\varphi(a_j v) = (-1)^{ij} a_j \varphi(v)$ for any $a_j \in A_j$, $v \in V$.

The set of all A-supermodule homomorphisms from V to W, denoted as $\mathrm{Hom}^{\mathfrak{s}}_A(V, W)$, is the A-supermodule with the following grading:

- $(\mathrm{Hom}^{\mathfrak{s}}_A(V, W))_0$ is the set of all A-supermodule homomorphisms from V to W of degree 0.
- $(\mathrm{Hom}^{\mathfrak{s}}_A(V, W))_1$ is the set of all A-supermodule homomorphisms from V to W of degree 1.

Remark 2.12
- Given an A-supermodule homomorphism $\varphi \in \mathrm{Hom}^{\mathfrak{s}}_A(V, W)$, its *kernel* $\ker\varphi$ and *image* $\mathrm{im}\varphi$ are defined in the same way as the ungraded case. If φ is homogeneous, then $\ker\varphi$ and $\mathrm{im}\varphi$ are subsupermodules of V and W respectively.
- Given two A-supermodules V, W, an A-supermodule homomorphism $\varphi \in \mathrm{Hom}^{\mathfrak{s}}_A(V, W)$ is called an ***A*-*supermodule isomorphism*** if it is homogeneous and bijective, and if such an A-supermodule homomorphism exists, we would say that V and W are ***super isomorphic***, and we would denote this as $V \cong^{\mathfrak{s}} W$.

Conventionally, given an A-supermodule V, $\mathrm{Hom}^{\mathfrak{s}}_A(V, V)$ will be denoted as $\mathrm{End}^{\mathfrak{s}}_A(V)$.

Lemma 2.13 *Suppose $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ are two A-supermodules such that there exists an ordinary A-module isomorphism $\varphi : V \to W$ such that $\varphi(V_0) = W_1$ and $\varphi(V_1) = W_0$. Then $V \cong^{\mathfrak{s}} W$.*

Proof. Let us show that the linear map $\tilde{\varphi}$ defined by: $\tilde{\varphi}(v_0) = \varphi(v_0)$ for any $v_0 \in V_0$, and $\tilde{\varphi}(v_1) = -\varphi(v_1)$ for any $v_1 \in V_1$ is an A-supermodule isomorphism. For any $v_i \in V_i$, note that for any $a_0 \in A_0$, $\tilde{\varphi}(a_0 v_i) = (-1)^i \varphi(a_0 v_i) = a_0(-1)^i \varphi(v_i) = a_0 \tilde{\varphi}(v_i)$, and for any $a_1 \in A_1$, $\tilde{\varphi}(a_1 v_i) = (-1)^{1-i} \varphi(a_1 v_i) = a_1(-1)^{1-i} \varphi(v_i) = -a_1(-1)^i \varphi(v_i) = -a_1 \tilde{\varphi}(v_i)$. Hence $\tilde{\varphi}$ is an A-supermodule isomorphism of degree 1, and $V \cong^{\mathfrak{s}} W$. ■

Definition 2.14 Let V be a non-zero A-supermodule. Then V is said to be ***super irreducible*** if the only subsupermodules of V are 0 and V.

Lemma 2.15 *If $V = V_0 \oplus V_1$ is a super irreducible A-supermodule, then V_0 and V_1 are irreducible as ungraded A_0-modules.*

Proof. Let $V = V_0 \oplus V_1$ be a super irreducible A-supermodule. Without loss of generality, suppose V_0 is not an irreducible ungraded A_0-module. Let $W_0 \subset V_0$ be a non-trivial A_0-submodule of V_0. Then $W_0 \oplus (A_1 \cdot W_0)$ would be a non-trivial A-subsupermodule of V, which is a contradiction. ■

We now come to the superalgebra analogue of Schur's Lemma.

Lemma 2.16 (Super Schur's Lemma) *If V and W are super irreducible A-supermodules, then every non-zero homogeneous A-supermodule homomorphism in $\mathrm{Hom}^{\mathfrak{s}}_A(V,W)$ has an inverse in $\mathrm{Hom}^{\mathfrak{s}}_A(W,V)$.*

Proof. Since $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ are super irreducible A-supermodules, then by Lemma 2.15, V_0, V_1, W_0 and W_1 are irreducible ungraded A_0-modules.

If $\varphi \in \mathrm{Hom}^{\mathfrak{s}}_A(V,W)$ is a non-zero homogeneous A-supermodule homomorphism of degree i, then the subsupermodule $\ker\varphi = \ker(\varphi|_{V_0}) \oplus \ker(\varphi|_{V_1})$ of V is not equal to V. Since V is super irreducible, then $\ker\varphi = 0$, implying $\ker(\varphi|_{V_0})$ and $\ker(\varphi|_{V_1})$ are both equal to 0, which tells us that $\varphi|_{V_0}$ and $\varphi|_{V_1}$ are non-zero elements of $\mathrm{Hom}_{A_0}(V_0, W_i)$ and $\mathrm{Hom}_{A_0}(V_1, W_{1-i})$ respectively. By the ungraded Schur's Lemma, $\varphi|_{V_0}$ and $\varphi|_{V_1}$ have inverses in $\mathrm{Hom}_{A_0}(W_i, V_0)$ and $\mathrm{Hom}_{A_0}(W_{1-i}, V_1)$. This will allow us to construct the inverse of φ: for any $w_0 \in W_0$, $w_1 \in W_1$, $\varphi^{-1}(w_0 + w_1) = (\varphi|_{V_i})^{-1}(w_0) + (\varphi|_{V_{1-i}})^{-1}(w_1)$. φ^{-1} is an A-supermodule homomorphism in $\mathrm{Hom}^{\mathfrak{s}}_A(W,V)$, and we are done. ■

Remark 2.17

- If V and W are super irreducible A-supermodules and $\varphi \in \mathrm{Hom}^{\mathfrak{s}}_A(V,W)$ is a non-zero homogeneous A-supermodule homomorphism of degree i, then φ^{-1} is also a non-zero homogeneous A-supermodule homomorphism of degree i.
- Given a super irreducible A-supermodule V, $\mathrm{End}^{\mathfrak{s}}_A(V)$ is a super division algebra. If $\mathrm{End}^{\mathfrak{s}}_A(V)$ has non-zero homogeneous elements of degree 1, then V_0 and V_1 are isomorphic as A_0-modules, and $\dim_F(V_0) = \dim_F(V_1)$.

Definition 2.18 Let V be an A-supermodule. Then V is ***completely super reducible*** if for every subsupermodule $W \subset V$, there exists another subsupermodule $U \subset V$ such that $V = W \oplus U$.

Definition 2.19 An F-superalgebra A is said to be ***super semisimple*** if its regular supermodule A° is completely super reducible as an A-supermodule.

Now we get to the super version of Maschke's Theorem.

Theorem 2.20 (Super Maschke's Theorem) *Let G be a finite group, and let G_0 be a subgroup of G with $|G : G_0| = 2$, and suppose that the characteristic of F does not divide $|G|$. Then every $F[G,G_0]$-supermodule is completely super reducible.*

Proof. Let V be an $F[G,G_0]$-supermodule, and let W be a subsupermodule of V. Since W is also an ungraded FG-submodule of V, then by the ungraded Maschke's Theorem, there exists an ungraded submodule U such that $W \oplus U = V$. We note that U is also an $F[G,G_0]$-subsupermodule of V, with grading $U = U_0 \oplus U_1$, where $U_0 = U \cap V_0$ and $U_1 = U \cap V_1$. ■

Theorem 2.21 *Let V be an A-supermodule. Then the following are equivalent:*

1. *V is completely super reducible.*
2. *V is a sum of super irreducible subsupermodules.*
3. *V is a direct sum of super irreducible subsupermodules.*

Proof. First, let us show that $1 \implies 2$.
Let V be a completely super reducible A-supermodule. Let W be the sum of all super irreducible subsupermodules of V. For the sake of contradiction, suppose $W \subsetneq V$. Since

V is completely super reducible, then $V = W \oplus U$ for some non-zero A-supermodule U. Since V is finite dimensional, U contains a super irreducible subsupermodule $U' \subset V$. Since W is the the sum of all super irreducible subsupermodules of V, $U' \subset W$, which implies $W \cap U \neq 0$, which is a contradiction. Thus $W = V$, and V is a sum of super irreducible subsupermodules.

Then, let us show that $2 \implies 1$.

Let V be a sum of super irreducible subsupermodules. Then $V = \sum V^\alpha$, where each V^α is super irreducible. Let $W \subset V$ be a subsupermodule of V. By finite dimensionality of V, choose a maximal subsupermodule $U \subset V$ with the property that $W \cap U = 0$. We claim that $W \oplus U = V$. If this was not true, then there would exist a V^α that $V^\alpha \not\subset W \oplus U$. This would mean that $V^\alpha \not\subset W$ and $V^\alpha \not\subset U$. Since V^α is super irreducible, then $V^\alpha \cap W = 0$ and $V^\alpha \cap U = 0$. Hence $U \oplus V^\alpha$ would be a subsupermodule that satisfies $W \cap (U \oplus V^\alpha) = 0$, which contradicts the maximality of U. Thus $W \oplus U = V$, and V is completely super reducible.

$3 \implies 2$ is immediate, so we just need to show that $2 \implies 3$.

Let V be a sum of super irreducible subsupermodules. Then $V = \sum V^\alpha$, where each V^α is super irreducible. We choose a maximal subsupermodule $W \subset V$ with the property that W is the direct sum of some of the V^α's. We aim to show that $W = V$. For the sake of contradiction, suppose $W \subsetneq V$. Then for some α, $V^\alpha \not\subset W$. Since V^α is super irreducible, then $V^\alpha \cap W = 0$, and $W \oplus V^\alpha \supsetneq W$. This contradicts the maximality of W. Thus we can conclude that $W = V$, and V is a direct sum of super irreducible subsupermodules. ■

Let A be an F-superalgebra with multiplication $*_A$. The ***super opposite algebra*** of A, denoted as $A^{\mathfrak{sop}}$, is the F-superalgebra that is A as a vector space, has the same grading as A, but with multiplication $*_{A^{\mathrm{op}}}$ defined by:

- $a_0 *_{A^{\mathfrak{sop}}} b_0 = b_0 *_A a_0$ for any $a_0, b_0 \in A_0$.
- $a_0 *_{A^{\mathfrak{sop}}} a_1 = a_1 *_A a_0$ and $b_1 *_{A^{\mathfrak{sop}}} b_0 = b_0 *_A b_1$ for any $a_0, b_0 \in A_0$, $a_1, b_1 \in A_1$.
- $a_1 *_{A^{\mathfrak{sop}}} b_1 = -(b_1 *_A a_1)$ for any $a_1, b_1 \in A_1$.

Definition 2.22 Let V be a completely super reducible A-supermodule and let M be a super irreducible A-supermodule. We define the ***M-homogeneous part*** of V as the sum of all the left subsupermodules of V that are super isomorphic to M. We denote this as $V^{\mathfrak{s}}(M)$.

Lemma 2.23 *Let $V = \bigoplus W_i$ be a direct sum of A-supermodules where each W_i is super irreducible. Let M be any super irreducible A-supermodule. Then the following holds:*

1. *$V^{\mathfrak{s}}(M)$ is an $\mathrm{End}^{\mathfrak{s}}_A(V)$-subsupermodule of V.*
2. *$V^{\mathfrak{s}}(M) = \sum \{W_i : W_i \cong^{\mathfrak{s}} M\}$.*
3. *The number of W_i that are super isomorphic to M is an invariant of V, and is independent of the given direct sum decomposition.*

Proof.

1. It is enough to show that for any homogeneous element $\varphi \in \mathrm{End}^{\mathfrak{s}}_A(V)$, $\varphi(V^{\mathfrak{s}}(M)) \subset V^{\mathfrak{s}}(M)$.

   Now let φ be a homogeneous element of $\mathrm{End}^{\mathfrak{s}}_A(V)$. To show that $\varphi(V^{\mathfrak{s}}(M)) \subset V^{\mathfrak{s}}(M)$, it is enough to show that for any subsupermodule $W \subset V$ that is super isomorphic to M, $\varphi(W) \subset V^{\mathfrak{s}}(M)$.

   Let W be a subsupermodule of V such that $W \cong^{\mathfrak{s}} M$. If $\varphi(W) = 0$, then immediately $\varphi(W) \subset V^{\mathfrak{s}}(M)$. Suppose $\varphi(W) \neq 0$. Since $W \cong^{\mathfrak{s}} M$ and M is super irreducible, W is also super irreducible. Therefore, $\ker(\varphi\mid_W)$ is either 0 or W. Since $\varphi(W) \neq 0$, $\ker(\varphi\mid_W) = 0$, which means the map $\varphi\mid_W : W \to \varphi(W)$ is injective, and hence bijective. Therefore, $W \cong^{\mathfrak{s}} \varphi(W)$, and since $W \cong^{\mathfrak{s}} M$, $\varphi(W) \cong^{\mathfrak{s}} M$. We conclude that $\varphi(W) \subset V^{\mathfrak{s}}(M)$.

2. It is clear that $\sum\{W_i : W_i \cong^{\mathfrak{s}} M\} \subset V^{\mathfrak{s}}(M)$. To show the reverse inclusion, it is enough to show that for any subsupermodule $W \subset V$ that is super isomorphic to M, $W \subset \sum\{W_i : W_i \cong^{\mathfrak{s}} M\}$.

   First, let π_i be the projection map from V onto W_i. Note that π_i is a homogeneous element of $\mathrm{Hom}^{\mathfrak{s}}_A(V, W_i)$ of degree 0. Now let $W \subset V$ be a subsupermodule such that $W \cong^{\mathfrak{s}} M$. We know that means W is super irreducible. For any i, $\ker(\pi_i\mid_W)$ is either 0 or W. If $\ker(\pi_i\mid_W) = 0$, then the map $\pi_i\mid_W : W \to W_i$ is injective. Since W_i is super irreducible, we also get that $\mathrm{im}(\pi_i\mid_W) = W_i$, hence $\pi_i(W) = W_i \subset \sum\{W_i : W_i \cong^{\mathfrak{s}} M\}$ for any i. However, since $W \subset \sum_i \pi_i(W)$, we get that $W \subset \sum\{W_i : W_i \cong^{\mathfrak{s}} M\}$, and we are done.

3. Let n denote the number of W_i that are super isomorphic to M. By 2), we get that $\dim_F(V^{\mathfrak{s}}(M)) = n \cdot \dim_F(M)$, and it is clear that n is an invariant of V.

■

Definition 2.24 Let V be an A-supermodule, and let $W \subset V$ be a subsupermodule of V. Then the factor supermodule $(V/W)^{\mathfrak{s}}$ is V/W as an ungraded A-module with the following grading: $(V/W)^{\mathfrak{s}}_0 = V_0/W_0$, and $(V/W)^{\mathfrak{s}}_1 = V_1/W_1$.

Lemma 2.25 *Let A be an F-superalgebra. Then the following holds:*

1. *Every super irreducible A-supermodule is super isomorphic to a factor supermodule of A°.*

2. *If A is super semisimple, then every super irreducible A-supermodule is super isomorphic to a subsupermodule of A°.*

Proof.

1. Let V be a super irreducible A-supermodule. Choose a non-zero element $v \in V_0$, and define the F-linear $\varphi : A \to V$ by $\varphi(x) = xv$. Note that for any $a_0 \in A_0$, $a_0 v \in V_0$, and for any $a_1 \in A_1$, $a_1 v \in V_1$. Hence $\varphi(A_0) \subseteq V_0$ and $\varphi(A_1) \subseteq V_1$. Also, for any $x, y \in A$, $\varphi(xy) = (xy)v = x(yv) = x\varphi(y)$. That means φ is a homogeneous element of $\mathrm{Hom}^{\mathfrak{s}}_A(A^\circ, V)$ of degree 0. Hence $\mathrm{im}\varphi$ is a subsupermodule of V, and since V is super irreducible, $\mathrm{im}\varphi$ is either 0 or V. Since $v \in \mathrm{im}\varphi$, $\mathrm{im}\varphi = V$, and this allows us to say that $V = \mathrm{im}\varphi \cong^{\mathfrak{s}} A^\circ/\ker\varphi$.

2. Let V be a super irreducible A-supermodule. From 1), $V \cong^{\mathfrak{s}} A^{\circ}/\ker\varphi$ where φ is the A-supermodule homomorphism defined in 1). If A is super semisimple, then since $\ker\varphi$ is a subsupermodule of A°, there exists a subsupermodule $U \subset A^{\circ}$ such that $A^{\circ} = \ker\varphi \oplus U$. This tells us that $U \cong^{\mathfrak{s}} A^{\circ}/\ker\varphi \cong^{\mathfrak{s}} V$. Hence V is super isomorphic to U, a subsupermodule of A°.

■

Given a super semisimple superalgebra A and a super irreducible A-supermodule M, it can be shown that $(A^{\circ})^{\mathfrak{s}}(M)$ is a two-sided superideal of A. We will denote $(A^{\circ})^{\mathfrak{s}}(M)$ seen as a two-sided superideal of A as just $A^{\mathfrak{s}}(M)$.

We can describe a representative set of super irreducible A-supermodules. We can describe a set $\mathfrak{M}^{\mathfrak{s}}(A)$ as a set of super irreducible A-supermodules such that every super irreducible A-supermodule is super isomorphic to exactly one element of $\mathfrak{M}^{\mathfrak{s}}(A)$.

We finally get to Wedderburn's theorem for superalgebras

Theorem 2.26 (Super Wedderburn's Theorem) *Let A be a super semisimple superalgebra and let M be a super irreducible A-supermodule. Then the following holds:*

1. *$A^{\mathfrak{s}}(M)$ is a two-sided superideal of A.*
2. *If V is a super irreducible A-supermodule, then V is annihilated by $A^{\mathfrak{s}}(M)$ unless $V \cong^{\mathfrak{s}} M$.*
3. *$A^{\mathfrak{s}}(M)$ is a minimal two-sided superideal of A.*
4. *$\mathfrak{M}^{\mathfrak{s}}(A)$ is a finite set.*

Proof.

1. Let us first prove that $A^{\mathfrak{s}}(M)$ is a two-sided superideal of A.

   First we note that $A^{\mathfrak{s}}(M)$ is a non-zero sum of left subsupermodules of A°. This is because, by Lemma 2.25(2), there is a subsupermodule $W \subset A^{\circ}$ such that $M \cong^{\mathfrak{s}} W$. Since left subsupermodules of A° are also left superideals of A, $A^{\mathfrak{s}}(M)$ is therefore a sum of left superideals of A, and hence is a left superideal of A.

   Next, we show that $A^{\mathfrak{s}}(M)$ is a right superideal. Let $V = V_0 \oplus V_1$ be a super irreducible left subsupermodule of A° that is super isomorphic to M. Once we show that for any $a \in A_0 \cup A_1$, $V \cdot a = (V_0 \cdot a) \oplus (V_1 \cdot a) \cong^{\mathfrak{s}} V$ when $V \cdot a \neq 0$, we can then say that for any $a \in A_0 \cup A_1$, $A^{\mathfrak{s}}(M) \cdot a$ is a sum of left subsupermodules of the form $V \cdot a$, and each non-zero $V \cdot a$ are all super isomorphic to M, allowing us to conclude that $A^{\mathfrak{s}}(M) \cdot a \subset A^{\mathfrak{s}}(M)$.

   When showing that $V \cdot a \cong^{\mathfrak{s}} V$ when $V \cdot a \neq 0$, we need to first note that since $V_0 \subset A_0$ and $V_1 \subset A_1$, $V_0 \cdot a \subset A_{(0+d(a))\text{mod}2}$ and $V_1 \cdot a \subset A_{(1+d(a))\text{mod}2}$, which means $(V_0 \cdot a) \cap (V_1 \cdot a) = 0$. This validates our grading of $V \cdot a = (V_0 \cdot a) \oplus (V_1 \cdot a)$. Let $\varphi : V \to V \cdot a$ be the super homomorphism defined by $\varphi(x) = xa$. Since V is super irreducible, then $\ker\varphi$ is either 0 or V. If $\ker\varphi = V$, then $V \cdot a = 0$. If $\ker\varphi = 0$, then φ is a bijection. It is also a homogeneous super homomorphism of degree 0, and because of associativity in A, for any $b \in A_0 \cup A_1$ and $x \in V$, $\varphi(bx) = (bx)a = b(xa) = b\varphi(x)$. Hence for any $a \in A_0 \cup A_1, V \cdot a \cong^{\mathfrak{s}} V \cong^{\mathfrak{s}} M$ when $V \cdot a \neq 0$. As stated in the previous paragraph, this allows us to conclude that $A^{\mathfrak{s}}(M) \cdot a \subset A^{\mathfrak{s}}(M)$, meaning $A^{\mathfrak{s}}(M)$ is a right superideal.

2. Let V be a super irreducible A-supermodule. First we will show that if $V \not\cong^{\mathfrak{s}} M$, then V is annihilated by $A^{\mathfrak{s}}(M)$.

   Let $V \not\cong^{\mathfrak{s}} M$. Since A is super semisimple, then A° is a direct sum of super irreducible subsupermodules, and so $A^{\circ} = \bigoplus W_i$. In particular, $A^{\circ} = (\bigoplus \{W_i : W_i \cong^{\mathfrak{s}} V\}) \oplus (\bigoplus \{W_i : W_i \cong^{\mathfrak{s}} M\}) \oplus (\bigoplus \{W_i : W_i \not\cong^{\mathfrak{s}} V \text{ and } W_i \not\cong^{\mathfrak{s}} M\})$. By Lemma 2.23(2), we have $A^{\mathfrak{s}}(V) = (\bigoplus \{W_i : W_i \cong^{\mathfrak{s}} V\})$ and $A^{\mathfrak{s}}(M) = (\bigoplus \{W_i : W_i \cong^{\mathfrak{s}} M\})$. Hence $A^{\mathfrak{s}}(V) \cap A^{\mathfrak{s}}(M) = 0$. Since $A^{\mathfrak{s}}(V)$ and $A^{\mathfrak{s}}(M)$ are two-sided superideals, $A^{\mathfrak{s}}(M)A^{\mathfrak{s}}(V) \subset A^{\mathfrak{s}}(V) \cap A^{\mathfrak{s}}(M) = 0$. Hence V is annihilated by $A^{\mathfrak{s}}(M)$.

   Now let $V \cong^{\mathfrak{s}} M$. We can express A° in the following way: $A^{\circ} = (\bigoplus \{W_i : W_i \cong^{\mathfrak{s}} M\}) \oplus (\bigoplus \{W_i : W_i \not\cong^{\mathfrak{s}} M\})$. From the previous paragraph, we can say that $(\bigoplus \{W_i : W_i \not\cong^{\mathfrak{s}} M\})$ annihilates M (and hence also V). As a consequence $(\bigoplus \{W_i : W_i \cong^{\mathfrak{s}} M\}) = A^{\mathfrak{s}}(M)$ does not annihilate M, and therefore $A^{\mathfrak{s}}(M)$ does not annihilate V.

3. Let $I \subsetneq A^{\mathfrak{s}}(M)$ be a two-sided superideal of A. We need to show that $I = 0$.

   Since $I \subsetneq A^{\mathfrak{s}}(M)$ and $A^{\mathfrak{s}}(M)$ is a sum of subsupermodules super isomorphic to M, there is a subsupermodule $N \subset A^{\mathfrak{s}}(M)$ such that $N \cong^{\mathfrak{s}} M$ and $N \not\subset I$. Since M is super irreducible, so is N. Since $N \cap I \subset N$ and $N \not\subset I$, $N \cap I = 0$. Since $IN \subset N \cap I$, $IN = 0$. Hence, for any homogeneous element $x \in I$, $x \cdot N = 0$. In fact, for any subsupermodule $V \subset A^{\mathfrak{s}}(M)$ that is super isomorphic to M, $x \cdot V = 0$, hence $x \cdot A^{\mathfrak{s}}(M) = 0$.

   To show that $x = 0$ for any homogeneous element $x \in I$, we first note that $A^{\circ} = (\bigoplus \{W_i : W_i \cong^{\mathfrak{s}} M\}) \oplus (\bigoplus \{W_i : W_i \not\cong^{\mathfrak{s}} M\}) = A^{\mathfrak{s}}(M) \oplus (\bigoplus \{W_i : W_i \not\cong^{\mathfrak{s}} M\})$. Then $1_A = m + n$ where $m \in A^{\mathfrak{s}}(M)$ and $n \in \bigoplus \{W_i : W_i \not\cong^{\mathfrak{s}} M\}$. Since $x \in A^{\mathfrak{s}}(M)$, by 2), x annihilates everything in $\bigoplus \{W_i : W_i \not\cong^{\mathfrak{s}} M\}$, hence $x \cdot 1_A = x(m+n) = xm + xn = xm$. But from the previous paragraph, $x \cdot A^{\mathfrak{s}}(M) = 0$, hence $x \cdot 1_A = xm = 0$, and we conclude that $x = 0$ for any homogeneous element $x \in I$. Hence $I = 0$, and $A^{\mathfrak{s}}(M)$ is a minimal two-sided superideal.

4. On the one hand, by 2.25(2), $A^{\mathfrak{s}}(V) \neq 0$ for any super irreducible A-supermodule V. On the other hand, since $A^{\circ} = \bigoplus W_i$ and each W_i is super isomorphic to one element of $\mathfrak{M}^{\mathfrak{s}}(A)$,
$$A^{\circ} = \bigoplus_{V \in \mathfrak{M}^{\mathfrak{s}}(A)} A^{\mathfrak{s}}(V)$$
   where each $A^{\mathfrak{s}}(V)$ is non-zero. Since A° is finite dimensional, there is a finite number of distinct $A^{\mathfrak{s}}(V)$ in the direct sum, hence $\mathfrak{M}^{\mathfrak{s}}(A)$ is a finite set.

■

Definition 2.27 Let A, B be two F-superalgebras. Then the super tensor product $A \hat{\otimes} B$ is $A \otimes_F B$ as an F-vector space with the following grading: $(A \hat{\otimes} B)_0 = (A_0 \otimes B_0) \oplus (A_1 \otimes B_1)$ and $(A \hat{\otimes} B)_1 = (A_0 \otimes B_1) \oplus (A_1 \otimes B_0)$. For any $a \in A$, $b \in B$, $a \hat{\otimes} b$ is identified with $a \otimes b$. Multiplication in $A \hat{\otimes} B$ is defined in the following way: for any $a_1, a_2 \in A_0 \cup A_1$, $b_1, b_2 \in B_0 \cup B_1$, $(a_1 \hat{\otimes} b_1) \cdot (a_2 \hat{\otimes} b_2) = (-1)^{d(a_2)d(b_1)}(a_1 a_2 \hat{\otimes} b_1 b_2)$.

While it is not obvious that the super tensor product is a commutative operation on F-superalgebras, let us show that for any F-superalgebras A, B, $A \hat{\otimes} B \cong B \hat{\otimes} A$.

For any homogeneous elements $a \in A_0 \cup A_1$, $b \in B_0 \cup B_1$, the required F-superalgebra isomorphism $\varphi : A \hat{\otimes} B \to B \hat{\otimes} A$ is defined by $\varphi(a \hat{\otimes} b) = (-1)^{d(a)d(b)}(b \hat{\otimes} a)$. Let us show that it is an F-superalgebra homomorphism.

For any $a, a' \in A_0 \cup A_1$, $b, b' \in B_0 \cup B_1$, let $i = d(a), j = d(a'), k = d(b)$, and $l = d(b')$. Then

$$\varphi((a \hat{\otimes} b) \cdot (a' \hat{\otimes} b')) = \varphi((-1)^{jk}(aa' \hat{\otimes} bb')) = (-1)^{jk}(-1)^{(i+j)(k+l)}(bb' \hat{\otimes} aa') = (-1)^{2jk+ik+il+jl}(bb' \hat{\otimes} aa') = (-1)^{ik+il+jl}(bb' \hat{\otimes} aa').$$

On the other hand,

$$\varphi(a \hat{\otimes} b)\varphi(a' \hat{\otimes} b') = (-1)^{ik}(b \hat{\otimes} a) \cdot (-1)^{jl}(b' \hat{\otimes} a') = (-1)^{ik+jl}(-1)^{il}(bb' \hat{\otimes} aa') = (-1)^{ik+il+jl}(bb' \hat{\otimes} aa').$$

It can be seen straight away that φ preserves grading, and so it is an F-superalgebra homomorphism. φ is also bijective, so $A \hat{\otimes} B \cong B \hat{\otimes} A$.

Next, let us show that the super tensor product is associative. In other words, let us show that for any F-superalgebras A, B, C, $(A \hat{\otimes} B) \hat{\otimes} C \cong A \hat{\otimes}(B \hat{\otimes} C)$.

For any homogeneous elements $a \in A_0 \cup A_1$, $b \in B_0 \cup B_1$, $c \in C_0 \cup C_1$, the required F-superalgebra isomorphism $\varphi : (A \hat{\otimes} B) \hat{\otimes} C \to A \hat{\otimes}(B \hat{\otimes} C)$ is defined by $\varphi((a \hat{\otimes} b) \hat{\otimes} c) = a \hat{\otimes}(b \hat{\otimes} c)$. Let us show that it is an F-superalgebra homomorphism.

For any $a, a' \in A_0 \cup A_1$, $b, b' \in B_0 \cup B_1$, $c, c' \in C_0 \cup C_1$, let i, j, k, l be $d(a'), d(b), d(b')$, and $d(c)$ respectively. Then

$$\varphi(((a \hat{\otimes} b) \hat{\otimes} c) \cdot ((a' \hat{\otimes} b') \hat{\otimes} c')) = \varphi((-1)^{(i+k)l+ij}(aa' \hat{\otimes} bb') \hat{\otimes} cc') = (-1)^{il+kl+ij} aa' \hat{\otimes}(bb' \hat{\otimes} cc').$$

On the other hand,

$$\varphi((a \hat{\otimes} b) \hat{\otimes} c)\varphi((a' \hat{\otimes} b') \hat{\otimes} c') = (a \hat{\otimes}(b \hat{\otimes} c)) \cdot (a' \hat{\otimes}(b' \hat{\otimes} c')) = (-1)^{(j+l)i+kl} aa' \hat{\otimes}(bb' \hat{\otimes} cc') = (-1)^{il+kl+ij} aa' \hat{\otimes}(bb' \hat{\otimes} cc').$$

It is an exercise to verify that φ preserves grading, and so it is an F-superalgebra homomorphism. φ is also bijective, so $(A \hat{\otimes} B) \hat{\otimes} C \cong A \hat{\otimes}(B \hat{\otimes} C)$.

Lemma 2.28 *Let A be a finite-dimensional super simple F-superalgebra. Then for some non-negative integers r, s, $A \cong M_{(r,s)}(F) \hat{\otimes} D$ for some super division algebra D whose centre contains F.*

Proof. We will prove this lemma in two steps.

- To set up the first step, we will let $I \subset A$ be a left superideal that is also a super irreducible A-supermodule. Then by the Super Schur's Lemma, $\mathrm{End}^{\mathfrak{s}}_A(I)$ is a super division algebra. Denoting $\mathrm{End}^{\mathfrak{s}}_A(I)$ as D, we can now describe our first step, which is to show that $A \cong \mathrm{End}^{\mathfrak{s}}_D(I)$ as F-superalgebras.

- To set up the second step, we first note that I_0 and I_1 are vector spaces over the ungraded division algebra D_0. If $D_1 = 0$, then we will let $r = \dim_{D_0}(I_0)$ and $s = \dim_{D_0}(I_1)$, and our second step would be to show that $\mathrm{End}^{\mathfrak{s}}_D(I) \cong M_{(r,s)}(F) \hat{\otimes} D^{\mathrm{op}}$. If $D_1 \neq 0$, then we will let $r = \dim_{D_0}(I_0)$, and $s = 0$, and our second would be to show that $\mathrm{End}^{\mathfrak{s}}_D(I) \cong M_{(r,0)}(F) \hat{\otimes} D^{\mathfrak{s}\mathrm{op}} = M_r(F) \hat{\otimes} D^{\mathfrak{s}\mathrm{op}}$.

Step 1. To show that $A \cong \mathrm{End}^{\mathfrak{s}}_D(I)$ we will consider the natural map $\iota : A \to \mathrm{End}^{\mathfrak{s}}_D(I)$, $a \mapsto \varphi_a$ where $\varphi_a(x) = ax$ for any $x \in I$. Let us quickly verify that this is indeed a map from A to $\mathrm{End}^{\mathfrak{s}}_D(I)$ that preserves grading.

Let $i \in \{0, 1\}$. Since, for any $a_i \in A_i$, $a_i \cdot I_j \subset I_{(i+j)\mathrm{mod}2}$ for any $j \in \{0, 1\}$, $\varphi_{a_i}(I_j) \subset I_{(i+j)\mathrm{mod}2}$ for any $j \in \{0, 1\}$.

Now let $i, j \in \{0,1\}$. For any $a_i \in A_i$ and $d_j \in D_j$, $d_j(a_i x) = (-1)^{ij} a_i d_j(x)$ for any $x \in I$, hence $\varphi_{a_i}(d_j(x)) = a_i d_j(x) = (-1)^{ij} d_j(a_i x) = (-1)^{ij} d_j(\varphi_{a_i}(x))$ for any $x \in I$. We can conclude that for any $i \in \{0,1\}$, $\iota(A_i) \subset (\mathrm{End}^{\mathfrak{s}}_D(I))_i$. Hence ι is a map from A to $\mathrm{End}^{\mathfrak{s}}_D(I)$ that preserves grading.

Next, let us show that ι is a homomorphism of F-superalgebras. Let us show that $\iota(ab) = \iota(a) \circ \iota(b)$. For any $x \in I$, $\varphi_{ab}(x) = (ab)x = a(bx) = a\varphi_b(x) = \varphi_a(\varphi_b(x))$, and hence we have shown that $\iota(ab) = \iota(a) \circ \iota(b)$.

It is clear that $\iota(1_A) = id$, the identity element of $\mathrm{End}^{\mathfrak{s}}_D(I)$. Also, since A is an F-algebra, ι is an F-linear transformation. We can conclude that ι is an F-superalgebra homomorphism.

Next, we show that ι is injective. Note that $\ker(\iota)$ is a two-sided superideal of A. Since A is super simple and $\iota(1_A) = id \neq 0$, $\ker(\iota) \neq A$, so $\ker(\iota) = 0$, and ι is injective.

Finally, we will show that ι is surjective. To do this, we show that $\iota(A)$ is a left superideal of $\mathrm{End}^{\mathfrak{s}}_D(I)$.

For this, we first show that $\iota(I)$ is a left superideal.

Note that for any $a \in I_0 \cup I_1$, the map $\psi_a : I \to I$ defined by $x \mapsto (-1)^{d(x)d(a)} xa$ for any $x \in I_0 \cup I_1$ is an element of $D = \mathrm{End}^{\mathfrak{s}}_A(I)$ of degree $d(a)$. Hence, for any homogeneous element $\varphi \in \mathrm{End}^{\mathfrak{s}}_D(I)$ of degree i, $\varphi(\psi_a(x)) = (-1)^{i \cdot d(a)} \psi_a(\varphi(x))$ for any $x \in I$. So for any $x, y \in I_0 \cup I_1$, $(\varphi \circ \iota(x))(y) = \varphi(\varphi_x(y)) = \varphi(xy) = \varphi((-1)^{d(x)d(y)} \psi_y(x)) = (-1)^{d(x)d(y)} \varphi(\psi_y(x)) = (-1)^{(d(x)+i)d(y)} \psi_y(\varphi(x)) = \varphi(x)y = \varphi_{\varphi(x)}(y) = (\iota \circ \varphi(x))(y)$. This tells us that for any $x \in I$, $\varphi \circ \iota(x) = \iota \circ \varphi(x)$. This applies for any homogeneous element $\varphi \in \mathrm{End}^{\mathfrak{s}}_D(I)$. We conclude that $\iota(I)$ is a left superideal of $\mathrm{End}^{\mathfrak{s}}_D(I)$, as desired.

Next, let us show that $\iota(A)$ is a left superideal of $\mathrm{End}^{\mathfrak{s}}_D(I)$. Note that since I is a left superideal of A, $I \cdot A$ is a two-sided superideal of A (with 0-component $I_0A_0 + I_1A_1$ and 1-component $I_0A_1 + I_1A_0$), and since A is super simple and IA is non-zero, $IA = A$. So $\iota(A) = \iota(IA) = \iota(I)\iota(A)$. Since $\iota(I)$ is a left superideal of $\mathrm{End}^{\mathfrak{s}}_D(I)$, so is $\iota(I)\iota(A)$, and hence $\iota(A)$ is a left superideal of $\mathrm{End}^{\mathfrak{s}}_D(I)$. Since $id \in \iota(A)$, $\iota(A) = \mathrm{End}^{\mathfrak{s}}_D(I)$, and hence ι is surjective. We have now shown that $A \cong \mathrm{End}^{\mathfrak{s}}_D(I)$.

Step 2. In this step, we show that if $D_1 = 0$, then $\mathrm{End}^{\mathfrak{s}}_D(I) \cong M_{(r,s)}(F) \hat{\otimes} D$ where $r = \dim_{D_0}(I_0)$ and $s = \dim_{D_0}(I_1)$. Since $D = D_0$ would be an ungraded division algebra when $D_1 = 0$, then, as an ungraded algebra, $\mathrm{End}^{\mathfrak{s}}_D(I) = \mathrm{End}_D(I)$, and by choosing a D-basis for I, we get $\mathrm{End}_D(I) \cong M_n(D^{\mathrm{op}})$ where $n = \dim_D(I)$. Since $\dim_D(I) = \dim_D(I_0) + \dim_D(I_1) = r + s$, then, as an ungraded algebra, $\mathrm{End}_D(I) \cong M_{r+s}(D^{\mathrm{op}})$.

Since as a D-super vector space, $I_0 \oplus I_1 \cong D^{(r,s)}$, then by choosing a D-basis for I_0 and I_1, we get that the required grading for $M_{r+s}(D^{\mathrm{op}})$ is $M_{(r,s)}(D^{\mathrm{op}}) \cong M_{(r,s)}(F) \hat{\otimes} D^{\mathrm{op}}$, and hence, as a superalgebra, $\mathrm{End}^{\mathfrak{s}}_D(I) = M_{(r,s)}(F) \hat{\otimes} D^{\mathrm{op}}$.

Now let us tackle the case where $D_1 \neq 0$. Let us first show that $(\mathrm{End}^{\mathfrak{s}}_D(I))_0 \cong M_n(D_0^{\mathrm{op}})$ where $n = \dim_{D_0}(I_0)$.

Let $v \in D_1$ be a non-zero element. Since $I_1 = v(I_0)$, then every element of $(\mathrm{End}^{\mathfrak{s}}_D(I))_0$ is determined by an element of $\mathrm{End}_{D_0}(I_0)$, and since, after choosing a D_0-basis for I_0, $\mathrm{End}_{D_0}(I_0) \cong M_n(D_0^{\mathrm{op}})$ where $n = \dim_{D_0}(I_0)$, we get that $(\mathrm{End}^{\mathfrak{s}}_D(I))_0 \cong M_n(D_0^{\mathrm{op}})$.

Now, we figure out $(\mathrm{End}^{\mathfrak{s}}_D(I))_1$. Let $\varphi \in (\mathrm{End}^{\mathfrak{s}}_D(I))_1$. Now let $a_0 \in I_0$ and $a_1 \in I_1$, and let $b_1 = va_0$ and $b_0 = va_1$. Note that $\varphi(a_0) = \varphi(v^{-1}b_1) = -v^{-1}\varphi(b_1) \in I_1$ and $\varphi(a_1) = \varphi(v^{-1}b_0) = -v^{-1}\varphi(b_0) \in I_0$. Note that for any $d_0 \in D_0$ and $x \in I$, $v^{-1}\varphi(d_0x) = v^{-1}d_0\varphi(x) = (d_0 *_{D^{\mathrm{op}}} (v^{-1}))\varphi(x)$. That means $v^{-1}\varphi$ is an element of $(\mathrm{End}^{\mathfrak{s}}_D(I))_0$, and hence, for any $\varphi \in (\mathrm{End}^{\mathfrak{s}}_D(I))_1$, $\varphi = v\psi$ for some $\psi \in (\mathrm{End}^{\mathfrak{s}}_D(I))_0 \cong M_n(D_0^{\mathrm{op}})$. Hence $(\mathrm{End}^{\mathfrak{s}}_D(I))_1 = v \cdot (\mathrm{End}^{\mathfrak{s}}_D(I))_0 \cong v \cdot M_n(D_0^{\mathrm{op}})$. Overall, we get that $\mathrm{End}^{\mathfrak{s}}_D(I) \cong M_n(F) \hat{\otimes} D^{\mathfrak{s}\mathrm{op}}$. In this superalgebra isomorphism, $(\mathrm{End}^{\mathfrak{s}}_D(I))_0 \cong M_n(F) \hat{\otimes} (D^{\mathfrak{s}\mathrm{op}})_0 \cong M_n(F) \hat{\otimes} (D_0^{\mathrm{op}})$ and $(\mathrm{End}^{\mathfrak{s}}_D(I))_1 \cong M_n(F) \hat{\otimes} (D^{\mathfrak{s}\mathrm{op}})_1$.

■

Lemma 2.29 *Let r, s be non-negative integers and let $D = D_0 \oplus D_1$ be a super division algebra over F such that $F \subset Z(D)$ and $D_1 \neq 0$. Then $M_{(r,s)}(F) \hat{\otimes} D \cong M_{r+s}(F) \hat{\otimes} D$.*

Proof. Let v be a non-zero element of D_1. For any $i, j \leq r+s$, we define $E_{i,j}$ to be the matrix in $M_{r+s}(F)$ with a 1 in the $(i,j)^{th}$ entry and 0 everywhere else. We define a linear map $\varphi : M_{(r,s)}(F) \hat{\otimes} D \to M_{r+s}(F) \hat{\otimes} D$ by:

- $\varphi(E_{i,j} \hat{\otimes} x) = E_{i,j} \hat{\otimes} x$ if $i, j \leq r$, and $x \in D_0 \cup D_1$,
- $\varphi(E_{i,j} \hat{\otimes} x) = (-1)^{d(x)} E_{i,j} \hat{\otimes} xv$ if $i \leq r$, $j > r$, and $x \in D_0 \cup D_1$,
- $\varphi(E_{i,j} \hat{\otimes} x) = E_{i,j} \hat{\otimes} v^{-1}x$ if $i > r$, $j \leq r$ and $x \in D_0 \cup D_1$,
- $\varphi(E_{i,j} \hat{\otimes} x) = (-1)^{d(x)} E_{i,j} \hat{\otimes} v^{-1}xv$ if $i, j > r$, and $x \in D_0 \cup D_1$.

One can check that this is a superalgebra homomorphism that preserves grading, and is bijective. ■

Lemma 2.28, in combination with the Super Wedderburn's Theorem, allows us to say that any super semisimple F-superalgebra A is a direct sum of super simple superalgebras of the form $M_{(r,s)}(F) \hat{\otimes} D$ where D is a super division algebra over F.

Corollary 2.30 *Suppose A is a super semisimple F-superalgebra and that char$(F) \neq 2$. Then $Z^{\mathfrak{s}}(A) = Z(A) \cap A_0$.*

Corollary 2.31 *Let $A = M_{(r,s)}(F) \hat{\otimes} D$ be a super simple F-superalgebra, where D is a super division algebra whose centre contains F. Then for any super irreducible A-supermodule V, $\mathrm{End}^{\mathfrak{s}}_A(V) \cong D^{\mathfrak{s}op}$.*

$\star$ Throughout the rest of this section, we will only cover the case where F is a field whose characteristic is not equal to 2.

Let A be a super simple F-superalgebra. Then $A \cong M_{(r,s)}(F) \hat{\otimes} D$ for some super division algebra D whose centre contains F. If $D_1 = 0$, then $Z^{\mathfrak{s}}(A) \cong Z(D) = Z^{\mathfrak{s}}(D)$, which is a field. If $D_1 \neq 0$, then $Z^{\mathfrak{s}}(A) \cong Z(D_0) \cap Z_{D_0}(D_1) = Z^{\mathfrak{s}}(D)$, and one can check that this is a field.

We now give the following important definition:

Definition 2.32 An F-superalgebra is said to be ***super central*** over F if $Z^{\mathfrak{s}}(A) = F$.

Now let $A = \bigoplus_i A^i$ be a super semisimple F-superalgebra such that each A^i is a super simple F-superalgebra. Suppose $A^i = M_{(r_i,s_i)} \hat{\otimes} D^i$. Then $Z^{\mathfrak{s}}(A) \cong \bigoplus_i Z^{\mathfrak{s}}(A^i) = \bigoplus_i Z^{\mathfrak{s}}(D^i)$. So the super centre of a super semisimple F-superalgebra is a direct product of fields. If a super semisimple superalgebra A is super central, then A is super simple.

Definition 2.33 An F-superalgebra is said to be ***super central simple*** if it is both super central over F, and super simple.

We now provide the superalgebra analogue of Theorem 1.25.

Theorem 2.34 *If A is a super central simple F-superalgebra and B is a super simple F-superalgebra, then $A \hat{\otimes} B$ is super simple.*

Proof. Let I be a non-zero two-sided superideal of $A \hat{\otimes} B$. We will show that $1_A \hat{\otimes} 1_B \in I$, which would imply that $I = A \hat{\otimes} B$.

First, let us show that if I contains a non-zero homogeneous element of the form $a \hat{\otimes} b$ where $a \in A_0 \cup A_1$ and $b \in B_0 \cup B_1$, then $1_A \hat{\otimes} 1_B \in I$. Since A is super simple, A has no non-trivial proper two-sided superideals, which means the two-sided superideal generated by a is the whole of A. This means 1_A is in the superideal generated by a. This means we can express 1_A in the form

$$1_A = \sum_{i=1}^{n} a_i a a_i' + \sum_{j=1}^{m} \tilde{a}_j a \tilde{a}_j'$$

for some $a_i \in A_0, a_i' \in A_{d(a)}, \tilde{a}_j \in A_1$ and $\tilde{a}_j \in A_{(1-d(a))}$. Thus

$$\sum_{i=1}^{n}(a_i \hat{\otimes} 1_B)(a \hat{\otimes} b)\big((-1)^{d(a)d(b)}(a_i' \hat{\otimes} 1_B)\big) + \sum_{i=1}^{m}(\tilde{a}_i \hat{\otimes} 1_B)(a \hat{\otimes} b)\big((-1)^{\left(1-d(a)\right)d(b)}(\tilde{a}_i' \hat{\otimes} 1_B)\big)$$

$$= \left(\sum_{i=1}^{n} a_i a a_i'\right) \hat{\otimes} b + \left(\sum_{j=1}^{m} \tilde{a}_j a \tilde{a}_j'\right) \hat{\otimes} b = \left(\sum_{i=1}^{n} a_i a a_i' + \sum_{j=1}^{m} \tilde{a}_j a \tilde{a}_j'\right) \hat{\otimes} b = 1_A \hat{\otimes} b$$

Similarly, since B is super simple, we can express 1_B in the form

$$1_B = \sum_{i=1}^{n'} b_i b b_i' + \sum_{j=1}^{m'} \tilde{b}_j b \tilde{b}_j'$$

for some $b_i \in B_0, b_i' \in B_{d(b)}, \tilde{b}_j \in B_1$ and $\tilde{b}_j \in B_{(1-d(b))}$. Thus

$$\sum_{i=1}^{n'}(1_A \hat{\otimes} b_i)(1_A \hat{\otimes} b)(1_A \hat{\otimes} b_i') + \sum_{i=1}^{m'}(1_A \hat{\otimes} \tilde{b}_i)(1_A \hat{\otimes} b)(1_A \hat{\otimes} \tilde{b}_i') = 1_A \hat{\otimes} \left(\sum_{i=1}^{n'} b_i b b_i'\right)$$

$$+1_A \hat{\otimes} \left(\sum_{j=1}^{m'} \tilde{b}_j b \tilde{b}_j'\right) = 1_A \hat{\otimes} \left(\sum_{i=1}^{n'} b_i b b_i' + \sum_{j=1}^{m'} \tilde{b}_j b \tilde{b}_j'\right) = 1_A \hat{\otimes} 1_B = 1_{A \hat{\otimes} B} \in I_0.$$

Therefore $I = A \hat{\otimes} B$.

Let us now show that I is guaranteed to have an element of the form $a \hat{\otimes} b$ where $a \in A_0 \cup A_1$ and $b \in B_0 \cup B_1$. Once we show this, then we can say that $1_A \hat{\otimes} 1_B \in I$, like before. Suppose, for the sake of contradiction, that $A \hat{\otimes} B$ does not have a homogeneous element of the form $a \hat{\otimes} b$ where a and b are homogeneous elements of A and B respectively. Suppose x is a homogeneous element that can be expressed as

$$x = \sum_{r=1}^{k} a_r \hat{\otimes} b_r$$

where $a_r \in A_0 \cup A_1, b_r \in B_0 \cup B_1, d(b_r) = (d(x) - d(a_r))\text{mod}2, k > 1$ and k is minimal over all homogeneous elements of I. Note that the b_r's would therefore be linearly independent over F. By our earlier argument, the two-sided ideal of A generated by a_k is the whole of A, meaning 1_A can be expressed in the following way:

$$1_A = \sum_{i=1}^{n} \alpha_i a_k \alpha'_i + \sum_{j=1}^{m} \tilde{\alpha}_j a_k \tilde{\alpha}'_j$$

for some $\alpha_i \in A_0, \alpha'_i \in A_{d(a_k)}, \tilde{\alpha}_j \in A_1$ and $\tilde{\alpha}_j \in A_{(1-d(a_k))}$. Thus

$$\sum_{i=1}^{n} (\alpha_i \hat{\otimes} 1_B) x \left((-1)^{d(a_k)d(b_k)} (\alpha'_i \hat{\otimes} 1_B)\right) + \sum_{j=1}^{m} (\tilde{\alpha}_j \hat{\otimes} 1_B) x \left((-1)^{\left((1-d(a_k))d(b_k)\right)} (\tilde{\alpha}'_j \hat{\otimes} 1_B)\right)$$

$$= \left(\sum_{r=1}^{k-1} c_r \hat{\otimes} b_r\right) + 1_A \hat{\otimes} b_k$$

where

$$c_r = \sum_{i=1}^{n} (-1)^{d(a_k)d(b_k)} \alpha_i a_r \alpha'_i + \sum_{j=1}^{m} (-1)^{\left((1-d(a_k))d(b_k)\right)} \tilde{\alpha}_i a_r \tilde{\alpha}_i{}'.$$

Denote $\left(\sum_{r=1}^{k-1} c_r \hat{\otimes} b_r\right) + 1_A \hat{\otimes} b_k$ as w. Note that, for each $r < k$, $c_r \in A_0 \cup A_1$ and $d(c_r \hat{\otimes} b_r) = d(b_k)$. So w is a homogeneous element of $A \hat{\otimes} B$ with k terms in its expression. Note that none the c_r's are 0, and the c_r's are linearly independent, otherwise it would contradict the minimality of k. Note that $c_{k-1} \notin F$, since if $c_{k-1} \in F$, $d(c_{k-1} \hat{\otimes} b_{k-1}) = d(b_{k-1}) = d(b_k)$. This means the last two terms can be expressed as $(c_{k-1} \hat{\otimes} b_{k-1}) + (1_A \hat{\otimes} b_k) = (c_{k-1} + 1_A) \hat{\otimes} (b_{k-1} + b_k)$, and we would get an expression of a homogeneous element of I with $k-1$ terms that satisfies the same conditions as x, which would contradict the minimality of k. Hence $c_{k-1} \notin Z^{\mathfrak{s}}(A)$ since $Z^{\mathfrak{s}}(A) = F$. Hence there exists a homogeneous element $c \in A_0 \cup A_1$ such that $cc_{k-1} \neq (-1)^{d(c_{k-1})d(c)} c_{k-1} c$. Therefore, the $(k-1)$th term in

$$(c \hat{\otimes} 1_B) w - (-1)^{\left(d(c_{k-1}) + d(b_{k-1})\right) d(c)} w (c \hat{\otimes} 1_B)$$

would be $\left(cc_{k-1} - (-1)^{d(c_{k-1})d(c)} c_{k-1} c\right) \hat{\otimes} b_{k-1}$, which is not equal to 0. The kth term would be $\left(c - (-1)^{\left(d(c_{k-1}) + d(b_{k-1}) + d(b_k)\right) d(c)} c\right) \hat{\otimes} b_k = (c - (-1)^{2d(b_k)d(c)} c) \hat{\otimes} b_k = (c - c) \hat{\otimes} b_k = 0$. Since the b_r's are linearly independent,

$$(c \hat{\otimes} 1_B) w - (-1)^{\left(d(c_{k-1}) + d(b_{k-1})\right) d(c)} w (c \hat{\otimes} 1_B)$$

is a non-zero homogeneous element of I that satisfies the same conditions as x, but has at most $k-1$ non-zero terms, which contradicts the minimality of k.

We can then say that there exists a homogeneous element of I in the form $a \hat{\otimes} b$ where $a \in A_0 \cup A_1$ and $b \in B_0 \cup B_1$, and from earlier, that means I contains $1_A \hat{\otimes} 1_B$, meaning $I = A \hat{\otimes} B$. We conclude that $A \hat{\otimes} B$ is super simple. ■

Theorem 2.35 *Let A and B be two F-superalgebras, and let A', B' be unital subsuperalgebras of A and B respectively. Then $Z^{\mathfrak{s}}_{A \hat{\otimes} B}(A' \hat{\otimes} B') = Z^{\mathfrak{s}}_A(A') \hat{\otimes} Z^{\mathfrak{s}}_B(B')$, and their 0- and 1-components coincide.*

Proof. It is an exercise to show that

$$\left(Z^{\mathfrak{s}}_A(A')\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')\right)_0 = Z^{\mathfrak{s}}_A(A')_0\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')_0 + Z^{\mathfrak{s}}_A(A')_1\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')_1 \subset \left(Z^{\mathfrak{s}}_{A\,\hat{\otimes}\,B}(A'\,\hat{\otimes}\,B')\right)_0$$

and

$$\left(Z^{\mathfrak{s}}_A(A')\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')\right)_1 = Z^{\mathfrak{s}}_A(A')_0\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')_1 + Z^{\mathfrak{s}}_A(A')_1\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')_0 \subset \left(Z^{\mathfrak{s}}_{A\,\hat{\otimes}\,B}(A'\,\hat{\otimes}\,B')\right)_1.$$

One can show that the reverse inclusion holds for both. We will outline the argument showing that $\left(Z^{\mathfrak{s}}_{A\,\hat{\otimes}\,B}(A'\,\hat{\otimes}\,B')\right)_0 \subset \left(Z^{\mathfrak{s}}_A(A')\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')\right)_0$, and a similar argument can be made to show that $\left(Z^{\mathfrak{s}}_{A\,\hat{\otimes}\,B}(A'\,\hat{\otimes}\,B')\right)_1 \subset \left(Z^{\mathfrak{s}}_A(A')\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')\right)_1$.

First, let $w \in \left(Z^{\mathfrak{s}}_{A\,\hat{\otimes}\,B}(A'\,\hat{\otimes}\,B')\right)_0$. Then $w \in A_0\,\hat{\otimes}\,B_0 + A_1\,\hat{\otimes}\,B_1$. We will show that $w \in \left(Z^{\mathfrak{s}}_A(A')\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')\right)_0$. Choose an F-basis for B_0, let us say $\{b_1, \dots b_n\}$, and choose an F-basis for B_1, let us say $\{\tilde{b}_1, \dots, \tilde{b}_{n'}\}$. Then

$$w = \sum_{i=1}^{n} x_i\,\hat{\otimes}\,b_i + \sum_{j=1}^{n'} \tilde{x}_j\,\hat{\otimes}\,\tilde{b}_j$$

for some $x_i \in A_0$ and $\tilde{x}_j \in A_1$. Since $w \in \left(Z^{\mathfrak{s}}_{A\,\hat{\otimes}\,B}(A'\,\hat{\otimes}\,B')\right)_0$, for any $a \in A'_0$, $w(a\,\hat{\otimes}\,1_B) = (a\,\hat{\otimes}\,1_B)w$. Hence

$$w(a\,\hat{\otimes}\,1_B) = \sum_{i=1}^{n} x_i a\,\hat{\otimes}\,b_i + \sum_{j=1}^{n'} \tilde{x}_j a\,\hat{\otimes}\,\tilde{b}_j = (a\,\hat{\otimes}\,1_B)w = \sum_{i=1}^{n} a x_i\,\hat{\otimes}\,b_i + \sum_{j=1}^{n'} a\tilde{x}_j\,\hat{\otimes}\,\tilde{b}_j.$$

Hence, for all i, $x_i a = a x_i$ and for all j, $x_j a = a x_j$. This applies for any $a \in A'_0$. Additionally, for any $\tilde{a} \in A'_1$, $w(\tilde{a}\,\hat{\otimes}\,1_B) = (\tilde{a}\,\hat{\otimes}\,1_B)w$. Hence

$$w(\tilde{a}\,\hat{\otimes}\,1_B) = \sum_{i=1}^{n} x_i\tilde{a}\,\hat{\otimes}\,b_i - \sum_{j=1}^{n'} \tilde{x}_j\tilde{a}\,\hat{\otimes}\,\tilde{b}_j = (\tilde{a}\,\hat{\otimes}\,1_B)w = \sum_{i=1}^{n} \tilde{a}x_i\,\hat{\otimes}\,b_i + \sum_{j=1}^{n'} \tilde{a}\tilde{x}_j\,\hat{\otimes}\,\tilde{b}_j.$$

Hence, for all i, $x_i\tilde{a} = \tilde{a}x_i$ and for all j, $x_j\tilde{a} = -\tilde{a}x_j$. This applies for any $\tilde{a} \in A'_1$. We have thus shown that for all i, $x_i \in Z^{\mathfrak{s}}_A(A')_0$ and for all j, $\tilde{x}_j \in Z^{\mathfrak{s}}_A(A')_1$, and hence $w \in Z^{\mathfrak{s}}_A(A')_0\,\hat{\otimes}\,B_0 + Z^{\mathfrak{s}}_A(A')_1\,\hat{\otimes}\,B_1$. Now we choose an F-basis for $Z^{\mathfrak{s}}_A(A')_0$, let us say $\{c_1, \dots c_m\}$, and choose an F-basis for $Z^{\mathfrak{s}}_A(A')_1$, let us say $\{\tilde{c}_1, \dots, \tilde{c}_{m'}\}$. Then

$$w = \sum_{i=1}^{m} c_i\,\hat{\otimes}\,y_i + \sum_{j=1}^{m'} \tilde{c}_j\,\hat{\otimes}\,\tilde{y}_j.$$

Now for any $b \in B_0$, $w(1_A\,\hat{\otimes}\,b) = (1_A\,\hat{\otimes}\,b)w$ and for any $\tilde{b} \in B_1$, $w(1_A\,\hat{\otimes}\,\tilde{b}) = (1_A\,\hat{\otimes}\,\tilde{b})w$. Using a similar argument as before, one can verify that these equations imply that for any i, $y_i \in Z^{\mathfrak{s}}_B(B')_0$ and for any j, $\tilde{y}_j \in Z^{\mathfrak{s}}_B(B')_1$. Hence $w \in Z^{\mathfrak{s}}_A(A')_0\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')_0 + Z^{\mathfrak{s}}_A(A')_1\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')_1$, and we get $\left(Z^{\mathfrak{s}}_{A\,\hat{\otimes}\,B}(A'\,\hat{\otimes}\,B')\right)_0 \subset \left(Z^{\mathfrak{s}}_A(A')\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')\right)_0$.

Similarly, one can show that $\left(Z^{\mathfrak{s}}_{A\,\hat{\otimes}\,B}(A'\,\hat{\otimes}\,B')\right)_1 \subset \left(Z^{\mathfrak{s}}_A(A')\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')\right)_1$.
In conclusion, we get $\left(Z^{\mathfrak{s}}_{A\,\hat{\otimes}\,B}(A'\,\hat{\otimes}\,B')\right)_0 = \left(Z^{\mathfrak{s}}_A(A')\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')\right)_0$ and $\left(Z^{\mathfrak{s}}_{A\,\hat{\otimes}\,B}(A'\,\hat{\otimes}\,B')\right)_1 = \left(Z^{\mathfrak{s}}_A(A')\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')\right)_1$, meaning $Z^{\mathfrak{s}}_{A\,\hat{\otimes}\,B}(A'\,\hat{\otimes}\,B') = Z^{\mathfrak{s}}_A(A')\,\hat{\otimes}\,Z^{\mathfrak{s}}_B(B')$, and their 0- and 1-components coincide. ∎

The following corollaries are now obvious:

Corollary 2.36 *If A and B are two F-superalgebras, then $Z^{\mathfrak{s}}(A \hat{\otimes} B) = Z^{\mathfrak{s}}(A) \hat{\otimes} Z^{\mathfrak{s}}(B)$.*

Corollary 2.37 *If A and B are two super central simple F-superalgebras, then $A \hat{\otimes} B$ is super central simple.*

Proof. Since A is super central simple, and B is super simple, then by Theorem 2.34, $A \hat{\otimes} B$ is super simple. Since A and B are super central, by Corollary 2.36, $Z^{\mathfrak{s}}(A \hat{\otimes} B) = Z^{\mathfrak{s}}(A) \hat{\otimes} Z^{\mathfrak{s}}(B) = F \cdot 1_A \hat{\otimes} F \cdot 1_B \cong F$, which means $A \hat{\otimes} B$ is also super central. Hence $A \hat{\otimes} B$ is super central simple. ∎

Lemma 2.38 *Let A be a super central simple F-superalgebra. Then $A^{\mathfrak{s}\mathrm{op}}$ is also super central simple.*

We now come to a key result in the classification of super central simple superalgebras.

Lemma 2.39 *Let D be a super central super division algebra over F.*

1. *If $D_1 = 0$, then $D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}} = D \otimes D^{\mathrm{op}} = M_n(F)$ where $n = \dim_F(D)$.*
2. *If $D_1 \neq 0$, then $D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}} \cong M_{(n,n)}(F)$ where $n = \dim_F(D_0)$.*

Proof. The first statement is proven by invoking Corollary 1.32. Now let us prove the second statement. If $D_1 \neq 0$, let us show that $D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}} \cong M_{(n,n)}(F)$, where $n = \dim_F(D_0)$. To do this, we will show that $D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}} \cong \mathrm{End}^{\mathfrak{s}}_F(D)$. First, let us denote the linear map $D \to D$, $x \mapsto axb$ as $g_{a,b}$. Then for any homogeneous elements $a \in D_0 \cup D_1$, $b \in D_0^{\mathfrak{s}\mathrm{op}} \cup D_1^{\mathfrak{s}\mathrm{op}}$, we can construct a linear map $\varphi : D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}} \to \mathrm{End}^{\mathfrak{s}}_F(D)$ where, for any $x \in D_0 \cup D_1$, $\varphi(a \hat{\otimes} b)(x) = (-1)^{d(x)d(b)} axb$.

We claim that φ is a superalgebra homomorphism. It is clear that φ preserves grading. For any $a, a', x \in D_0 \cup D_1$, $b, b' \in D_0^{\mathfrak{s}\mathrm{op}} \cup D_1^{\mathfrak{s}\mathrm{op}}$, let i, j, k, l be $d(b), d(a'), d(b')$ and $d(x)$ respectively. Then $\varphi((a \hat{\otimes} b)(a' \hat{\otimes} b'))(x) = \varphi((-1)^{ij}(aa' \hat{\otimes} b *_{D^{\mathfrak{s}\mathrm{op}}} b'))(x) = \varphi((-1)^{ij+ik}(aa' \hat{\otimes} b'b))(x) = (-1)^{ij+ik}(-1)^{(i+k)l} aa'xb'b = (-1)^{i(j+k+l)+kl} aa'xb'b$.

On the other hand,
$(\varphi(a \hat{\otimes} b) \circ \varphi(a' \hat{\otimes} b'))(x) = \varphi(a \hat{\otimes} b)((-1)^{kl} a'xb') = (-1)^{i(j+k+l)}(-1)^{kl} aa'xb'b$. Hence $\varphi((a \hat{\otimes} b)(a' \hat{\otimes} b'))(x) = (\varphi(a \hat{\otimes} b) \circ \varphi(a' \hat{\otimes} b'))(x)$, and $\varphi : D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}} \to \mathrm{End}^{\mathfrak{s}}_F(D)$ is a superalgebra homomorphism.

Since φ is a superalgebra homomorphism, $\ker\varphi$ is a two-sided superideal. Since D is super central simple, so is $D^{\mathfrak{s}\mathrm{op}}$ by Lemma 2.38. By Corollary 2.37, $D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}}$ is super central simple, hence $\ker\varphi$ is either 0 or all of $D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}}$. However, $\ker\varphi \neq D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}}$ since $\varphi(1_D \hat{\otimes} 1_{D^{\mathfrak{s}\mathrm{op}}}) = \mathrm{id}_D$. Hence $\ker\varphi = 0$, which means $\varphi : D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}} \to \mathrm{End}^{\mathfrak{s}}_F(D)$ is injective.

Since $\dim_F(\mathrm{im}\varphi) = \dim_F(D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}}) = \dim_F(D)^2 = \dim_F(\mathrm{End}^{\mathfrak{s}}_F(D))$, we get $\mathrm{im}(\varphi) = \mathrm{End}^{\mathfrak{s}}_F(D)$, which means φ is surjective. We conclude that φ is a superalgebra isomorphism, and $D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}} \cong \mathrm{End}^{\mathfrak{s}}_F(D) \cong M_{(n,n)}(F)$ where $n = \dim_F(D_0)$, as desired. ∎

For super opposite superalgebras, we have:

Lemma 2.40 *Let $A \cong M_{(r,s)}(F) \hat{\otimes} D$ be a super central simple F-superalgebra, where D is a super central super division algebra. Then $A^{\mathfrak{s}\mathrm{op}} \cong M_{(r,s)}(F) \hat{\otimes} D^{\mathfrak{s}\mathrm{op}}$.*

Proof. If $D_1 = 0$, then the required superalgebra isomorphism $\varphi : A^{\mathfrak{s}\mathrm{op}} \to M_{(r,s)}(F) \hat{\otimes} D^{\mathfrak{s}\mathrm{op}}$ is the linear map defined by:

- $\varphi(E_{i,j} \hat{\otimes} x) = E_{j,i} \hat{\otimes} x$ whenever $i > r$ or $i, j \leq r$ for any $x \in D_0 \cup D_1$,

- $\varphi(E_{i,j} \hat{\otimes} x) = -E_{j,i} \hat{\otimes} x$ whenever $i \leq r$ and $j > r$ for any $x \in D_0 \cup D_1$.

If $D_1 \neq 0$, then we know, by Lemma 2.29, $M_{(r,s)}(F) \hat{\otimes} D \cong M_{r+s}(F) \hat{\otimes} D$ and $M_{(r,s)}(F) \hat{\otimes} D^{\mathfrak{s}\mathrm{op}} \cong M_{r+s}(F) \hat{\otimes} D^{\mathfrak{s}\mathrm{op}}$, and the required superalgebra isomorphism $\varphi : (M_{r+s}(F) \hat{\otimes} D)^{\mathfrak{s}\mathrm{op}} \to M_{r+s}(F) \hat{\otimes} D^{\mathfrak{s}\mathrm{op}}$ is the linear map defined by $\varphi(E_{i,j} \hat{\otimes} x) = E_{j,i} \hat{\otimes} x$ for any $x \in A_0 \cup A_1$. ■

The following lemma comes from [4, Proposition 2.10 in pg. 102].

Lemma 2.41 *Let m, n, r, s be non-negative integers such that $m + n > 0$ and $r + s > 0$. Then $M_{(m,n)}(F) \hat{\otimes} M_{(r,s)}(F) \cong M_{(mr+ns,ms+nr)}(F)$.*

Proof. Let U and V be F-vector spaces of dimension $m + n$ and $r + s$ respectively, and let $W = U \otimes_F V$. We know that, once we choose a graded F-basis for $U^{(m,n)}$ and $V^{(r,s)}$, $\mathrm{End}_F^{\mathfrak{s}}(U^{(m,n)}) \cong M_{(m,n)}(F)$ and $\mathrm{End}_F^{\mathfrak{s}}(V^{(r,s)}) \cong M_{(r,s)}(F)$. So proving the lemma is equivalent to proving that $\mathrm{End}_F^{\mathfrak{s}}(U^{(m,n)}) \hat{\otimes} \mathrm{End}_F^{\mathfrak{s}}(V^{(r,s)}) \cong \mathrm{End}_F^{\mathfrak{s}}(W^{(mr+ns,ms+nr)})$.

Note that the F-super vector space $W^{(mr+ns,ms+nr)}$ can be constructed by grading the F-module tensor product $U^{(m,n)} \otimes V^{(r,s)}$ in the following way:

$$\left(U^{(m,n)} \otimes V^{(r,s)}\right)_0 = \left(U_0^{(m,n)} \otimes V_0^{(r,s)}\right) \oplus \left(U_1^{(m,n)} \otimes V_1^{(r,s)}\right) \cong (F^m \otimes F^r) \oplus (F^n \otimes F^s) \cong F^{mr} \oplus F^{ns} \cong F^{mr+ns}, \text{ and } \left(U^{(m,n)} \otimes V^{(r,s)}\right)_1 = \left(U_0^{(m,n)} \otimes V_1^{(r,s)}\right) \oplus \left(U_1^{(m,n)} \otimes V_0^{(r,s)}\right) \cong (F^m \otimes F^s) \oplus (F^n \otimes F^r) \cong F^{ms} \oplus F^{nr} \cong F^{ms+nr}.$$

We now construct $\varphi : \mathrm{End}_F^{\mathfrak{s}}(U^{(m,n)}) \hat{\otimes} \mathrm{End}_F^{\mathfrak{s}}(V^{(r,s)}) \to \mathrm{End}_F^{\mathfrak{s}}(W^{(mr+ns,ms+nr)})$. Let f, g, x and y be arbitrary homogeneous elements of $\mathrm{End}_F^{\mathfrak{s}}(U^{(m,n)})$, $\mathrm{End}_F^{\mathfrak{s}}(V^{(r,s)})$, $U^{(m,n)}$ and $V^{(r,s)}$ respectively. We define $\varphi(f \hat{\otimes} g)$ as follows:

$$\varphi(f \hat{\otimes} g)(x \otimes y) = (-1)^{d(x)d(g)} f(x) \otimes g(y).$$

One can check that φ is well defined, and is an F-superalgebra isomorphism. ■

Just like the algebra tensor product induces a group operation on equivalence classes of central simple algebras, the super tensor product induces a group operation on equivalence classes of super central simple superalgebras, as we see next.

2.2 The Brauer-Wall Group of F

The Brauer-Wall group, named after R. Brauer and C.T.C. Wall, is the super version of the Brauer group. It gives a group structure on the isomorphism classes of super central simple superalgebras.

Let A and B be two super central simple F-superalgebras. From the Super Wedderburn's Theorem, as superalgebras, $A \cong M_{(r,s)}(F) \hat{\otimes} D_A$ and $B \cong M_{(r',s')}(F) \hat{\otimes} D_B$ for some super central super division algebras D_A and D_B. We define the equivalence relation on super central simple F-superalgebras as $A \sim B$ if $D_A \cong D_B$ as superalgebras. One can check that this satisfies the conditions of an equivalence relation on the set of super central simple F-superalgebras. Let F be a field. We denote the set of equivalence classes of super central simple F-superalgebras as $BW(F)$. We use $[A]$ to denote the equivalence class containing A. Now define a binary operation on $BW(F)$ in the following way:

For any two equivalence classes $[A], [B]$, we set $[A] \cdot [B] := [A \hat{\otimes} B]$. Let us verify that this operation on equivalence classes is well defined. Let $A = M_{(m,n)}(F) \hat{\otimes} D_A$ and

$B = M_{(r,s)}(F) \hat{\otimes} D_B$ for some super central super division algebras D_A, D_B. For any $A' \sim A$, $A' \cong M_{(m',n')}(F) \hat{\otimes} D_A$ and for any $B' \sim B$, $B' = M_{(r',s')}(F) \hat{\otimes} D_B$. Our task is to show that $[A' \hat{\otimes} B'] = [A \hat{\otimes} B]$. Suppose $D_A \hat{\otimes} D_B = M_{(a,b)}(F) \hat{\otimes} D$ where D is a super central super division algebra. Then $A \hat{\otimes} B \cong M_{(mr+ns,ms+nr)}(F) \hat{\otimes}(D_A \hat{\otimes} D_B) \cong M_{(mr+ns,ms+nr)}(F) \hat{\otimes}(M_{(a,b)}(F) \hat{\otimes} D) \cong M_{(c,d)}(F) \hat{\otimes} D$, where $c = a(mr+ns)+b(ms+nr)$ and $d = a(ms+nr)+b(mr+ns)$.

On the other hand, $A' \hat{\otimes} B' \cong M_{(m'r'+n's',m's'+n'r')}(F) \hat{\otimes}(D_A \hat{\otimes} D_B) \cong M_{(c',d')}(F) \hat{\otimes} D$ where $c' = a(m'r'+n's')+b(m's'+n'r')$ and $d' = a(m's'+n'r')+b(m'r'+n's')$. This has the same super central super division algebra part as $A \hat{\otimes} B$, and so $A \hat{\otimes} B \sim A' \hat{\otimes} B'$, hence $[A' \hat{\otimes} B'] = [A \hat{\otimes} B]$ and the binary operation on the set of equivalence classes is well defined.

Since $\hat{\otimes}$ is an associative operation, then the operation $\cdot$ we have defined is associative. Note that for any super central simple F-superalgebra A, $[A] \cdot [F] = [A \hat{\otimes} F] = [A]$. So the equivalence class $[F]$ acts as the identity element under the operation. Also for any super central simple F-superalgebra $A = M_{(r,s)}(F) \hat{\otimes} D$, $A^{\mathfrak{s}\mathrm{op}} \cong M_{(r,s)}(F) \hat{\otimes} D^{\mathfrak{s}\mathrm{op}}$ by Lemma 2.40, and $[A] \cdot [A^{\mathfrak{s}\mathrm{op}}] = [D] \cdot [D^{\mathfrak{s}\mathrm{op}}] = [D^{\mathfrak{s}\mathrm{op}}] \cdot [D] = [D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}}] = [F]$, since $D \hat{\otimes} D^{\mathfrak{s}\mathrm{op}} \cong M_{n,n}(F)$ where $n = \dim_F(D_0)$ by Lemma 2.39. So for any equivalence class $[D]$ (where D is a super central super division algebra), $[D^{\mathfrak{s}\mathrm{op}}]$ is its inverse. We can now define the Brauer-Wall group of F.

Definition 2.42 Given a field F, the ***Brauer-Wall group of*** F is the set $BW(F)$ of equivalence classes of super central simple F-superalgebras, with group operation $[A] \cdot [B] = [A \hat{\otimes} B]$ for any $[A], [B] \in BW(F)$.

Note that $BW(F)$ is abelian, as $\hat{\otimes}$ is commutative.

It must be emphasised that the Brauer-Wall group depends on the field F. Note that given a field F, the order of $Br(F)$ is equal to the number of distinct super central super division algebras over F.

To study the Brauer-Wall group of a given field F, we need to understand some further properties of super central simple F-superalgebras. We will now go through some of the main results from [5].

$\star$ In this subsection, A will refer to a super central simple F-superalgebra. We maintain the assumption that char$F \neq 2$ throughout this subsection.

Lemma 2.43 *The following holds:*

1. *If $A_1 \neq 0$, then $A_1 \cdot A_1 = A_0$.*

2. *If I is a non-zero two-sided ordinary ideal of A_0, $I + A_1 \cdot I \cdot A_1 = A_0$ and $A_1 \cdot I + I \cdot A_1 = A_1$.*

Proof.

1. Suppose $A_1 \cdot A_1 \neq A_0$. Then $(A_1 \cdot A_1) + A_1$ would be a proper two-sided superideal, which is a contradiction.

2. Let I be a two-sided ordinary ideal of A_0. Then $(I + A_1 \cdot I \cdot A_1) + (A_1 \cdot I + I \cdot A_1)$ is a two-sided superideal of A, with 0-component $I + A_1 \cdot I \cdot A_1$ and 1-component $A_1 \cdot I + I \cdot A_1$. Since A is super simple, this superideal is either A or 0. If $I \neq 0$, we get $I + A_1 \cdot I \cdot A_1 = A_0$ and $A_1 \cdot I + I \cdot A_1 = A_1$, as required.

∎

Lemma 2.44 *If J is a proper two-sided ordinary ideal of A, then the projections $\pi_0 : J \to A_0$ and $\pi_1 : J \to A_1$ are bijective.*

Proof. We will first show that π_0 is surjective. To begin, we note that $J \cap A_0$ and $\pi_0(J)$ are two-sided ideals in A_0. Let us show that $J \cap A_0$ and $\pi_0(J)$ are not equal.

Suppose, for the sake of contradiction, that $J \cap A_0 = \pi_0(J)$. Then $\pi_0(J) \subset J$, which means, for any $j = j_0 + j_1 \in J$ (where $j_i \in A_i$), we have $j_0 \in J$, which implies $j_1 \in J$, and so $J = (J \cap A_0) \oplus (J \cap A_1)$, which means J is also a proper two-sided superideal, which is a contradiction since A is a super simple superalgebra.

Now we will show that both $J \cap A_0$ and $\pi_0(J)$ are improper two-sided ideals of A_0. We first note that, because J is a two-sided ideal of A, $A_1 \cdot (J \cap A_0) \cdot A_1 \subset J \cap A_0$. Similarly, $A_1 \cdot \pi_0(J) \cdot A_1 \subset \pi_0(J)$. If we assume $J \cap A_0$ is a proper two-sided ideal of A_0, then $J \cap A_0$ is non-zero, and $(J \cap A_0) + A_1 \cdot (J \cap A_0) \cdot A_1 = J \cap A_0 = A_0$ by Lemma 2.43, which is a contradiction. Similarly, if we assume $\pi_0(J)$ is a proper two-sided ideal of A_0, then $\pi_0(J)$ is non-zero, and $\pi_0(J) + A_1 \cdot \pi_0(J) \cdot A_1 = \pi_0(J) = A_0$ by Lemma 2.43, which is again a contradiction.

So both $J \cap A_0$ and $\pi_0(J)$ are improper in A_0. We cannot have $J \cap A_0 = A_0$, otherwise $A_0 \subset J$ and $\pi_0(J) = A_0$. That means we can only have $J \cap A_0 = 0$ and $\pi_0(J) = A_0$. Thus we have shown that π_0 is surjective.

Next, let us now show that π_1 is surjective. Since we just showed that $J \cap A_0 = 0$, $J \cap A_1 = A_0 \cdot (J \cap A_1) = A_1 \cdot A_1 \cdot (J \cap A_1) \subset A_1 \cdot (J \cap A_0) = A_1 \cdot 0 = 0$, and $\pi_1(J)$ contains $A_1 \cdot \pi_0(J) = A_1 \cdot A_0 = A_1$. Thus the π_1 is surjective.

Finally, let us show that π_0 and π_1 are injective. Since $J \cap A_1 = 0$, $\ker(\pi_0) = J \cap A_1 = 0$, and since $J \cap A_0 = 0$, $\ker(\pi_1) = J \cap A_0 = 0$. Hence both π_0 and π_1 are injective, and we can conclude that both are bijective linear maps. ∎

Lemma 2.45 *If A is not ordinary simple, then A_0 is ordinary simple and $A_1 = A_0 \cdot v$, with $v \in Z(A) \cap A_1$ and $v^2 = 1_A$.*

Proof. Suppose A is not ordinary simple. Then it has a proper two-sided ideal J, and by Lemma 2.44, the projections $\pi_0 : J \to A_0$ and $\pi_1 : J \to A_1$ are bijective linear maps. Consider the element $x = \pi_0^{-1}(1_A) \in J$. Since $\pi_0(x) = 1_A$, $x = 1_A + v$, where $v = \pi_1(x) \in A_1$. Since J is a two-sided ideal, J contains the element $v(1_A + v) = v^2 + v$. Since π_1 is bijective and $\pi_1(1_A + v) = v = \pi_1(v^2 + v)$, $1_A + v = v^2 + v$, meaning $v^2 = 1_A$.

Let us show that $v \in Z(A)$ and $A_1 = A_0 \cdot v$. For any $y \in (A_0 \cup A_1)\backslash\{0\}$, J contains the elements $xy = (1_A + v)y = y + vy$ and $yx = y(1_A + v) = y + yv$. Let $i = d(y)$. Since π_i is bijective and $\pi_i(y + vy) = y = \pi_i(y + yv)$, $y + vy = y + yv$, which means $vy = yv$ for any $y \in (A_0 \cup A_1)\backslash\{0\}$. As a consequence, $v \in Z(A)$. That means $A_1 = A_1 \cdot 1_A = A_1 \cdot v^2 \subset A_0 \cdot v$, and so $A_1 = A_0 \cdot v$.

Finally, if I is a two-sided ideal of A_0, we have

$$A_1 \cdot I \cdot A_1 = (A_0 \cdot v) \cdot I \cdot (A_0 \cdot v) = A_0 \cdot v \cdot I \cdot v = A_0 \cdot I \cdot v^2 = A_0 \cdot I = I,$$

and if we assumed I is a non-trivial proper two-sided ideal, then, by Lemma 2.43, $I + A_1 \cdot I \cdot A_1 = I = A_0$, which is a contradiction. Therefore, I is improper, and A_0 is ordinary simple. ∎

Lemma 2.46 *If A is not ordinary central simple, then A_0 is ordinary central simple, and A is not ordinary central.*

Proof. We will first prove that if A is a super central simple F-superalgebra that is not ordinary central simple, then it is not ordinary central. To show this, we will show that A not being ordinary simple implies it is not ordinary central. We will prove the contrapositive.

Let A be ordinary central. We can show that A being ordinary central implies it is ordinary simple. If we assumed A was not ordinary simple, then, by Lemma 2.45, we can find a $v \in Z(A) \cap A_1$ such that $v^2 = 1$ and $A_1 = A_0 \cdot v$. But $Z(A) = F \cdot 1 \subset A_0$, which means $Z(A) \cap A_1 = 0$ and $v^2 = 0^2 = 0 = 1_A$, a contradiction. Thus A being ordinary central implies it is also ordinary simple.

Hence if A is not ordinary central simple, then A is not ordinary central. Let us now show that A_0 is ordinary central simple.

Since A is super central, $Z(A) \cap A_0 = F \cdot 1 \subsetneq Z(A)$, which means $Z(A) \cap A_1$ is a non-zero subspace of $Z(A)$. For any $z \in Z(A) \cap A_1, z^2 \in Z(A) \cap A_0 = F \cdot 1$. To show that A_0 is ordinary central simple, let us first show that there exists an element $v \in Z(A) \cap A_1$ such that $A_1 = A_0 \cdot v$.

To show this, we will first propose that there exists an element $v \in Z(A) \cap A_1$ such that $v^2 \neq 0$. For the sake of contradiction, suppose $v^2 = 0$ for all $v \in Z(A) \cap A_1$. That means $Z(A)$ has zero-divisors, and as a consequence, it would mean $Z(A)$ is not a field. Therefore A is not ordinary simple, and by Lemma 2.45, we can find a $v \in Z(A) \cap A_1$ such that $v^2 = 1_A$, which is a contradiction. Hence there exists a $v \in Z(A) \cap A_1$ such that $v^2 \in (F \cdot 1_A)\backslash\{0\}$. Let $u = v^2$. Then $A_1 = A_1 \cdot u = A_1 \cdot v^2 \subset A_0 \cdot v$, which means $A_1 = A_0 \cdot v$.

Now let us show that A_0 is central. Since $v \in Z(A)$ and $A_1 = A_0 \cdot v$, then any element of A is in the form $a + bv$ where $a, b \in A_0$, and for any $z \in Z(A_0)$, $z(a + bv) = za + zbv = az + bvz = (a + bv)z$ for any $a, b \in A_0$, which means $z \in Z(A)$, and hence $Z(A_0) \subset Z(A) \cap A_0 = F \cdot 1_A$. Therefore, $Z(A_0) = F \cdot 1_A$, and A_0 is central.

To show that A_0 is ordinary simple, we first note that if I is a two-sided ideal of A_0, then $I + I \cdot u$ is a two-sided superideal of A. Since A is super simple, $I + I \cdot u$ is either 0 or all of A. Therefore I can only be either 0 or all of A_0. Hence A_0 is ordinary simple, as required. ■

Corollary 2.47 *If A is not ordinary central simple, then $Z(A) = F \cdot 1_A + F \cdot v$ where v is an element of $Z(A) \cap A_1$ such that v^2 is a non-zero element of $F \cdot 1_A$.*

Proof. The proof of lemma 2.46 tells us that there exists a $v \in Z(A) \cap A_1$, such that $v^2 = a \in (F \cdot 1_A)\backslash\{0\}$, and hence $A_1 = A_0 \cdot v$.

So far, we know that $F \cdot 1_A + F \cdot v \subset Z(A)$. Let us show that we have an equality. To do this, we show that $Z(A) \cap A_1 = F \cdot v$.

Let z be a non-zero element of $\in Z(A) \cap A_1$. Then $z = cv$ for some $c \in A_0\backslash\{0\}$. Note that $zv = cv^2$ is a non-zero element of $Z(A) \cap A_0$, since c is non-zero and v^2 is a non-zero element of $F \cdot 1_A$. Since A is super central, $Z(A) \cap A_0 = F \cdot 1_A$, which means $zv = b \in F \cdot 1_A\backslash\{0\}$. That means $zv^2 = za = bv$, hence $z = (b/a)v$, and since $(b/a) \in F \cdot 1_A$, $z \in F \cdot v$. Hence we have shown that $Z(A) \cap A_1 = F \cdot v$, and since $Z(A) \cap A_0 = F \cdot 1_A$ and $Z(A)$ is a subsuperalgebra, we can conclude that $Z(A) = F \cdot 1_A + F \cdot v$. ■

Remark 2.48 We further claim that $Z_A(A_0) = Z(A)$ in the case where A is super central simple, but not ordinary central simple. The inclusion $Z(A) \subset Z_A(A_0)$ is obvious. To show the reverse inclusion, first note that in this situation, $A_1 = A_0 \cdot v$ where $v \in Z(A) \cap A_1$. Hence any element of A can be expressed in the form $x + yv$ where $x, y \in A_0$. Now let $a \in Z_A(A_0)$. Since $v \in Z(A) \cap A_1$, then for any $x + yv \in A$ (where $x, y \in A_0$), $a(x + yv) = ax + ayv = xa + yav = xa + yva = (x + yv)a$. Hence $a \in Z(A)$, and we conclude that $Z_A(A_0) = Z(A)$. Overall we have $Z_A(A_0) = Z(A) = F \cdot 1_A + F \cdot v$.

We also note that in the case where A is super central simple, but not ordinary central simple, A can be expressed as $A = A_0 \otimes Z(A) = A_0 \otimes (F \cdot 1_A + F \cdot v)$.

Lemma 2.49 *Let $A = M_{(r,s)}(F) \hat{\otimes} D$ be a super central simple F-superalgebra, where D is a super central super division algebra. Then A is ordinary central simple if and only if D is ordinary central simple.*

Proof. Suppose D is ordinary central simple. In the case where $D_1 = 0$, $D = D_0$ is an ordinary central division algebra, and since D is purely even, then, as an F-algebra, $A = M_{(r,s)}(F) \hat{\otimes} D \cong M_{(r,s)}(F) \otimes D = M_{r+s}(F) \otimes D \cong M_{r+s}(D)$. Hence A is ordinary central simple in the case where $D_1 = 0$.

If $D_1 \neq 0$, then $M_{(r,s)}(F) \hat{\otimes} D \cong M_{r+s}(F) \hat{\otimes} D$, and since $M_{r+s}(F)$ is purely even, $M_{r+s}(F) \hat{\otimes} D \cong M_{r+s}(F) \otimes D$. Since $M_{r+s}(F)$ and D are ordinary central simple, then $A \cong M_{r+s}(F) \otimes D$ is ordinary central simple.

Now suppose A is ordinary central simple. For the sake of contradiction, suppose D is not ordinary central simple. Then $D_1 \neq 0$, because otherwise D would be an ordinary central division algebra. Since $D_1 \neq 0$, $M_{(r,s)}(F) \hat{\otimes} D \cong M_{r+s}(F) \hat{\otimes} D \cong M_{r+s}(F) \otimes D$.

If D is not ordinary central, then $Z(A) \cong Z\left(M_{r+s}(F) \otimes D\right) \cong Z(D)$, which would mean A is not ordinary central, which is a contradiction.

If D is not ordinary simple, then it contains a non-trivial proper two-sided ideal I, and the subspace $M_{r+s}(F) \otimes I$ in A would be a non-trivial proper two-sided ideal of A, which is again a contradiction. Thus A being ordinary central simple implies that D is ordinary central simple. ■

We summarise our analysis thus:

Let A be a super central simple F-superalgebra that is not ordinary central simple.

- A and D are not ordinary central.
- $A_1 \neq 0$, and $D_1 \neq 0$.
- D_0 is a central division algebra.
- $A_0 \cong M_n(D_0)$ for some $n > 0$, and A_0 is ordinary central simple.
- $Z(A) = Z_A(A_0) = F \cdot 1_A + F \cdot v$ where $v \in Z(A) \cap A_1$ and v^2 is a non-zero element of $F \cdot 1_A$.
- $A \cong A_0 \otimes (F \cdot 1_A + F \cdot v) \cong M_n(D_0) \otimes (F \cdot 1_A + F \cdot v) \cong M_n(F) \otimes D$.
- If D is also not ordinary simple:
  - v^2 is a square in $F \cdot 1_A$.
  - A is not ordinary simple.
  - $A \cong M_n(D_0) \times M_n(D_0)$.
- If D is ordinary simple:
  - v^2 is not a square in $F \cdot 1_A$.
  - A is ordinary simple.

Next, we deal with the case where A is ordinary central simple.

Lemma 2.50 *Suppose A is ordinary central simple. If $A_1 \neq 0$, then A_0 is not ordinary central simple.*

Proof. Let $A = M_{(r,s)}(F) \hat{\otimes} D$ be a super central simple F-superalgebra that is also ordinary central simple. If $D_1 = 0$, then $A_0 = M_{(r,s)}(F)_0 \otimes D$. Since $A_1 \neq 0$, then both r and s are positive integers, meaning $A_0 = M_{(r,s)}(F)_0 \otimes D \cong M_r(D) \times M_s(D)$, which would mean A_0 is neither ordinary central or ordinary simple. In this case $Z(A_0) \cong F \times F \cong \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) : a, b \in F\}$.

Now suppose $D_1 \neq 0$. Then $A = M_{(r,s)}(F) \hat{\otimes} D \cong M_{r+s}(F) \hat{\otimes} D$. Letting $n = r + s$, then $A_0 \cong M_n(D_0)$, which means A_0 is ordinary simple. Let us show that A_0 is not ordinary central.

To begin, we give a construction that is applicable in both the $D_1 = 0$ and the $D_1 \neq 0$ case. Consider the involution T of A defined by: $T(x) = x$ whenever $x \in A_0$, and $T(y) = -y$ whenever $y \in A_1$. If $\mathrm{char} F \neq 2$, then it can be easily checked that the set of elements of A fixed by T is A_0. It can also be easily checked that T is a non-trivial F-algebra automorphism. Since A is an ordinary central simple F-algebra, then by the Skolem–Noether theorem, T is an inner automorphism. So there exists an element $v \in A^\times$ such that for any $x \in A$, $T(x) = v^{-1}xv$.

Note that $T(v) = v^{-1}vv = v$, and since the elements of A fixed by T is A_0, $v \in A_0$. Since for any $x \in A_0$, $v^{-1}xv = x$, $v \in Z(A_0)$. Since T is a non-trivial automorphism, $v \notin F \cdot 1_A$. We further note that for any $x \in A$, $T(T(x)) = x$, meaning for any $x \in A$, $v^{-1}(v^{-1}xv)v = (v^2)^{-1}xv^2 = x$, hence v^2 is a non-zero element of $Z(A) = F \cdot 1_A$. Therefore, $F \cdot 1_A + F \cdot v$ is a unital subalgebra of A_0, and it is isomorphic to $\{\left(\begin{smallmatrix} a & ub \\ b & a \end{smallmatrix}\right) : a, b \in F, u = v^2\}$. One can check that this is a field if and only if $v^2 = u$ is not a square in $F^\times$. Also note that $F \cdot 1_A + F \cdot v \subset Z(A_0)$.

In the case where $D_1 = 0$, $Z(A_0) = F \times F$ would not be a field, and so $v^2 = u$ would be a square in $F^\times$ if $D_1 = 0$. If $D_1 \neq 0$, then $Z(A_0) \cong Z(D_0)$, which is a field. Since $F \cdot 1_A + F \cdot v \subset Z(A_0)$, then $F \cdot 1_A + F \cdot v$ would also need to be a field, which means $v^2 = u$ is not a square in $F^\times$. We have thus shown that A_0 is not central. ■

Corollary 2.51 *Suppose A is ordinary central simple. If $A_1 \neq 0$, then $Z(A_0) = F \cdot 1_A + F \cdot v$ where v is an element of $A_0 \backslash (F \cdot 1_A)$ such that v^2 is a non-zero element of $F \cdot 1_A$.*

Proof. Let $A = M_{(r,s)}(F) \hat{\otimes} D$ be a super central simple F-superalgebra that is also ordinary central simple. We have previously seen that if $D_1 = 0$, then $A_0 \cong M_r(D) \times M_s(D)$, which means $Z(A_0) \cong F \cdot I_r \times F \cdot I_s$ where I_r and I_s are the identity elements of $M_r(D)$ and $M_s(D)$ respectively. Note that $Z_(A_0) \cong F \cdot I_r \times F \cdot I_s = F \cdot (I_r, I_s) + F \cdot (I_r, -I_s)$. Hence $Z_(A_0) \cong F \cdot 1_A + F \cdot v$ where $v = (I_r, -I_s)$. Note that $v^2 = (I_r, I_s) = 1_A$, which is a non-zero element of $F \cdot 1_A$.

Now let us investigate the case where $D_1 \neq 0$. From the previous lemma, we showed that there is an element $v \in A_0 \backslash (F \cdot 1_A)$ such that $v^{-1}xv = (-1)^{d(x)}x$ for any $x \in A_0 \cup A_1$. We also noted that when $D_1 \neq 0$, v^2 is an element of $F \cdot 1_A$ that is not a square, and $F \cdot 1_A + F \cdot v \subset Z(A_0)$. We can show that $F \cdot 1_A + F \cdot v \supset Z(A_0)$ by showing that $F \cdot 1_A + F \cdot v = Z_A(A_0) \supset Z(A_0)$.

In order to show that $F \cdot 1_A + F \cdot v = Z_A(A_0)$, we first note that $F \cdot 1_A + F \cdot v$ is a field, since $F \cdot 1_A + F \cdot v \subset Z(A_0) \cong Z(D_0)$, and we know $Z(D_0)$ is a field. Hence $F \cdot 1_A + F \cdot v$ is a simple unital subalgebra of A. By Corollary 1.30, $Z_A(Z_A(F \cdot 1_A + F \cdot v)) = F \cdot 1_A + F \cdot v$. Note that $x \in Z_A(F \cdot 1_A + F \cdot v)$ if and only if $v^{-1}xv = x$ if and only if $x \in A_0$. That means $Z_A(F \cdot 1_A + F \cdot v) = A_0$, hence $Z_A(Z_A(F \cdot 1_A + F \cdot v)) = Z_A(A_0) = F \cdot 1_A + F \cdot v$.

Since we always have $Z(A_0) \subset Z_A(A_0)$, we get $Z(A_0) \subset F \cdot 1_A + F \cdot v$, and hence we can conclude that $Z(A_0) = F \cdot 1_A + F \cdot v$. ■

We summarise our analysis thus:

Let A be a super central simple F-superalgebra that is also ordinary central simple.

- D is an ordinary central simple algebra.
- If $A_1 = 0$, then:
  - $D_1 = 0$ and $D_0 = D$ is an ordinary central division algebra.
  - $A_0 = A$ is ordinary central simple.
- If $A_1 \neq 0$, then:
  - $Z(A_0) = Z_A(A_0) = F \cdot 1_A + F \cdot v$ where $v \in A_0 \backslash (F \cdot 1_A)$, v^2 is a non-zero element of $F \cdot 1_A$, and for any $x \in A_1$, $v^{-1}xv = -x$.
  - If $D_1 = 0$, then:
    * $D_0 = D$ is an ordinary central division algebra.
    * $A_0 \cong M_r(D) \times M_s(D)$ for some $r, s > 0$, and A_0 is neither ordinary central or ordinary simple.
    * v^2 is a square in $F \cdot 1_A$.
  - If $D_1 \neq 0$, then:
    * D_0 is a division algebra that is not ordinary central.
    * $A_0 \cong M_n(D_0)$ for some $n > 0$, and A_0 is ordinary simple but not ordinary central.
    * v^2 is not a square in $F \cdot 1_A$.

We will now work towards describing the properties of $BW(F)$. We remind the reader that we are assuming $\mathrm{char}F \neq 2$ throughout the rest of this subsection.

Theorem 2.52 *Let A and B be two super central simple F-superalgebras which are ordinary central simple. Then $A \hat{\otimes} B$ is ordinary central simple.*

Proof. First we will cover the case where at least one of A and B is purely even. If one of A and B is purely even, then $A \hat{\otimes} B \cong A \otimes B$, and since A and B are ordinary central simple, then $A \otimes B$ is ordinary central simple, hence $A \hat{\otimes} B$ is ordinary central simple.

Now let us cover the case where both A and B have a none-zero 1-component. If A_1 and B_1 are both non-zero, then $Z(A_0) = F \cdot 1_A + F \cdot v$ and $Z(B_0) = F \cdot 1_B + F \cdot v'$ where $v \notin F \cdot 1_A$, $v' \notin F \cdot 1_B$, $v^{-1}xv = -x$ for any $x \in A_1$ and $v'^{-1}yv' = -y$ for any $y \in B_1$. To show that $A \hat{\otimes} B$ is ordinary central simple, we will show that $(A \hat{\otimes} B)_0$ is not central. We first note that for any $a_0 \in A_0$ and $b_0 \in B_0$, $(v \hat{\otimes} v')(a_0 \hat{\otimes} b_0) = va_0 \hat{\otimes} v'b_0 = a_0v \hat{\otimes} b_0v' = (a_0 \hat{\otimes} b_0)(v \hat{\otimes} v')$, and for any $a_1 \in A_1$ and $b_1 \in B_1$, $(v \hat{\otimes} v')(a_1 \hat{\otimes} b_1) = va_1 \hat{\otimes} v'b_1 = (-a_1v) \hat{\otimes} (-b_1v') = a_1v \hat{\otimes} b_1v' = (a_1 \hat{\otimes} b_1)(v \hat{\otimes} v')$. Hence $(v \hat{\otimes} v') \in Z((A \hat{\otimes} B)_0)$, and $F \cdot (1_A \hat{\otimes} 1_B) + F \cdot (v \hat{\otimes} v') \subset Z((A \hat{\otimes} B)_0)$. Since $(v \hat{\otimes} v') \notin F \cdot (1_A \hat{\otimes} 1_B)$, $(A \hat{\otimes} B)_0$ is not central, which implies $A \hat{\otimes} B$ is ordinary central simple. ■

Remark 2.53 We note that $Z((A \hat{\otimes} B)_0) = F \cdot (1_A \hat{\otimes} 1_B) + F \cdot (v \hat{\otimes} v')$ since $A \hat{\otimes} B$ is ordinary central simple, meaning $\dim_F(Z((A \hat{\otimes} B)_0))$ is at most 2. Also, for any $a_0 \in A_0$ and $b_1 \in B_1$, $(v \hat{\otimes} v')(a_0 \hat{\otimes} b_1) = va_0 \hat{\otimes} v'b_1 = a_0v \hat{\otimes} (-b_1v') = -(a_0v \hat{\otimes} b_1v') =$

$-(a_0 \hat{\otimes} b_1)(v \hat{\otimes} v')$, and for any $a_1 \in A_1$ and $b_0 \in B_0$, $(v \hat{\otimes} v')(a_1 \hat{\otimes} b_0) = va_1 \hat{\otimes} v'b_0 = (-a_1 v) \hat{\otimes} b_0 v' = -(a_1 v \hat{\otimes} b_0 v') = -(a_1 \hat{\otimes} b_0)(v \hat{\otimes} v')$. This means for any $x \in (A \hat{\otimes} B)_1$, $(v \hat{\otimes} v')x = -x(v \hat{\otimes} v')$.

Let us denote the subset of $BW(F)$ consisting of classes of ordinary central simple algebras as $P(F)$. Then we have the following lemma:

Corollary 2.54 *$P(F)$ is a subgroup of $BW(F)$.*

Proof. Theorem 2.52 tells us that the set of super central simple F-superalgebras that are also ordinary central simple is closed under the super tensor product operation. This means the subset $P(F)$ is closed under the group operation of $BW(F)$. It is clear that the identity element $[F]$ of $BW(F)$ is an element of $P(F)$. Let us show that for any element $[A] \in P(F)$, $[A]^{-1} = [A^{\mathrm{sop}}] \in P(F)$. It is enough to show that for any super central simple F-superalgebra that is ordinary central simple, A^{sop} is also ordinary central simple.

If A is purely even, then $A^{\mathrm{sop}} = A^{\mathrm{op}}$, and since A^{op} has the same two-sided ideals as A, and $Z(A) = Z(A^{\mathrm{op}})$, $A^{\mathrm{op}} = A^{\mathrm{sop}}$ is also ordinary central simple.

If $A_1 \neq 0$, then $Z(A_0) = F \cdot 1_A + F \cdot v$ where v is an element in $Z(A_0)$ such that, for any $x \in A_1$, $v^{-1}xv = -x$. Note that $Z((A^{\mathrm{sop}})_0) = F \cdot 1_A + F \cdot v$, hence A^{sop} is ordinary central simple. We can conclude that $P(F)$ is a subgroup of $BW(F)$. ■

Theorem 2.55 *$BW(F)/P(F) \cong C_2$.*

Proof. First we will show that the set $BW(F) \backslash P(F)$ is non-empty for any field F not of characteristic 2. We simply note that $D = F \times F$ can be turned into a super division algebra by giving it the following grading: $D_0 = F \cdot (1_F, 1_F)$, and $D_1 = F \cdot (1_F, -1_F)$. This is a super central simple F-superalgebra that is not ordinary central simple, hence the set $BW(F) \backslash P(F)$ is non-empty.

To show that $BW(F)/P(F) \cong C_2$, it is enough to show that if A and B are super central simple F-superalgebras that are not ordinary central simple, then $A \hat{\otimes} B$ is ordinary central simple. If A and B are super central simple F-superalgebras that are not ordinary central simple, then $Z(A) = F \cdot 1_A + F \cdot v$ and $Z(B) = F \cdot 1_A + F \cdot v'$, where $v \in Z(A) \cap A_1$ and $v' \in Z(B) \cap B_1$. To show that $A \hat{\otimes} B$ is ordinary central simple, we will show that $F \cdot (1_A \hat{\otimes} 1_B) + F \cdot (v \hat{\otimes} v') \subset (A \hat{\otimes} B)_0$, which would mean $(A \hat{\otimes} B)_0$ is not ordinary central. Note that for any $a_0 \in A_0$ and $b_0 \in B_0$, $(v \hat{\otimes} v')(a_0 \hat{\otimes} b_0) = va_0 \hat{\otimes} v'b_0 = a_0 v \hat{\otimes} b_0 v' = (a_0 \hat{\otimes} b_0)(v \hat{\otimes} v')$, and for any $a_1 \in A_1$ and $b_1 \in B_1$, $(v \hat{\otimes} v')(a_1 \hat{\otimes} b_1) = -(va_1 \hat{\otimes} v'b_1) = -(a_1 v \hat{\otimes} b_1 v') = (a_1 \hat{\otimes} b_1)(v \hat{\otimes} v')$. Hence $(v \hat{\otimes} v') \in Z((A \hat{\otimes} B)_0)$ and $F \cdot (1_A \hat{\otimes} 1_B) + F \cdot (v \hat{\otimes} v') \subset (A \hat{\otimes} B)_0$. Since $(v \hat{\otimes} v') \notin F \cdot (1_A \hat{\otimes} 1_B)$, $(A \hat{\otimes} B)_0$ is not central. Hence $A \hat{\otimes} B$ is ordinary central simple, and for any equivalence classes $[A], [B] \in BW(F) \backslash P(F)$, $[A] \cdot [B] = [A \hat{\otimes} B] \in P(F)$. We conclude that $BW(F)/P(F) \cong C_2$. ■

Hence $BW(F)$ is an extension of $P(F)$ by C_2. We note that $Br(F)$ can be identified with a subgroup of $BW(F)$, namely the classes of $BW(F)$ that contain the purely even super central super division algebras. Since a purely even super central super division algebra is a central simple algebra, $Br(F)$ is a subgroup of $P(F)$, and we have $Br(F) \trianglelefteq P(F) \trianglelefteq BW(F)$.

Theorem 2.56 *$P(F)/Br(F) \cong F^{\times}/F^{\times 2}$.*

Proof. First, we will show that $P(F)/Br(F)$ is trivial if and only if $F^\times/F^{\times 2}$ is trivial.

First, let $P(F)/Br(F)$ be trivial. Suppose, for the sake of contradiction, that $F^\times/F^{\times 2}$ is non-trivial. Then $F^\times$ has an element u that is not a square. For any $x \in F^\times$, $F^{\mathfrak{s}}(\sqrt{x})$ will denote the algebra $\{\left(\begin{smallmatrix} a & xb \\ b & a \end{smallmatrix}\right) : a, b \in F\}$ with the following grading: $F^{\mathfrak{s}}(\sqrt{x})_0 = F \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $F^{\mathfrak{s}}(\sqrt{x})_1 = F \cdot \left(\begin{smallmatrix} 0 & x \\ 1 & 0 \end{smallmatrix}\right)$. One can express $F^{\mathfrak{s}}(\sqrt{x})$ as $F \cdot 1 \oplus F \cdot v_x$ where $v_x^2 = x$.

Note that $F^{\mathfrak{s}}(\sqrt{x})$ is a super central super division algebra that is not ordinary central simple for any $x \in F^\times$. Also note if there exists an element $u \in F^\times$ that is not a square, then $A = F^{\mathfrak{s}}(\sqrt{u}) \hat{\otimes} F^{\mathfrak{s}}(\sqrt{-1})$ would be a super division algebra; one can check that $A_0 \cong F^{\mathfrak{s}}(\sqrt{u})$, which is a field, and $A_1 = A_0 \cdot (v_x \hat{\otimes} 1)$.

Since $F^{\mathfrak{s}}(\sqrt{u})$ and $F^{\mathfrak{s}}(\sqrt{-1})$ are not ordinary central simple, A is a super central simple superalgebra that is ordinary central simple. However, since it is a super central super division algebra that is not purely even, $[A] \in P(F)\backslash Br(F)$, which is a contradiction.

Now let $F^\times/F^{\times 2}$ be trivial, and suppose, for the sake of contradiction, that $P(F)/Br(F)$ is non-trivial. Then there is a super central super division algebra D that is ordinary central simple, but is not purely even. Then $Z(D_0) = F \cdot 1_D + F \cdot v$ where $v \in D_0\backslash(F \cdot 1_D)$, and v^2 is a non-zero element of $F \cdot 1_D$. Since D is ordinary central simple and $D_1 \neq 0$, $Z(D_0)$ is a field, which means v^2 is not a square in $F \cdot 1_D$. Hence there are non-squares in $F^\times$, which is a contradiction.

Now let us construct an isomorphism between $P(F)/Br(F)$ and $F^\times/F^{\times 2}$ when they are both non-trivial. We will denote our required isomorphism as φ. For any $[A] \in Br(F)$, $\varphi([A]) = [1]'$ (for this proof, prime will denote an equivalence class in $F^\times/F^{\times 2}$). For any $[A] \in P(F)\backslash Br(F)$, we know that $Z(A_0) = F \cdot 1_A + F \cdot v_A$ where $v_A \in A_0\backslash(F \cdot 1_A)$, and $v_A^2 = u_A$ is not a square in $F \cdot 1_A$. Using the notation from before, $Z(A_0) = F^{\mathfrak{s}}(\sqrt{u_A})$, where u_A is not a square in $F^\times$. Then for any $[A] \in P(F)\backslash Br(F)$, we will let $\varphi([A]) = [u_A]'$ Let us show that this map is well defined.

Let A be a super central simple superalgebra whose equivalence class $[A] \in BW(F)$ is contained in $P(F)\backslash Br(F)$, and let B be a super central simple superalgebra such that $[B] \in Br(F)$. If $B_1 = 0$, then $(A \hat{\otimes} B)_0 = A_0 \otimes B_0 = A_0 \otimes B$, hence $Z(A \hat{\otimes} B)_0 \cong Z(A_0 \otimes B) \cong Z(A_0) \otimes Z(B) = (F \cdot 1_A + F \cdot v_A) \otimes F \cong F \cdot 1_A + F \cdot v_A$, and since $v_A^2 = u_A$, $\varphi([A \hat{\otimes} B]) = [u_A]'$. If $B_1 \neq 0$, then $Z(B_0) = F \cdot 1_B + F \cdot v_B$ where $v_B \in B_0\backslash(F \cdot 1_B)$ and $v_B^2 = u_B$ is a square in $F^\times$. Note that $F \cdot (1_A \hat{\otimes} 1_B) + F \cdot (v_A \hat{\otimes} v_B) \subset Z((A \hat{\otimes} B)_0)$, and since $\dim_F Z((A \hat{\otimes} B)_0) \leq 2$, $Z((A \hat{\otimes} B)_0) = F \cdot (1_A \hat{\otimes} 1_B) + F \cdot (v_A \hat{\otimes} v_B)$. Squaring $(v_A \hat{\otimes} v_B)$, we get $(v_A^2 \hat{\otimes} v_B^2) = (u_A \hat{\otimes} u_B) = u_A u_B (1_A \hat{\otimes} 1_B)$, hence $\varphi([A \hat{\otimes} B]) = [u_A u_B]'$, and since u_B is a square in $F^\times$, $\varphi([A \hat{\otimes} B]) = [u_A u_B]' = [u_A]'$, and we have shown that the map $\varphi : P(F)/Br(F) \to F^\times/F^{\times 2}$ is well defined.

Let us show that φ is a group homomorphism. In the previous paragraph, we showed that if $[A] \in P(F)\backslash Br(F)$ and $[B] \in Br(F)$, then $\varphi([A][B]) = \varphi([A \hat{\otimes} B]) = [u_A]' = [u_A]'[1]' = \varphi([A])\varphi([B])$. Let us now show that $\varphi([A][B]) = \varphi([A])\varphi([B])$ for any $[A], [B] \in P(F)\backslash Br(F)$.

Let $[A], [B] \in P(F)\backslash Br(F)$. Then $Z(A_0) = F \cdot 1_A + F \cdot v_A$ and $Z(B_0) = F \cdot 1_B + F \cdot v_B$ where $v_A^2 = u_A$ and $v_B^2 = u_B$ are not squares in $F^\times$. We note that $Z((A \hat{\otimes} B)_0) = F \cdot (1_A \hat{\otimes} 1_B) + F \cdot (v_A \hat{\otimes} v_B)$, and $(v_A \hat{\otimes} v_B)^2 = (v_A^2 \hat{\otimes} v_B^2) = (u_A \hat{\otimes} u_B) = u_A u_B (1_A \hat{\otimes} 1_B)$. If $u_A u_B$ is not a square in $F^\times$, then $[A \hat{\otimes} B] \in P(F)\backslash Br(F)$, and hence $\varphi([A][B]) = \varphi([A \hat{\otimes} B]) = [u_A u_B]' = [u_A]'[u_B]' = \varphi([A])\varphi([B])$. If $u_A u_B$ is a square in $F^\times$, then the super division algebra part of $A \hat{\otimes} B$ is purely even, hence $[A \hat{\otimes} B] \in Br(F)$, and $\varphi([A][B]) = \varphi([A \hat{\otimes} B]) = [1]' = [u_A u_B]' = [u_A]'[u_B]' = \varphi([A])\varphi([B])$. We conclude that φ is a group homomorphism.

It can be easily seen that φ is injective, since we have defined it in such a way that for any $[A] \in P(F)\backslash Br(F)$, $\varphi([A]) \neq F^{\times 2}$. Let us show that φ is surjective. For any element $u \in F^{\times}$ that is not a square, we previously said that the superalgebra $A = F^{\mathfrak{s}}(\sqrt{u}) \hat{\otimes} F^{\mathfrak{s}}(\sqrt{-1})$ has the property that $Z(A_0) \cong F^{\mathfrak{s}}(\sqrt{u}) = F \cdot 1_A + F \cdot v$ where $v \in A_0 \backslash (F \cdot 1_A)$ and $v^2 = u$. Thus, $\varphi([F^{\mathfrak{s}}(\sqrt{u}) \hat{\otimes} F^{\mathfrak{s}}(\sqrt{-1})]) = [u]'$, and we have shown that φ is surjective.

We can finally conclude that $\varphi : P(F)/Br(F) \to F^{\times}/F^{\times 2}$ is an isomorphism. ■

Hence $P(F)$ is an extension of $Br(F)$ by $F^{\times}/F^{\times 2}$.

Corollary 2.57 *Suppose both $Br(F)$ and $F^{\times}/F^{\times 2}$ are finite. Then $BW(F)$ is finite, and $|BW(F)| = 2|Br(F)||F^{\times}/F^{\times 2}|$.*

Corollary 2.58 *Suppose F is algebraically closed. Then $BW(F) \cong C_2$.*

Proof. If F is algebraically closed, then every element of $F^{\times}$ is a square, which means $F^{\times}/F^{\times 2}$ is trivial. In addition, by Corollary 1.35, $Br(F)$ is also trivial. By Corollary 2.57, $|BW(F)| = 2$, hence $BW(F) \cong C_2$. ■

Our discussion above shows that if F is a field not of characteristic 2, $Br(F)$ and $BW(F)$ are related by the exact sequences:

$$1 \to P(F) \to BW(F) \to C_2 \to 1, \text{ and}$$

$$1 \to Br(F) \to P(F) \to F^{\times}/F^{\times 2} \to 1.$$

2.3 The Brauer-Wall Group of $\mathbb{C}$

Since $\mathbb{C}$ is algebraically closed, then by Corollary 2.58, $BW(\mathbb{C}) \cong C_2$. This also means there are only two super central super division algebras over $\mathbb{C}$. They are $\mathbb{C} \oplus 0$ (which is $\mathbb{C}$ as a purely even algebra), and $\mathbb{C} \times \mathbb{C}$ equipped with the following grading: $(\mathbb{C} \times \mathbb{C})_0 = \{(a, a) : a \in \mathbb{C}\}$ and $(\mathbb{C} \times \mathbb{C})_1 = \{(a, -a) : a \in \mathbb{C}\}$. We will denote the latter super division algebra as $\mathbb{C} \oplus \mathbb{C}v$, where $\mathbb{C}v$ is the 1-component of the super division algebra. We note that $Z(\mathbb{C} \oplus \mathbb{C}v) = \mathbb{C} \oplus \mathbb{C}v$. We also note that both $\mathbb{C} \oplus 0$ and $\mathbb{C} \oplus \mathbb{C}v$ are super division algebras over $\mathbb{R}$, but they are not super central over $\mathbb{R}$. Both of their super centres are isomorphic to $\mathbb{C}$.

In $BW(\mathbb{C})$, we simply have that $[\mathbb{C} \oplus \mathbb{C}v]^2 = [\mathbb{C} \oplus 0]$. In fact, $(\mathbb{C} \oplus \mathbb{C}v) \hat{\otimes} (\mathbb{C} \oplus \mathbb{C}v) \cong M_{(1,1)}(\mathbb{C})$.

2.4 The Brauer-Wall Group of $\mathbb{R}$

Let us construct all of the super central super division algebras over $\mathbb{R}$. We know that if D is a purely even super central super division algebras over $\mathbb{R}$, then it is an ordinary central division algebra over $\mathbb{R}$. Hence the purely even super central super division algebras over $\mathbb{R}$ are $\mathbb{R} \oplus 0$ and $\mathbb{H} \oplus 0$. Their equivalence classes $[\mathbb{R} \oplus 0]$ and $[\mathbb{H} \oplus 0]$ in $BW(\mathbb{R})$ make up the subgroup $Br(\mathbb{R})$ contained in $BW(\mathbb{R})$. Now let us construct the super central super division algebras over $\mathbb{R}$ that are ordinary central simple, and thus find the subgroup $P(\mathbb{R})$.

2.4.1 Super central super division algebras over $\mathbb{R}$ that are ordinary central simple

We know from Theorem 2.56 that $P(\mathbb{R})/Br(\mathbb{R}) \cong \mathbb{R}^\times/\mathbb{R}^{\times 2}$. Since $\mathbb{R}^\times/\mathbb{R}^{\times 2} = \{[1], [-1]\} \cong C_2$, then $|P(\mathbb{R})| = |Br(\mathbb{R})| \cdot |\mathbb{R}^\times/\mathbb{R}^{\times 2}| = 2 \cdot 2 = 4$. So there are four super central super division algebras over $\mathbb{R}$ that are ordinary central simple, and two of them are purely even. Let us find the two super central super division algebras that are not purely even.

If $D = D_0 \oplus D_1$ is a super central super division algebra that is ordinary central simple, and $D_1 \neq 0$, then D_0 is an ordinary division algebra that is non-central, and $Z(D_0) = \mathbb{R} \cdot 1_A + \mathbb{R} \cdot v$ where v^2 is not a square in $\mathbb{R} \cdot 1_A$. Hence $u = v^2$ is a negative real number, and we can show that this means $Z(D_0) = \mathbb{R} \cdot 1_A + \mathbb{R} \cdot v \cong \mathbb{C}$. If $v^2 = -a$ for some positive real number a, then $(\frac{1}{\sqrt{a}}v)^2 = -1$. Letting $\tilde{v} = \frac{1}{\sqrt{a}}v$, then $\mathbb{R} \cdot 1_A + \mathbb{R} \cdot v = \mathbb{R} \cdot 1_A + \mathbb{R} \cdot \tilde{v}$, and it is clear that this is isomorphic to $\mathbb{C}$.

However, we know that D_0 is a division algebra that is not ordinary central, and by Frobenius' Theorem, the only non-central division algebra over $\mathbb{R}$ is $\mathbb{C}$. Hence $D_0 \cong \mathbb{C}$ and $D_0 = Z(D_0) = \mathbb{R} \cdot 1_A + \mathbb{R} \cdot \tilde{v}$ whenever D is a super central super division algebra over $\mathbb{R}$ that is ordinary central simple, and not purely even. Let us show that, in this situation, for any non-zero $x \in D_1$, $x^2 \in (\mathbb{R} \cdot 1_D)\backslash\{0\}$.

For any non-zero $x \in D_1$, $x^2 \in D_0$, so $x^2 = a + b\tilde{v}$ for some real numbers a, b. Since $(x^2)x = x(x^2)$, $(x^2)x = (a + b\tilde{v})x = ax + b\tilde{v}x = xa - xb\tilde{v} = x(a - b\tilde{v}) = x(x^2) = x(a + b\tilde{v})$, hence $a - b\tilde{v} = a + b\tilde{v}$, implying $b = 0$ and $x^2 = a \neq 0$. Hence, for any non-zero $x \in D_1$, $x^2 \in (\mathbb{R} \cdot 1_D)\backslash\{0\}$.

Let us show that there are two possibilities: either, for every non-zero $x \in D_1$, x^2 is a positive real number, or, for every non-zero $x \in D_1$, x^2 is a negative real number. To show this, it is enough to show that for any two non-zero elements $x, y \in D_1$, x^2/y^2 is positive. For any two non-zero elements $x, y \in D_1$, $xy^{-1} \in D_0$, so there exists real numbers a, b such that $x = (a + b\tilde{v})y$. Note that $x^2 = (a + b\tilde{v})y(a + b\tilde{v})y = (a + b\tilde{v})(a - b\tilde{v})y^2 = (a^2 + b^2)y^2$. Since x^2 is non-zero, $(a^2 + b^2)$ is non-zero, which additionally means it is positive, and $x^2/y^2 = (a^2 + b^2) > 0$, which means x^2 and y^2 are either both positive or both negative.

If, for some arbitrary non-zero element $x \in D_1$, $x^2 = a > 0$, then we will let $v^+ = \frac{1}{\sqrt{a}}x$. We can see that $(v^+)^2 = 1$, and hence $D = D_0 \oplus D_0 \cdot v^+$ where $D_0 \cong \mathbb{C}$. For any super central super division algebra D' over $\mathbb{R}$ whose 0-component is isomorphic to $\mathbb{C}$ and every non-zero degree 1 element squares to a positive number, $D' \cong D_0 \oplus D_0 v^+$. We will denote this super division algebra as $\mathbb{C} \oplus \mathbb{C}v^+$.

Similarly, if every element of D_1 squares to a negative number, then we can find an element $v^- \in D_1$ that squares to -1, and any super central super division algebra over $\mathbb{R}$ that is in that situation is isomorphic to $D_0 \oplus D_0 v^-$ where $D_0 \cong \mathbb{C}$. We will denote this super division algebra as $\mathbb{C} \oplus \mathbb{C}v^{-1}$. One can check that $\mathbb{C} \oplus \mathbb{C}v^+ \ncong \mathbb{C} \oplus \mathbb{C}v^-$. Also, both of these super division algebras are not isomorphic to $\mathbb{C} \oplus \mathbb{C}v$, since $Z(\mathbb{C} \oplus \mathbb{C}v) = \mathbb{C} \oplus \mathbb{C}v \cong \mathbb{C} \times \mathbb{C}$, whereas $Z(\mathbb{C} \otimes \mathbb{C}v^+) \cong Z(\mathbb{C} \otimes \mathbb{C}v^-) \cong \mathbb{C}$.

Hence we have constructed the two super central super division algebras over $\mathbb{R}$ that are ordinary central simple, and are not purely even. One can check that $\mathbb{C} \oplus \mathbb{C}v^+ \cong M_2(\mathbb{R})$, and $\mathbb{C} \oplus \mathbb{C}v^- \cong \mathbb{H}$. Now, let us find the four remaining super central super division algebras over $\mathbb{R}$ that are not ordinary central simple.

2.4.2 Super central super division algebras over $\mathbb{R}$ that are not ordinary central simple

In this situation, the super division algebra is not purely even, D_0 is a central division algebra over $\mathbb{R}$, and $D \cong D_0 \otimes (\mathbb{R} \cdot 1_D \oplus \mathbb{R} \cdot v)$ where v is a non-zero element of $Z(D) \cap D_1$ such that v^2 is a non-zero element of $\mathbb{R} \cdot 1_D$. If $v^2 = a > 0$, then we will let $v^+ = \frac{1}{\sqrt{a}} v$, and we can see that $(v^+)^2 = 1$. If $v^2 = -a < 0$ for some positive number a, then we will let $v^- = \frac{1}{\sqrt{a}} v$, and we can see that $(v^+)^2 = -1$. We can now construct the four super central super division algebras over $\mathbb{R}$ that are not ordinary central simple:

- $\mathbb{R} \otimes (\mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot v^+) = \mathbb{R} \oplus \mathbb{R}v^+$,
- $\mathbb{R} \otimes (\mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot v^-) = \mathbb{R} \oplus \mathbb{R}v^-$,
- $\mathbb{H} \otimes (\mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot v^+) = \mathbb{H} \oplus \mathbb{H}v^+$, and
- $\mathbb{H} \otimes (\mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot v^-) = \mathbb{H} \oplus \mathbb{H}v^-$.

2.4.3 Group Structure of $BW(\mathbb{R})$

So far, we only know that $|BW(\mathbb{R})| = 2|Br(\mathbb{R})| \cdot |\mathbb{R}^\times/\mathbb{R}^{\times 2}| = 2 \cdot 2 \cdot 2 = 8$. Let us explore the group structure of $BW(\mathbb{R})$.

We note that $(\mathbb{R}\oplus\mathbb{R}v^+) \hat{\otimes} (\mathbb{R}\oplus\mathbb{R}v^+) = \mathbb{C}\oplus\mathbb{C}v^+$, and $(\mathbb{C}\oplus\mathbb{C}v^+) \hat{\otimes} (\mathbb{R}\oplus\mathbb{R}v^+) = \mathbb{H}\oplus\mathbb{H}v^-$. Hence, in $BW(\mathbb{R})$, $[\mathbb{R} \oplus \mathbb{R}v^+]^2 = [\mathbb{C} \oplus \mathbb{C}v^+]$ and $[\mathbb{R} \oplus \mathbb{R}v^+]^3 = [\mathbb{H} \oplus \mathbb{H}v^-]$. Note that $\mathbb{H} \oplus \mathbb{H}v^- = (\mathbb{H} \oplus 0) \hat{\otimes} (\mathbb{R} \oplus \mathbb{R}v^-)$, hence, in $BW(\mathbb{R})$, $[\mathbb{R} \oplus \mathbb{R}v^+]^4 = [\mathbb{R} \oplus \mathbb{R}v^+]^3[\mathbb{R} \oplus \mathbb{R}v^+] = [\mathbb{H}\oplus\mathbb{H}v^-][\mathbb{R}\oplus\mathbb{R}v^+] = [\mathbb{H}\oplus 0][\mathbb{R}\oplus\mathbb{R}v^-][\mathbb{R}\oplus\mathbb{R}v^+] = [\mathbb{H}\oplus 0][\mathbb{R}\oplus\mathbb{R}v^+]^{-1}[\mathbb{R}\oplus\mathbb{R}v^+] = [\mathbb{H}\oplus 0]$.

Also note that $(\mathbb{R} \oplus \mathbb{R}v^-) \hat{\otimes} (\mathbb{R} \oplus \mathbb{R}v^-) = \mathbb{C} \oplus \mathbb{C}v^-$, and $(\mathbb{C} \oplus \mathbb{C}v^-) \hat{\otimes} (\mathbb{R} \oplus \mathbb{R}v^-) = \mathbb{H}\oplus\mathbb{H}v^+$. Note also that $(\mathbb{H}\oplus\mathbb{H}v^+) \cong (\mathbb{H}\oplus 0) \hat{\otimes} (\mathbb{R}\oplus\mathbb{R}v^+)$, hence, in $BW(\mathbb{R})$, $[\mathbb{R}\oplus\mathbb{R}v^+]^5 = [\mathbb{R}\oplus\mathbb{R}v^+]^4[\mathbb{R}\oplus\mathbb{R}v^+] = [\mathbb{H}\oplus 0][\mathbb{R}\oplus\mathbb{R}v^+] = [\mathbb{H}\oplus\mathbb{H}v^+]$. We can compute the higher powers of $[\mathbb{R} \oplus \mathbb{R}v^+]$ in $BW(\mathbb{R})$:

$[\mathbb{R}\oplus\mathbb{R}v^+]^6 = [\mathbb{R}\oplus\mathbb{R}v^+]^5[\mathbb{R}\oplus\mathbb{R}v^+] = [\mathbb{H}\oplus\mathbb{H}v^+][\mathbb{R}\oplus\mathbb{R}v^+] = [\mathbb{C}\oplus\mathbb{C}v^-][\mathbb{R}\oplus\mathbb{R}v^-][\mathbb{R}\oplus \mathbb{R}v^+] = [\mathbb{C} \oplus \mathbb{C}v^-][\mathbb{R} \oplus \mathbb{R}v^+]^{-1}[\mathbb{R} \oplus \mathbb{R}v^+] = [\mathbb{C} \oplus \mathbb{C}v^-]$.
$[\mathbb{R}\oplus\mathbb{R}v^+]^7 = [\mathbb{R}\oplus\mathbb{R}v^+]^6[\mathbb{R}\oplus\mathbb{R}v^+] = [\mathbb{C}\oplus\mathbb{C}v^-][\mathbb{R}\oplus\mathbb{R}v^+] = [\mathbb{R}\oplus\mathbb{R}v^-][\mathbb{R}\oplus\mathbb{R}v^-][\mathbb{R}\oplus\mathbb{R}v^+] = [\mathbb{R}\oplus\mathbb{R}v^-][\mathbb{R}\oplus\mathbb{R}v^+]^{-1}[\mathbb{R}\oplus\mathbb{R}v^+] = [\mathbb{R}\oplus\mathbb{R}v^-] = [\mathbb{R}\oplus\mathbb{R}v^+]^{-1}$. Hence $[\mathbb{R}\oplus\mathbb{R}v^+]^8 = [\mathbb{R}\oplus 0]$, and hence $BW(\mathbb{R}) \cong C_8$, and $\mathbb{R} \oplus \mathbb{R}v^+$ generates the entire group.

In summary, the powers of $[\mathbb{R} \oplus \mathbb{R}v^+]$ are:

- $[\mathbb{R} \oplus \mathbb{R}v^+]^1 = [\mathbb{R} \oplus \mathbb{R}v^+]$,
- $[\mathbb{R} \oplus \mathbb{R}v^+]^2 = [\mathbb{C} \oplus \mathbb{C}v^+]$,
- $[\mathbb{R} \oplus \mathbb{R}v^+]^3 = [\mathbb{H} \oplus \mathbb{H}v^-]$,
- $[\mathbb{R} \oplus \mathbb{R}v^+]^4 = [\mathbb{H} \oplus 0]$,
- $[\mathbb{R} \oplus \mathbb{R}v^+]^5 = [\mathbb{H} \oplus \mathbb{H}v^+]$,
- $[\mathbb{R} \oplus \mathbb{R}v^+]^6 = [\mathbb{C} \oplus \mathbb{C}v^-]$,
- $[\mathbb{R} \oplus \mathbb{R}v^+]^7 = [\mathbb{R} \oplus \mathbb{R}v^-]$,
- $[\mathbb{R} \oplus \mathbb{R}v^+]^8 = [\mathbb{R} \oplus 0]$.

So we now have a complete list of super central simple super division algebras over $\mathbb{R}$:

- $\mathbb{R} \oplus 0$. It is a purely even algebra.
- $\mathbb{R} \oplus \mathbb{R}v^+ \cong \mathbb{R} \times \mathbb{R}$. We graded $(\mathbb{R} \times \mathbb{R})$ in the following way: $(\mathbb{R} \times \mathbb{R})_0 = \mathbb{R} \cdot (1,1) \cong \mathbb{R}$, and $(\mathbb{R} \times \mathbb{R})_1 = \mathbb{R} \cdot (1,-1)$. Also, $v^+ \in Z_D(D_0)$ and $(v^+)^2 = 1$.
- $\mathbb{C} \oplus \mathbb{C}v^+ \cong M_2(\mathbb{R})$. We grade $M_2(\mathbb{R})$ in the following way: $M_2(\mathbb{R})_0 = \{\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) : a, b \in \mathbb{R}\} \cong \mathbb{C}$, and $M_2(\mathbb{R})_1 = \{\left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right) : a, b \in \mathbb{R}\}$. Also, $v^+ x (v^+)^{-1} = \overline{x}$ for any $x \in D_0$ and $(v^+)^2 = 1$
- $\mathbb{H} \oplus \mathbb{H}v^- \cong \mathbb{H} \otimes \mathbb{C} \cong M_2(\mathbb{C})$. We grade $M_2(\mathbb{C})$ in the following way: $M_2(\mathbb{C})_0 = \{\left(\begin{smallmatrix} z & -\overline{w} \\ w & \overline{z} \end{smallmatrix}\right) : z, w \in \mathbb{C}\} \cong \mathbb{H}$, and $M_2(\mathbb{C})_1 = \{\left(\begin{smallmatrix} z & \overline{w} \\ w & -\overline{z} \end{smallmatrix}\right) : z, w \in \mathbb{C}\}$. Also, $v^- \in Z_D(D_0)$ and $(v^-)^2 = -1$.
- $\mathbb{H} \oplus 0$. It is purely even.
- $\mathbb{H} \oplus \mathbb{H}v^+ \cong \mathbb{H} \otimes (\mathbb{R} \oplus \mathbb{R}v^+) \cong \mathbb{H} \otimes (\mathbb{R} \times \mathbb{R}) \cong \mathbb{H} \times \mathbb{H}$. We graded $(\mathbb{H} \times \mathbb{H})$ in the following way: $(\mathbb{H} \times \mathbb{H})_0 = \mathbb{H} \cdot (1,1) \cong \mathbb{H}$, and $(\mathbb{H} \times \mathbb{H})_1 = \mathbb{H} \cdot (1,-1)$. Also, $v^+ \in Z_D(D_0)$ and $(v^+)^2 = 1$
- $\mathbb{C} \oplus \mathbb{C}v^- \cong \mathbb{H}$. We grade $\mathbb{H}$ in the following way: $\mathbb{H}_0 = \{a + bi : a, b \in \mathbb{R}\} \cong \mathbb{C}$, and $\mathbb{H}_1 = \{aj + bk : a, b \in \mathbb{R}\}$. Also, $(v^-) x (v^-)^{-1} = \overline{x}$ for any $x \in D_0$ and $(v^-)^2 = -1$.
- $\mathbb{R} \oplus \mathbb{R}v^- \cong \mathbb{C}$. We grade $\mathbb{C}$ in the following way: $\mathbb{C}_0 = \mathbb{R}$, and $\mathbb{H}_1 = \{ai : a \in \mathbb{R}\}$. Also, $v^- \in Z_D(D_0)$ and $(v^-)^2 = -1$.

In total, there are ten finite-dimensional super division algebras over $\mathbb{R}$. Eight of them are super central, while two of them ($\mathbb{C} \oplus 0$ and $\mathbb{C} \oplus \mathbb{C}v$) are not super central over $\mathbb{R}$.

3 Character Theory of finite groups

3.1 Representation and Character Theory preliminaries

Throughout this section, G is a finite group and F is a field. We may restrict the characteristic of F as required.

Definition 3.1 An F***-representation*** of G is a group homomorphism $\mathfrak{X} : G \to \mathrm{GL}_n(F)$ for some $n \geq 1$. The integer n is the degree of the representation $\mathfrak{X}$.

Given an F-representation $\mathfrak{X}$ of G, its linear extension is the F-algebra homomorphism $\tilde{\mathfrak{X}} : FG \to M_n(F)$ defined by:

$$\tilde{\mathfrak{X}}\left(\sum_{g\in G} c_g e_g\right) := \sum_{g\in G} c_g \mathfrak{X}(g).$$

By convention, we will use the same symbol to denote both an F-representation and its linear extension.

Note that an F-representation $\mathfrak{X}$ of G of degree n determines an n-dimensional FG-module in the following way: given an n-dimensional F-vector space V, we can identify $\mathrm{End}_F(V)$ with $M_n(F)$ (the codomain of $\mathfrak{X}$) since $M_n(F) \cong \mathrm{End}_F(V)$. The corresponding FG-module of $\mathfrak{X}$ is V with the left FG-action defined by:

$$FG \times V \to V, \ \left(\sum_{g\in G} c_g e_g, v\right) \mapsto \mathfrak{X}\left(\sum_{g\in G} c_g e_g\right) v.$$

Conversely an n-dimensional FG-module V determines an F-representation $\mathfrak{X}$ in the following way: first, we define $\mathrm{GL}_F(V)$ to be the group of invertible F-endomorphisms of V. Note that $\mathrm{GL}_F(V)$ and $\mathrm{GL}_n(V)$ are isomorphic as groups, so we can identify $\mathrm{GL}_F(V)$ with $\mathrm{GL}_n(F)$. We define the F-representation $\mathfrak{X} : G \to \mathrm{GL}_F(V)$ corresponding to V by:

$$\mathfrak{X}(g) := (v \mapsto e_g v \ \text{ for any } v \in V).$$

We say that an F-representation of G is ***irreducible*** if its corresponding FG-module is irreducible.

Definition 3.2 Let $\mathfrak{X}$ be an F-representation of G. Then the F***-character*** of $\mathfrak{X}$ is a function $\chi : G \to F$ defined by $\chi(g) = \mathrm{tr}(\mathfrak{X}(g))$, where tr is the matrix trace. If $\mathfrak{X}$ has degree n, we say χ is a character of degree n.

We can see immediately that, when $\mathrm{char}(F) = 0$, $\deg(\chi) = \chi(1)$.

Definition 3.3 A function $f : G \to F$ is called a ***class function*** if $f(x) = f(y)$ whenever x and y are conjugate in G.

Lemma 3.4 *Let G be a finite group and let F be a field. The following holds:*

1. *Equivalent F-representations of G have the same F-characters.*
2. *Characters are class functions.*

A character χ is said to be ***irreducible*** if it is a character of an irreducible F-representation of G. We denote the set of all the distinct irreducible F-characters of G as $\mathrm{Irr}_F(G)$. The trivial character of the trivial representation of G is denoted as $\mathbb{1}_G$.

We now specialise to the case where $F = \mathbb{C}$.

Definition 3.5 Let φ and ψ be $\mathbb{C}$-valued class functions on G. Then the ***inner product of*** φ ***and*** ψ is defined as

$$\langle \varphi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\psi(g)}.$$

Theorem 3.6 (First Orthogonality Relation) *Let $\chi_1, \chi_2 \in \mathrm{Irr}_{\mathbb{C}}(G)$. Then:*

$$\langle \chi_1, \chi_2 \rangle = \begin{cases} 1 & \text{if } \chi_1 = \chi_2, \\ 0 & \text{if } \chi_1 \neq \chi_2. \end{cases}$$

[1, Corollary 2.14]

Corollary 3.7 *Let φ and ψ be any $\mathbb{C}$-characters of G. Then $\langle \varphi, \psi \rangle = \langle \psi, \varphi \rangle$ is a non-negative integer. In addition, $\varphi \in \mathrm{Irr}_{\mathbb{C}}(G)$ if and only if $\langle \varphi, \varphi \rangle = 1$. [1, Corollary 2.17]*

The inner product allows us to quickly check if a $\mathbb{C}$-character of G is irreducible.

Now, given any finite group G, we can use the set $\mathrm{Irr}_{\mathbb{C}}(G)$ to decompose $\mathbb{C}G$ as a direct sum of simple subalgebras:

$$\mathbb{C}G \cong \bigoplus_{\chi \in \mathrm{Irr}_{\mathbb{C}}(G)} M_{\chi(1)}(\mathbb{C}).$$

So for any irreducible $\mathbb{C}G$-module V, let χ be its corresponding $\mathbb{C}$-character. Then we have $\mathbb{C}G(V) \cong M_{\chi(1)}(\mathbb{C})$.

We also hope to express $\mathbb{R}G$ as a direct sum of simple algebras, using information about $\mathrm{Irr}_{\mathbb{C}}(G)$. To do this, we will need to introduce the Frobenius-Schur indicator.

Given a class function φ of a group G, set $\varphi^{(2)}(g) := \varphi(g^2)$. It is straightforward to check that $\varphi^{(2)} : G \to F$ is also a class function.

Definition 3.8 Let χ be a $\mathbb{C}$-character. The ***Frobenius-Schur indicator*** $\epsilon(\chi)$ of χ is defined as

$$\epsilon(\chi) := \frac{1}{|G|} \sum_{x \in G} \chi(g^2) = \langle \chi^{(2)}, \mathbb{1}_G \rangle.$$

Theorem 3.9 (Frobenius-Schur) *Let $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$. Then we have the following:*

1. *$\chi^{(2)}$ is a difference of characters.*
2. *$\epsilon(\chi) = 1, 0$ or -1.*
3. *$\epsilon(\chi) \neq 0$ if and only if χ is real-valued.*

[1, Theorem 4.5]

Definition 3.10 Given a $\mathbb{C}$-representation $\mathfrak{X} : G \to \mathrm{GL}_n(\mathbb{C})$, we say that $\mathfrak{X}$ is a ***real representation*** if there exists an invertible matrix P such that $P^{-1}\mathfrak{X}(g)P$ is a real $n \times n$ matrix for all $g \in G$. Given a $\mathbb{C}$-character χ of G, we say that χ is ***realisable over*** $\mathbb{R}$ if it is a character of a real representation.

Given a $\mathbb{C}$-character χ of G, we say χ is ***real*** (or ***real-valued***) if $\chi(g)$ is a real number for all $g \in G$. We note that every $\mathbb{C}$-character that is realisable over $\mathbb{R}$ is a real character, but not all real characters are realisable over $\mathbb{R}$.

Definition 3.11 Let χ be a real character of G. Then χ is said to be ***quaternionic*** if it is not realisable over $\mathbb{R}$.

We call a $\mathbb{C}$-character of G that is not real a ***complex*** character.

Theorem 3.12 *Let $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$. Then:*

1. *$\epsilon(\chi) = 0$ if and only if χ is a complex character.*
2. *$\epsilon(\chi) = 1$ if and only if χ is realisable over $\mathbb{R}$.*
3. *$\epsilon(\chi) = -1$ if and only if χ is a quaternionic character.*

[1, Theorem 4.5, Corollary 4.15 + Theorem 4.19]

We can now begin to explore $\mathbb{R}G$-modules by taking irreducible $\mathbb{C}G$-modules, and restricting them down to an $\mathbb{R}G$-module.

Before we begin our analysis, let us state a quick definition. Given an irreducible $\mathbb{R}G$-module V, we say that V is of type $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ if $\mathrm{End}_{\mathbb{R}G}(V) = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ respectively.

Now, given a finite group G, let V_χ be an irreducible $\mathbb{C}G$-module with character χ.

- If $\epsilon(\chi) = 1$, then $(V_\chi)_{\downarrow\mathbb{R}} = \tilde{V}_\chi \oplus \tilde{V}_\chi$ where $\tilde{V}_\chi$ is an irreducible $\mathbb{R}G$-module of type $\mathbb{R}$.
- If $\epsilon(\chi) = 0$, then $(V_\chi)_{\downarrow\mathbb{R}} = \tilde{V}_\chi$ where $\tilde{V}_\chi$ is an irreducible $\mathbb{R}G$-module of type $\mathbb{C}$.
- If $\epsilon(\chi) = -1$, then $(V_\chi)_{\downarrow\mathbb{R}} = \tilde{V}_\chi$ where $\tilde{V}_\chi$ is an irreducible $\mathbb{R}G$-module of type $\mathbb{H}$.

Let us go on the other direction; we will outline what can happen when we take an irreducible $\mathbb{R}G$-module, and induce it up to a $\mathbb{C}G$-module.

Let V be an irreducible $\mathbb{R}G$-module.

- If V is of type $\mathbb{R}$, $V^{\uparrow\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V = V_\chi$, where V_χ is an irreducible $\mathbb{C}G$-module with character χ, and $\epsilon(\chi) = 1$.
- If V is of type $\mathbb{C}$, $V^{\uparrow\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V = V_\chi \oplus V_{\overline{\chi}}$, where V_χ and $V_{\overline{\chi}}$ are irreducible $\mathbb{C}G$-modules with characters χ and $\overline{\chi}$ respectively, and $\epsilon(\chi) = \epsilon(\overline{\chi}) = 0$.
- If V is of type $\mathbb{H}$, $V^{\uparrow\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V = V_\chi \oplus V_\chi$, where V_χ is an irreducible $\mathbb{C}G$-module with character χ, and $\epsilon(\chi) = -1$.

We can also begin to describe the simple subalgebra decomposition of $\mathbb{R}G$. First, let us introduce some notation.

Let S be any finite set, and let n be a non-negative integer such that $n \leq |S|$. Then we define $\mathcal{P}_n(S)$ as the set of subsets of S of cardinality n.

Now, we give the decomposition of $\mathbb{R}G$:

$$\mathbb{R}G \cong \bigoplus_{\substack{\chi \in \mathrm{Irr}_{\mathbb{C}}(G) \\ \epsilon(\chi)=1}} M_{\chi(1)}(\mathbb{R}) \oplus \bigoplus_{\substack{\chi \in \mathrm{Irr}_{\mathbb{C}}(G) \\ \epsilon(\chi)=-1}} M_{\frac{\chi(1)}{2}}(\mathbb{H}) \oplus \bigoplus_{\substack{\{\chi,\chi'\} \in \mathcal{P}_2(\mathrm{Irr}_{\mathbb{C}}(G)) \\ \chi'=\overline{\chi} \\ \epsilon(\chi)=\epsilon(\chi')=0}} M_{\chi(1)}(\mathbb{C}).$$

3.2 Introduction to Clifford Theory

Let H be a subgroup of a group G. Given an irreducible $\mathbb{C}$-character χ of H, the restriction of χ to H, denoted as $\chi_{\downarrow H}$, is a character of H. In the same vein, given an irreducible $\mathbb{C}$-character ϑ of H, there is a character of G called the induced character of ϑ. Recall that, given a class function φ of a subgroup H of G, the induced class function $\varphi^{\uparrow G}$ on G is given by

$$\varphi^{\uparrow G}(g) = \frac{1}{|H|}\sum_{x\in G}\varphi^{\circ}\left(xgx^{-1}\right),$$

where φ° is defined as

$$\varphi^{\circ}(g) = \begin{cases}\varphi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H.\end{cases}$$

In 1937, Alfred H. Clifford developed a detailed and effective theory that explored restrictions and inductions of characters when H is a normal subgroup of G.

Definition 3.13 Let N be a normal subgroup of G, and let ϑ be a complex-valued class function of N. Given $g \in G$, we define a function ϑ^g on N by

$$\vartheta^g(n) := \vartheta\left(gng^{-1}\right)$$

for all $n \in N$. Then ϑ^g is called a ***G-conjugate*** of ϑ.

Lemma 3.14 *With G, N and ϑ as above,*

(1) ϑ^g is a class function of N.

(2) $(\vartheta^g)^h = \vartheta^{gh}$ for all g and h in G.

(3) If φ and ϑ are class functions of N, then

$$\langle\varphi^g, \vartheta^g\rangle_N = \langle\varphi, \vartheta\rangle_N.$$

(4) For any class function χ of G,

$$\langle\chi_{\downarrow N}, \vartheta^g\rangle_N = \langle\chi_{\downarrow N}, \vartheta\rangle_N.$$

(5) If ϑ is an ordinary character of N, then so is ϑ^g.

(6) The subset of all $g \in G$ satisfying $\vartheta^g = \vartheta$ is a subgroup of G containing N. This subgroup is called the stabilizer of ϑ in G, and we denote it as G_ϑ.

[1, Lemma 6.1]

Given a normal subgroup $N \trianglelefteq G$ and a class function ϑ of N, we have

$$G = \bigcup_{i=1}^{m} G_\vartheta g_i,$$

where $G_\vartheta g_1, \ldots, G_\vartheta g_m$ are the right cosets of G_ϑ in G and $m = |G : G_\vartheta|$. Then, given $g \in G$, it can be expressed as sg_i for some $s \in G_\vartheta$. We observe that

$$\vartheta^g = \vartheta^{sg_i} = (\vartheta^s)^{g_i} = \vartheta^{g_i}.$$

We additionally have $\vartheta^{g_i} = \vartheta^{g_j} \iff i = j$. Thus ϑ has $m = |G : G_\vartheta|$ different G-conjugates.

Theorem 3.15 (Clifford's Theorem - Part 1) *Let N be a normal subgroup of G and let χ be an irreducible $\mathbb{C}$-character of G. Let ϑ be an irreducible constituent of $\chi_{\downarrow N}$. Let $\vartheta_1 = \vartheta, \ldots, \vartheta_m$ be the different G-conjugates of ϑ. Then*

$$\chi_{\downarrow N} = e\left(\vartheta_1 + \cdots + \vartheta_m\right),$$

where e is some positive integer. [1, Theorem 6.2]

We can actually expand on Theorem 3.15 by taking into account the stabilizer subgroup G_ϑ of ϑ. Since $\langle \chi_{\downarrow N}, \vartheta \rangle = e > 0$, there must be some irreducible character φ of G_ϑ that satisfies $\langle \chi_{\downarrow G_\vartheta}, \varphi \rangle \neq 0$ and $\langle \varphi_{\downarrow N}, \vartheta \rangle \neq 0$. In other words, $\chi_{\downarrow G_\vartheta}$ has an irreducible constituent φ such that ϑ is an irreducible constituent of $\varphi_{\downarrow N}$. It turns out that $\chi = \varphi^{\uparrow G}$, so that χ is induced from G_ϑ.

Theorem 3.16 (Clifford's Theorem - Part 2) *Using the notation introduced, if φ is an irreducible character of G_ϑ that satisfies*

$$\langle \chi_{\downarrow G_\vartheta}, \varphi \rangle \neq 0 \quad \textit{and} \quad \langle \varphi_{\downarrow N}, \vartheta \rangle \neq 0,$$

then $\chi = \varphi^{\uparrow G}$ and $\varphi_{\downarrow N} = e\vartheta$.

Proof. Since G_ϑ is the stabilizer subgroup of ϑ in G and $\langle \varphi_{\downarrow N}, \vartheta \rangle \neq 0$, it follows that $\varphi_{\downarrow N} = f\vartheta$ for some integer f, by Theorem 3.15 (since φ is an irreducible character of G_ϑ, and the only G_ϑ-conjugate of ϑ is itself). In addition, since $\langle \chi_{\downarrow N}, \vartheta \rangle = e$ and $\varphi_{\downarrow N}$ is a constituent of $\chi_{\downarrow N}$ (which is implied from $\langle \chi_{\downarrow G_\vartheta}, \varphi \rangle \neq 0$), we must have that $f \leq e$. Let us look at $\chi(1)$. Since $\chi_{\downarrow N} = e\left(\vartheta_1 + \cdots + \vartheta_m\right)$, $m = |G : G_\vartheta|$ and $\vartheta^g(1) = \vartheta(g1g^{-1}) = \vartheta(1)$ for all $g \in G$, we get $\chi(1) = e|G : G_\vartheta|\vartheta(1)$.

Now since $\langle \chi_{\downarrow G_\vartheta}, \varphi \rangle_{G_\vartheta} \neq 0$, we have $\langle \chi, \varphi^{\uparrow G} \rangle_G \neq 0$ by Frobenius reciprocity. Hence χ is an irreducible constituent of $\varphi^{\uparrow G}$, which means $\chi(1) \leq \varphi^{\uparrow G}(1) = |G : G_\vartheta|\varphi(1) = |G : G_\vartheta| f\vartheta(1)$. So $\chi(1) = e\,|G : G_\vartheta|\,\vartheta(1) \leq |G : G_\vartheta| f\vartheta(1) \implies e \leq f$. But we already have $f \leq e$, and so $e = f$ and $\chi(1) = e|G : G_\vartheta|\vartheta(1) = |G : G_\vartheta|\varphi(1)$. We can now conclude that $\chi = \varphi^{\uparrow G}$ and $\varphi_{\downarrow N} = e\vartheta$. ■

We note that φ would in fact be the unique irreducible character of G_ϑ with the properties described in Theorem 3.16. Also, we have that $\langle \chi_{\downarrow G_\vartheta}, \varphi \rangle = e$.

Corollary 3.17 *Let N be a normal subgroup of G such that $|G : N| = p$, where p is a prime. Then, for each $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$, either $\chi_{\downarrow N} \in \mathrm{Irr}_{\mathbb{C}}(N)$ or N has an irreducible character ϑ with the property that $\vartheta^{\uparrow G} = \chi$. In that case, $\chi_{\downarrow N} = \vartheta_1 + \cdots + \vartheta_p$, where the ϑ_i's are the conjugates of ϑ.*

Let us discuss this corollary further. In the case where $\chi_{\downarrow N} = \vartheta \in \mathrm{Irr}_{\mathbb{C}}(G)$, it is straightforward to prove the following: Let $\mathbb{1}_G = \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p$ be the different linear characters of G/N inflated to G. Then the characters $\chi\varepsilon_1 = \chi, \chi\varepsilon_2, \ldots, \chi\varepsilon_p$ are all distinct and they are the extensions of ϑ to G. We also get that $\vartheta^{\uparrow G} = \chi + \chi\varepsilon_2 + \cdots + \chi\varepsilon_p$.

Let us now specialise in the case where $|G : N| = 2$. Given any $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ we have the following two possibilities:

1. If $\chi_{\downarrow N} \notin \mathrm{Irr}_{\mathbb{C}}(N)$, then for some $\vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$, $\chi_{\downarrow N} = \vartheta + \vartheta^g$ for some $g \in G \backslash N$. In this case, $G_\vartheta = N$, and $\vartheta^{\uparrow G} = (\vartheta^g)^{\uparrow G} = \chi$.

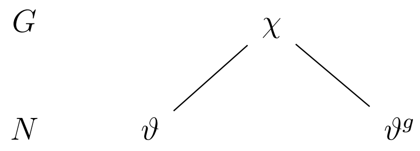
2. If $\chi_{\downarrow N} = \vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$, then $G_\vartheta = G$, and $\vartheta^{\uparrow G} = \chi + \mathrm{sgn}\chi$ where $\mathrm{sgn}\chi$ is the $\mathbb{C}$-character defined by:

$$\mathrm{sgn}\chi(g) := \begin{cases} \chi(g) & \text{if } g \in N, \\ -\chi(g) & \text{if } g \in G \backslash N. \end{cases}$$

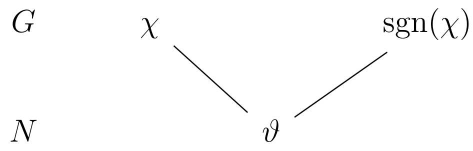
Note that $(\mathrm{sgn}\chi)_{\downarrow N} = \chi_{\downarrow N} = \vartheta$.

We can visualise the induction and restriction relationships present in these two cases using the poset diagrams below:

Case 1:



Case 2:



3.3 Super representations and super characters

In section 3.1, we saw how an F-representation of a finite group G determines an FG-module, and vice versa. We now want to define a super representation in such a way that we can readily construct a supermodule from a super representation, and vice versa.

Recall that given an FG-module V, the map $FG \to \mathrm{End}_F(V)$ defined by $x \mapsto (v \mapsto xv \text{ for all } v \in V)$ is an F-algebra homomorphism. This can be turned into an F-representation by restriction to the basis elements $\{e_g : g \in G\}$. We want to say something similar regarding $F[G,N]$-supermodules.

Let G be a group, and let $N \lhd G$ be a normal subgroup of index 2. Let $V = V_0 \oplus V_1$ be a finite dimensional $F[G,N]$-supermodule. Let us first note that $\dim_F(V_0) = \dim_F(V_1)$ since, for any $g \in G\backslash N$, the linear map $V_0 \to V_1$, $v_0 \mapsto e_g v_0$ is bijective.

Now let $n = \dim_F(V_0)$. We claim that the map $\varphi : F[G,N] \to \mathrm{End}^{\mathfrak{s}}_F(V^{(n,n)})$ defined by: $\varphi(x) := (v \mapsto xv \text{ for all } v \in V)$ is an F-superalgebra homomorphism. Since V is an FG-module, φ is an ordinary F-algebra homomorphism. We just need to show that φ preserves grading. Note that for any $x \in F[G,N]_0$, $x \cdot V_0 \subset V_0$ and $x \cdot V_1 \subset V_1$. This implies that $\varphi(F[G,N]_0) \subset \mathrm{End}^{\mathfrak{s}}_F(V^{(n,n)})_0$. Also, for any $x \in F[G,N]_1$, $x \cdot V_0 \subset V_1$ and $x \cdot V_1 \subset V_0$. This means $\varphi(F[G,N]_1) \subset \mathrm{End}^{\mathfrak{s}}_F(V^{(n,n)})_1$, and thus φ is an F-superalgebra homomorphism.

By restricting φ to the basis elements $\{e_g : g \in G\}$, we begin to see how we could define an F-super representation. We identify $M_{(n,n)}(F)$ with $\mathrm{End}^{\mathfrak{s}}_F(V^{(n,n)})$. We can see that for any $g \in N$, $\varphi(e_g) \in M_{(n,n)}(F)_0 \cap \mathrm{GL}_{2n}(F)$, and for any $g \in G\backslash N$, $\varphi(e_g) \in M_{(n,n)}(F)_1 \cap \mathrm{GL}_{2n}(F)$. We can now define F-super representations.

Definition 3.18 Let G be a finite group, and let $N \trianglelefteq G$ be a normal subgroup of index 2. Then an ***F-super representation*** of the pair (G,N) is an F-representation $\mathfrak{X} : G \to \mathrm{GL}_{2n}(F)$ that additionally satisfies $\mathfrak{X}(N) \subset M_{(n,n)}(F)_0$ and $\mathfrak{X}(G\backslash N) \subset M_{(n,n)}(F)_1$ for some $n \geq 1$.

This definition of an F-super representation allows us to readily construct the corresponding $F[G,N]$-supermodule.

An ***F-super character*** of (G,N) is an F-character of an F-super representation of (G,N).

Let $g \in G\backslash N$, and let $\mathfrak{X}$ be an F-representation of N of degree n. We let $\mathfrak{X}^g : N \to \mathrm{GL}_n(F)$ be the F-representation defined by $\mathfrak{X}^g(n) := \mathfrak{X}(g^{-1}ng)$ for any $n \in N$. Given an FN-module V with corresponding representation $\mathfrak{X}$, we let V^g denote the FN-module with corresponding representation $\mathfrak{X}^\mathfrak{g}$. It is an exercise to show that, for any two elements $g, h \in G\backslash N$ and for any FN-module V, $V^g \cong V^h$. For any $g \in G\backslash N$, we call V^g a G-conjugate module of V.

Theorem 3.19 *Let $V = V_0 \oplus V_1$ be an $F[G,N]$-supermodule. Then, for any $g \in G\backslash N$, $V_1 \cong V_0^g$ as FN-modules.*

Proof. Let $\mathfrak{X}$ denote the representation of G corresponding to V. Additionally, let $\mathfrak{X}_0$ and $\mathfrak{X}_1$ denote the representations of N corresponding to V_0 and V_1 respectively so that for any $n \in N$, $\mathfrak{X}(e_n)v_0 = \mathfrak{X}_0(e_n)v_0$ for any $v_0 \in V_0$ and $\mathfrak{X}(e_n)v_1 = \mathfrak{X}_1(e_n)v_1$ for any $v_1 \in V_1$. Now let $g \in G\backslash N$. Note that $V_1 = \mathfrak{X}(e_g) \cdot V_0$. Let $\varphi : V_1 \to V_0^g$ be the linear map defined by: $\varphi(v_1) := \mathfrak{X}(e_g^{-1})v_1$.

Let us show that φ is an FN-module isomorphism. Note that, for any $n \in N$, $\varphi(\mathfrak{X}_1(e_n)v_1) = \mathfrak{X}(e_g^{-1})\mathfrak{X}_1(e_n)v_1 = \mathfrak{X}(e_g^{-1})\mathfrak{X}(e_n)v_1 = \mathfrak{X}(e_g^{-1}e_n)v_1 = \mathfrak{X}(e_g^{-1}e_n e_g e_g^{-1})v_1 = \mathfrak{X}(e_g^{-1}e_n e_g)\mathfrak{X}(e_g^{-1})v_1 = \mathfrak{X}_0(e_g^{-1}e_n e_g)\varphi(v_1) = \mathfrak{X}^g(e_n)\varphi(v_1)$.

Hence $V_1 \cong V_0^g$ as FN-modules. ∎

From this, we can also say that $V_0 \cong V_1^g$ for any $g \in G\backslash N$. We deduce from Theorem 3.19 that, for any $F[G,N]$-supermodule $V = V_0 \oplus V_1$, $V = V_0^{\uparrow G} = V_1^{\uparrow G}$. This gives us a criteria outlining when an $\mathbb{C}$-character is a super character.

Theorem 3.20 *An F-character χ of G is an F-super character of (G,N) if and only if there exists an F-character ϑ of N such that $\vartheta^{\uparrow G} = \chi$.*

Note that by Lemma 2.15, an $F[G,N]$-supermodule $V = V_0 \oplus V_1$ is super irreducible if and only if V_0 and V_1 are irreducible FN-modules.

We say that an F-super character χ is ***super irreducible*** if its corresponding $F[G,N]$-supermodule is super irreducible. It is immediate that a super character χ of (G,N) is super irreducible if and only if there exists $\vartheta \in \mathrm{Irr}_F(N)$ such that $\vartheta^{\uparrow G} = \chi$.

3.4 The Gow indicator and the Super Frobenius-Schur indicator

Recall that given a $\mathbb{C}$-character $\chi \in \mathrm{Irr}_\mathbb{C}(G)$, the Frobenius-Schur indicator $\epsilon(\chi)$ is the average value of $\chi(g^2)$ as g runs over all of G. R. Gow explored a related indicator in [6]:

Definition 3.21 Let N be a subgroup of index 2 in a finite group G and let ϑ be an irreducible character of N. Then the ***Gow indicator*** $\eta(\vartheta)$ of ϑ is:

$$\eta(\vartheta) := \frac{1}{|N|} \sum_{g \in G\backslash N} \vartheta(g^2).$$

Lemma 3.22 *It turns out that $\eta(\vartheta) = 0, 1$ or -1. Moreover, $\eta(\chi) \neq 0$ if and only if ϑ is G-conjugate to $\overline{\vartheta}$.*

Let us explore the different possibilities given the setting of Definition 3.21.

If $G_\vartheta = N$, then ϑ would have one other G-conjugate. Let $g \in G\backslash N$. Then ϑ^g would be that other G-conjugate, and $\vartheta \neq \vartheta^g$. By Clifford Theory (Theorem 3.16), $\vartheta^{\uparrow G}$ would be an irreducible character of G. Since we have $\epsilon(\vartheta) = \epsilon(\vartheta^g)$ and $\eta(\vartheta) = \eta(\vartheta^g)$, we also get $\epsilon(\vartheta^{\uparrow G}) = \epsilon(\vartheta) + \eta(\vartheta)$. Table 1 below lists the different possible values of $\epsilon(\vartheta^{\uparrow G})$, $\epsilon(\vartheta)$ and $\eta(\vartheta)$.

If $G_\vartheta = G$, then ϑ can be extended to an irreducible character φ of G. In this case, we would have $2\epsilon(\varphi) = \epsilon(\vartheta) + \eta(\vartheta)$. Table 2 below lists the different possible values of $\epsilon(\varphi)$, $\epsilon(\vartheta)$ and $\eta(\vartheta)$.

Table 1: When $G_\vartheta = N$

$\epsilon\left(\vartheta^{\uparrow G}\right)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
1	0	1
1	1	0
0	0	0
-1	0	-1
-1	-1	0

Table 2: When $G_\vartheta = G$

$\epsilon(\varphi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
1	1	1
-1	-1	-1
0	0	0
0	1	-1
0	-1	1

Let G be a group, and let $N \triangleleft G$ be a normal subgroup of index 2. Let ϑ be an irreducible $\mathbb{C}$-character of N. Then the ***Super Frobenius-Schur indicator*** of the $\mathbb{C}$-super character $\vartheta^{\uparrow G}$ is defined as follows:

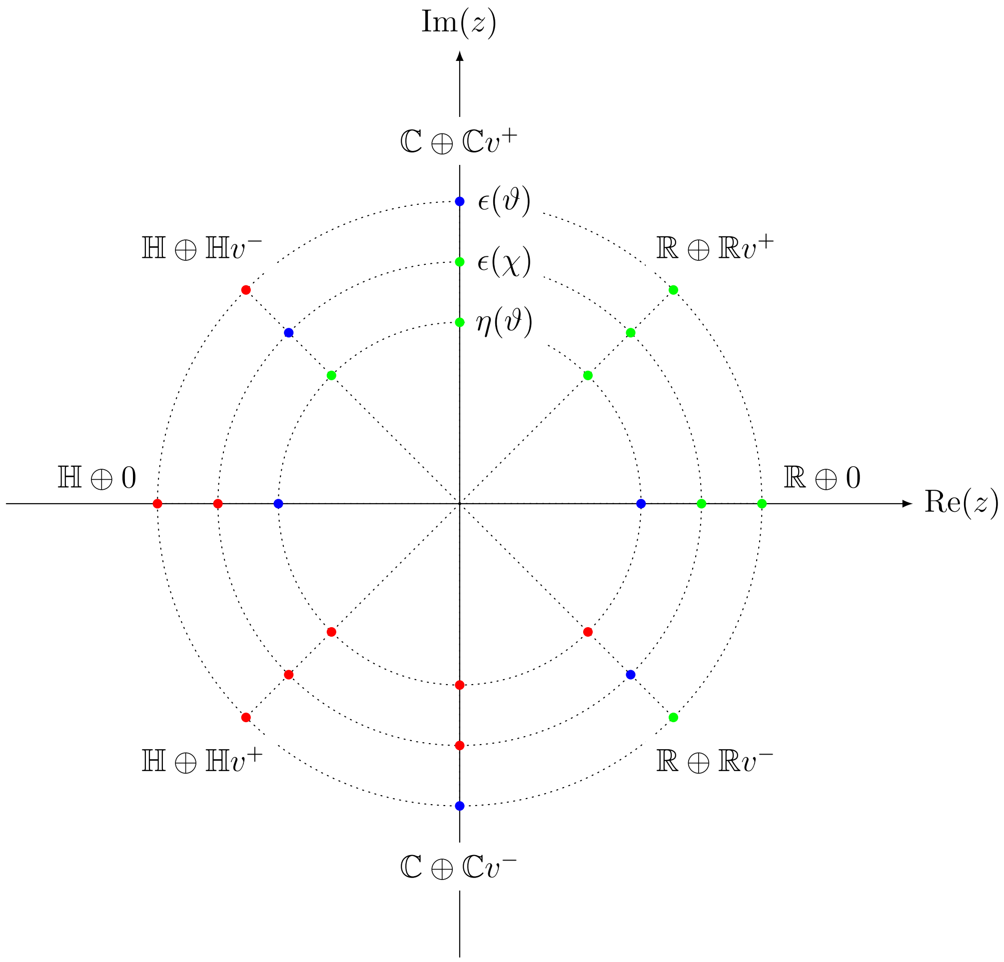
1. If both $\epsilon(\vartheta)$ and $\eta(\vartheta)$ are equal to zero, then $\mathcal{S}(\vartheta^{\uparrow G}) = 0$.
2. If at least one of $\epsilon(\vartheta)$ and $\eta(\vartheta)$ is non-zero, then $\mathcal{S}(\vartheta^{\uparrow G}) = \frac{\epsilon(\vartheta)+i\eta(\vartheta)}{|\epsilon(\vartheta)+i\eta(\vartheta)|}$.

The possible values of $\mathcal{S}(\vartheta)$ are the eighth roots of unity, and 0. While the Super Frobenius-Schur indicator alone doesn't distinguish the two separate cases where the indicator is 0, we can distinguish them by checking if $\vartheta^{\uparrow G}$ is an irreducible $\mathbb{C}$-character of G or not.

We can create a new table that lists the possible values of $\mathcal{S}(\vartheta)$, $\epsilon(\vartheta)$ and $\eta(\vartheta)$, with the help of our two previous tables. We shall also list the ten possible values of $\epsilon(\chi)$, where $\chi = \vartheta^{\uparrow G}$ if $G_\vartheta = N$ and $\chi = \varphi$ if $G_\vartheta = G$. Here, $\omega = e^{\frac{2\pi i}{8}}$. Following the notation from [7], we will define a variable $q(\vartheta)$ to be equal to 0 if $G_\vartheta = N$, and 1 if $G_\vartheta = G$.

$\mathcal{S}(\vartheta)$	$q(\vartheta)$	$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
0	0	0	0	0
0	1	0	0	0
1	0	1	1	0
ω	1	1	1	1
$\omega^2 = i$	0	1	0	1
ω^3	1	0	-1	1
$\omega^4 = -1$	0	-1	-1	0
ω^5	1	-1	-1	-1
$\omega^6 = -i$	0	-1	0	-1
ω^7	1	0	1	-1

We can visualize the last eight cases using the diagram below. The eight roots of unity each have three coloured dots associated with them. These dots tell you the values of $\epsilon(\vartheta)$, $\epsilon(\chi)$ and $\eta(\vartheta)$ that occur. Green, blue and red dots represent the values 1, 0 and -1 respectively. We will also write down the corresponding super central super division algebra of $\mathbb{R}$.



Let us give an example of each of the ten possibilities, for specific G and N:

1.

$\mathcal{S}(\vartheta)$	$q(\vartheta)$	$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
0	0	0	0	0

Let $G = S_3 \times C_3$ and $N = C_3 \times C_3$. The presentation of S_3 is $\langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2)^3 = 1\rangle$, while the presentation of C_3 is $\langle g : g^3 = 1\rangle$. Let ψ_1 be the character of $S_3 \times \{1\}$ with the given character values:

	$\{(1,1)\}$	$\{(s_1,1)\}$	$\{(s_1 s_2, 1)\}$
χ	2	0	-1

Let $\zeta = e^{\frac{2\pi i}{3}}$, and let ψ_2 be the (linear) character of $\{1\} \times C_3$ defined by $\psi_2(1, g) = \zeta$. Now let $\chi = \psi_1 \otimes \psi_2$. This is an irreducible character of G. When we restrict χ to N, $\chi_{\downarrow N}$ would be a sum of two G-conjugate characters of N.

To describe these two G-conjugate characters, let ϕ be the (linear) character of $C_3 \times \{1\} \leq S_3 \times \{1\}$ defined by $\phi(s_1 s_2, 1) = \zeta$. Then the two G-conjugate characters of N that make up $\chi_{\downarrow N}$ are $\vartheta = \phi \otimes \psi_2$ and $\vartheta^g = \phi^2 \otimes \psi_2$. So $\chi_{\downarrow N} = \vartheta + \vartheta^g$, and we are in the situation where $q(\vartheta) = 0$. χ is a complex character of G, while ϑ and ϑ^g are complex characters of N. Hence $\epsilon(\chi) = 0$ and $\epsilon(\vartheta) = \epsilon(\vartheta^g) = 0$. This also gives us the value of the Gow indicator, which is $\eta(\vartheta) = 0$.

2.

$\mathcal{S}(\vartheta)$	$q(\vartheta)$	$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
0	1	0	0	0

Let $G = C_8$ and $N = C_4$. The presentation of C_8 is $\langle g : g^8 = 1\rangle$. Let $\zeta = e^{\frac{2\pi i}{8}}$, and consider the (linear) character χ of C_8 defined by $\chi(g) = \zeta$. When we restrict χ to C_4 (where C_4 is the subgroup generated by g^2), we get a single character $\vartheta = \chi_{\downarrow N}$, which means $q(\vartheta) = 1$. χ and ϑ are $\mathbb{C}$-characters of G and N respectively, and hence $\epsilon(\chi) = 0$ and $\epsilon(\vartheta) = 0$. This also gives us the value of the Gow indicator, which is $\eta(\vartheta) = 0$.

3.

$\mathcal{S}(\vartheta)$	$q(\vartheta)$	$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
1	0	1	1	0

Let $G = D_8$ and $N = K_4$. The presentation of D_8 is $\langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. Let χ be the character of G with the given character values:

	$\{1\}$	$\{a^2\}$	$\{a, a^3\}$	$\{b, a^2b\}$	$\{ab, a^3b\}$
χ	2	-2	0	0	0

The subgroup K_4 contained in D_8 is the subgroup generated by a^2 and b. When we restrict χ to N, $\chi_{\downarrow N}$ would be a sum of two G-conjugate characters of N. So $\chi_{\downarrow N} = \vartheta + \vartheta^g$, where $\chi_{\downarrow N}$, ϑ and ϑ^g have the following character values:

	$\{1\}$	$\{a^2\}$	$\{b\}$	$\{a^2b\}$
$\chi_{\downarrow N}$	2	-2	0	0
ϑ	1	-1	1	-1
ϑ^g	1	-1	-1	1

We therefore are in the situation where $q(\vartheta) = 0$, and we can compute that $\epsilon(\chi) = 1$, $\epsilon(\vartheta) = 1$ and $\eta(\vartheta) = 0$.

4.

$\mathcal{S}(\vartheta)$	$q(\vartheta)$	$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
ω	1	1	1	1

Let $G = C_2$ and $N = \{1\}$. The presentation of C_2 is $\langle g : g^2 = 1\rangle$. Consider the (linear) character χ of C_2 defined by $\chi(g) = -1$. When we restrict χ to N, we get a single character $\vartheta = \chi_{\downarrow N}$, which means $q(\vartheta) = 1$. χ and ϑ are real characters of G and N respectively, and hence $\epsilon(\chi) = 1$ and $\epsilon(\vartheta) = 1$. This also gives us the value of the Gow indicator, which is $\eta(\vartheta) = 1$.

5.

$\mathcal{S}(\vartheta)$	$q(\vartheta)$	$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
$\omega^2 = i$	0	1	0	1

Let $G = D_8$ and $N = C_4$. The presentation of D_8 is $\langle a, b : a^4 = b^2 = 1, bab^{-1} = a^{-1}\rangle$. Let χ be the character of G with the given character values:

	$\{1\}$	$\{a^2\}$	$\{a, a^3\}$	$\{b, a^2b\}$	$\{ab, a^3b\}$
χ	2	-2	0	0	0

The subgroup C_4 contained in D_8 is the subgroup generated by a. When we restrict χ to N, $\chi_{\downarrow N}$ would be a sum of two G-conjugate characters in N. So $\chi_{\downarrow N} = \vartheta + \vartheta^g$, where $\chi_{\downarrow N}$, ϑ and ϑ^g have the following character values:

	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
$\chi_{\downarrow N}$	2	0	-2	0
ϑ	1	i	-1	-i
ϑ^g	1	-i	-1	i

We therefore are in the situation where $q(\vartheta) = 0$, and we can compute that $\epsilon(\chi) = 1$, $\epsilon(\vartheta) = 0$ and $\eta(\vartheta) = 1$.

6.

$\mathcal{S}(\vartheta)$	$q(\vartheta)$	$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
ω^3	1	0	-1	1

Let $G = SD_{16}$ and let $N = Q_8$. The presentation of SD_{16} is $\langle a, d : a^8 = b^2 = 1, bab^{-1} = a^3\rangle$. Let χ be the character of G with the given character values:

	$\{1\}$	$\{a^4\}$	$\{a^2, a^6\}$	$\{a, a^3\}$	$\{a^5, a^7\}$	$\{b, a^2b, a^4b, a^6b\}$,	$\{ab, a^3b, a^5b, a^7b\}$
χ	2	-2	0	$i\sqrt{2}$	-$i\sqrt{2}$	0	0

The subgroup Q_8 contained in SD_{16} is the subgroup generated by a^2 and ab. When we restrict χ to Q_8, we get a single character $\vartheta = \chi_{\downarrow N}$, which means $q(\vartheta) = 1$. χ is a complex character of G, while ϑ is a quaternionic character of N, and hence $\epsilon(\chi) = 0$ and $\epsilon(\vartheta) = -1$. This also gives us the value of the Gow indicator, which is $\eta(\vartheta) = 1$.

7.

$\mathcal{S}(\vartheta)$	$q(\vartheta)$	$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
$\omega^4 = -1$	0	-1	-1	0

Let $G = Q_8 \wr C_2$ and $N = Q_8 \times Q_8$. Within N, there is a subgroup $H = Q_8 \times \{1\} \cong Q_8$. The presentation of Q_8 is $\langle a, b : a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1}\rangle$. Let ψ be the character of H with the given character values:

	$\{1\}$	$\{a^2\}$	$\{a, a^3\}$	$\{b, b^3\}$	$\{ab, a^3b\}$
ψ	2	-2	0	0	0

We let $\vartheta = \psi \otimes \mathbb{1}_{Q_8}$, and let $\chi = \vartheta^{\uparrow G}$. ϑ and χ turn out to be irreducible characters of N and G respectively. When we restrict χ to N, $\chi_{\downarrow N}$ would be a sum of two G-conjugate characters in N. So $\chi_{\downarrow N} = \vartheta + \vartheta^g$. Hence we have $q(\vartheta) = 0$. In their respective groups, χ and ϑ are both quaternionic characters, so $\epsilon(\chi) = -1$ and

$\epsilon(\vartheta) = -1$. We also get $\eta(\vartheta) = 0$.

8.

$\mathcal{S}(\vartheta)$	$q(\vartheta)$	$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
ω^5	1	-1	-1	-1

Let $G = Q_8 \times C_2$ and $N = Q_8$. The presentation of Q_8 is $\langle a, b : a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1}\rangle$. Let ϑ be the character of N with the given character values:

	$\{1\}$	$\{a^2\}$	$\{a, a^3\}$	$\{b, b^3\}$	$\{ab, a^3b\}$
ϑ	2	-2	0	0	0

We let $\chi = \mathrm{Inf}(\vartheta)$. It is straightforward to show that χ is an irreducible character of G. We also immediately get $\chi_{\downarrow N} = \vartheta$ and hence $q(\vartheta) = 1$. χ and ϑ are quaternionic characters of G and N respectively, and so $\epsilon(\chi) = -1$ and $\epsilon(\vartheta) = -1$. We also get $\eta(\vartheta) = -1$.

9.

$\mathcal{S}(\vartheta)$	$q(\vartheta)$	$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
$\omega^6 = -i$	0	-1	0	-1

Let $G = Q_8$ and $N = C_4$. The presentation of Q_8 is $\langle a, b : a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1}\rangle$. Let χ be the character of G with the given character values:

	$\{1\}$	$\{a^2\}$	$\{a, a^3\}$	$\{b, b^3\}$	$\{ab, a^3b\}$
χ	2	-2	0	0	0

The subgroup C_4 contained in Q_8 is the subgroup generated by a When we restrict χ to N, $\chi_{\downarrow N}$ would be a sum of two G-conjugate characters in N. So $\chi_{\downarrow N} = \vartheta + \vartheta^g$, where $\chi_{\downarrow N}$, ϑ and ϑ^g have the following character values:

	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
$\chi_{\downarrow N}$	2	0	-2	0
ϑ	1	i	-1	-i
ϑ^g	1	-i	-1	i

We therefore are in the situation where $q(\vartheta) = 0$, and we can compute that $\epsilon(\chi) = -1$, $\epsilon(\vartheta) = 0$ and $\eta(\vartheta) = -1$.

10.

$\mathcal{S}(\vartheta)$	$q(\vartheta)$	$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
ω^7	1	0	1	-1

Let $G = C_4$ and $N = C_2$. The presentation of C_8 is $\langle g : g^4 = 1\rangle$. Consider the (linear) character χ of C_4 defined by $\chi(g) = i$. When we restrict χ to C_2, we get a single character $\vartheta = \chi_{\downarrow N}$, which means $q(\vartheta) = 1$. χ is a complex character of G, while ϑ is a real character of N, and hence $\epsilon(\chi) = 0$ and $\epsilon(\vartheta) = 1$. This also gives us the value of the Gow indicator, which is $\eta(\vartheta) = -1$.

4 The Superalgebras $\mathbb{C}[G,N]$ and $\mathbb{R}[G,N]$

Throughout this section, G is a finite group, and $N \triangleleft G$ is a normal subgroup of index 2. We aim to describe the decomposition of the superalgebras $\mathbb{C}[G,N]$ and $\mathbb{R}[G,N]$. We will also describe the super irreducible $\mathbb{C}[G,N]$- and $\mathbb{R}[G,N]$-supermodules.

4.1 Simple subalgebras of $\mathbb{C}G$ and $\mathbb{R}G$

In this subsection, we will first recap how we obtain the simple subalgebras of $\mathbb{C}G$ and $\mathbb{R}G$ for arbitrary G. We will use a construction that comes from [8, Theorem 3.15].

Let χ be an irreducible $\mathbb{C}$-character of G. Its corresponding (primitive) central idempotent e_χ is:

$$e_\chi = \frac{\chi(1)}{|G|}\sum_{g\in G}\overline{\chi(g)}e_g \in Z(\mathbb{C}G).$$

It is known that for any $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$, $e_\chi\mathbb{C}G \cong M_{\chi(1)}(\mathbb{C})$, which means $e_\chi\mathbb{C}G$ is a (central) simple $\mathbb{C}$-subalgebra of $\mathbb{C}G$. Note that if χ is a real character, then e_χ can also be seen as a central idempotent in $\mathbb{R}G$.

Lemma 4.1 *Let χ be an irreducible $\mathbb{C}$-character of G. We have the following:*

- *χ is realisable over $\mathbb{R}$ if and only if $e_\chi\mathbb{R}G \cong M_{\chi(1)}(\mathbb{R})$.*
- *χ is quaternionic if and only if $e_\chi\mathbb{R}G \cong M_{\frac{\chi(1)}{2}}(\mathbb{H})$.*
- *χ is complex if and only if $(e_\chi + e_{\overline{\chi}})\mathbb{R}G \cong M_{\chi(1)}(\mathbb{C})$.*

[8, Theorem 8.16]

All simple $\mathbb{R}$-subalgebras of $\mathbb{R}G$ are of the form $(e_\chi + e_{\overline{\chi}})\mathbb{R}G$ for some $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$.

4.2 Super simple subsuperalgebras of $\mathbb{C}[G,N]$

Let χ be an irreducible $\mathbb{C}$-character of G. Previously we saw that we have two possible situations regarding $\chi_{\downarrow N}$:

- either $\chi_{\downarrow N}$ is a sum of two irreducible characters of N that are G-conjugates of each other, or
- $\chi_{\downarrow N}$ is an irreducible character of N.

These two cases allow us to find the two different possible types of super simple subsuperalgebras of $\mathbb{C}[G,N]$:

- If $\chi_{\downarrow N} = \vartheta + \vartheta^g$ where $\vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$ and ϑ^g is its G-conjugate, then $\vartheta^{\uparrow G} = \chi$. However, for any $g \in G\backslash N$,

  $$\chi(g) = \vartheta^{\uparrow G}(g) = \frac{1}{|N|}\sum_{x\in G}\vartheta^\circ\left(xgx^{-1}\right) = 0$$

  since $xgx^{-1} \in G\backslash N$ for all $x \in G$ (which means $\vartheta^\circ\left(xgx^{-1}\right) = 0$). Hence the central idempotent e_χ corresponding to χ is contained in the subalgebra $\mathbb{C}N$. With this in mind, we can use e_χ to create a super simple subsuperalgebra of $\mathbb{C}[G, N]$.

The superalgebra $e_\chi\mathbb{C}[G, N]$ is $e_\chi\mathbb{C}G$ as an algebra, but graded in the following way:

– Its 0-component is $e_\chi\mathbb{C}N$.

– Its 1-component is $e_\chi\mathbb{C}[G\backslash N]$.

Note that since $e_\chi \in \mathbb{C}N$, then $e_\chi\mathbb{C}N$ is contained in the subspace $\mathbb{C}N$, while $e_\chi\mathbb{C}[G\backslash N]$ is contained in the subspace $\mathbb{C}[G\backslash N]$.
As a superalgebra, $e_\chi\mathbb{C}[G, N] \cong M_{(\vartheta(1),\vartheta(1))}(\mathbb{C})$.

- If $\chi_{\downarrow N} = \vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$, then $\vartheta^{\uparrow G} = \chi + \mathrm{sgn}\chi$ (where $\mathrm{sgn}\chi(g) = \chi(g)$ whenever $g \in N$, and $\mathrm{sgn}\chi(g) = -\chi(g)$ whenever $g \in G\backslash N$). Note that $\vartheta^{\uparrow G}(g) = 0$ whenever $g \in G\backslash N$, so $e_\chi + e_{\mathrm{sgn}\chi}$ is contained in the subalgebra $\mathbb{C}N$. We can use $e_\chi + e_{\mathrm{sgn}\chi}$ to create a super simple subsuperalgebra of $\mathbb{C}[G, N]$.

  The superalgebra $(e_\chi + e_{\mathrm{sgn}\chi})\mathbb{C}[G, N]$ is $(e_\chi + e_{\mathrm{sgn}\chi})\mathbb{C}G$ as an algebra, but graded in the following way:

  – Its 0-component is $(e_\chi + e_{\mathrm{sgn}\chi})\mathbb{C}N = e_\vartheta\mathbb{C}N$.

  – Its 1-component is $(e_\chi + e_{\mathrm{sgn}\chi})\mathbb{C}[G\backslash N]$.

  As a superalgebra, $(e_\chi + e_{\mathrm{sgn}\chi})\mathbb{C}[G, N] \cong M_{\vartheta(1)}(\mathbb{C}) \hat{\otimes} (\mathbb{C} \oplus \mathbb{C}v)$.

We can express $\mathbb{C}[G, N]$ as a direct sum of super simple subsuperalgebras in the following way:

$$\mathbb{C}[G, N] \cong \bigoplus_{\substack{\chi\in\mathrm{Irr}_{\mathbb{C}}(G)\\ \chi_{\downarrow N}=\vartheta+\vartheta^g\\ \vartheta,\vartheta^g\in\mathrm{Irr}_{\mathbb{C}}(N)}} M_{(\frac{\chi(1)}{2},\frac{\chi(1)}{2})}(\mathbb{C}) \oplus \bigoplus_{\substack{\{\chi,\chi'\}\in\mathcal{P}_2(\mathrm{Irr}_{\mathbb{C}}(G))\\ \chi'=\mathrm{sgn}(\chi),\\ \chi_{\downarrow N}=\chi'_{\downarrow N}\in\mathrm{Irr}_{\mathbb{C}}(N)}} M_{\chi(1)}(\mathbb{R}) \hat{\otimes} (\mathbb{C} \oplus \mathbb{C}v).$$

4.3 Super simple subsuperalgebras of $\mathbb{R}[G, N]$

With G, N and χ as in the previous subsection, let us revisit the two possibilities regarding $\chi_{\downarrow N}$:

- If $\chi_{\downarrow N} = \vartheta + \vartheta^g$, then we are able to use e_χ to construct a super central simple subsuperalgebra of $\mathbb{R}[G, N]$ if χ is a real character. If χ is real, then the superalgebra $e_\chi\mathbb{R}[G, N]$ is $e_\chi\mathbb{R}G$ as an algebra, but graded in the following way:

  – Its 0-component is $e_\chi\mathbb{R}N$.

  – Its 1-component is $e_\chi\mathbb{R}[G\backslash N]$.

  From our discussion of the Gow indicator, there are four possibilities if χ is real:

$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
1	1	0
1	0	1
-1	-1	0
-1	0	-1

Let us look at these four possibilities:

- If $\epsilon(\chi)=1$ and $\epsilon(\vartheta)=1$, then $e_\chi\mathbb{R}[G,N]\cong M_{(\vartheta(1),\vartheta(1))}(\mathbb{R})$.
- If $\epsilon(\chi)=1$ and $\epsilon(\vartheta)=0$, then $e_\chi\mathbb{R}[G,N]\cong M_{\vartheta(1)}(\mathbb{R})\,\hat{\otimes}(\mathbb{C}\oplus\mathbb{C}v^+)$.
- If $\epsilon(\chi)=-1$ and $\epsilon(\vartheta)=-1$, then $e_\chi\mathbb{R}[G,N]\cong M_{\left(\frac{\vartheta(1)}{2},\frac{\vartheta(1)}{2}\right)}(\mathbb{H})$.
- If $\epsilon(\chi)=-1$ and $\epsilon(\vartheta)=0$, then $e_\chi\mathbb{R}[G,N]\cong M_{\vartheta(1)}(\mathbb{R})\,\hat{\otimes}(\mathbb{C}\oplus\mathbb{C}v^-)$.

If χ is complex, then the superalgebra $(e_\chi+e_{\overline{\chi}})\mathbb{R}[G,N]$ is $(e_\chi+e_{\overline{\chi}})\mathbb{R}G$ as an algebra, but graded in the following way:

- Its 0-component is $(e_\chi+e_{\overline{\chi}})\mathbb{R}N$.
- Its 1-component is $(e_\chi+e_{\overline{\chi}})\mathbb{R}[G\backslash N]$.

As a superalgebra, $(e_\chi+e_{\overline{\chi}})\mathbb{R}[G,N]\cong M_{\left(\frac{\chi(1)}{2},\frac{\chi(1)}{2}\right)}(\mathbb{C})$

- If $\chi_{\downarrow N}=\vartheta\in\mathrm{Irr}_{\mathbb{C}}(N)$ then we can use $e_\chi+e_{\mathrm{sgn}(\chi)}$ to construct a super central simple subsuperalgebra of $\mathbb{R}[G,N]$ if ϑ is real. If ϑ is real, then the superalgebra $(e_\chi+e_{\mathrm{sgn}(\chi)})\mathbb{R}[G,N]$ is $(e_\chi+e_{\mathrm{sgn}(\chi)})\mathbb{R}G$ as an algebra, but graded in the following way:

  - Its 0-component is $(e_\chi+e_{\mathrm{sgn}(\chi)})\mathbb{R}N=e_\vartheta\mathbb{R}N$.
  - Its 1-component is $(e_\chi+e_{\mathrm{sgn}(\chi)})\mathbb{R}[G\backslash N]$.

  From our discussion of the Gow indicator, there are four possibilities if ϑ is real:

$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
1	1	1
0	-1	1
-1	-1	-1
0	1	-1

  Let us look at these four possibilities:

  - If $\epsilon(\chi)=1$ and $\epsilon(\vartheta)=1$, then $(e_\chi+e_{\mathrm{sgn}(\chi)})\mathbb{R}[G,N]\cong M_{\vartheta(1)}(\mathbb{R})\,\hat{\otimes}(\mathbb{R}\oplus\mathbb{R}v^+)$.
  - If $\epsilon(\chi)=0$ and $\epsilon(\vartheta)=-1$, then $(e_\chi+e_{\mathrm{sgn}(\chi)})\mathbb{R}[G,N]\cong M_{\frac{\vartheta(1)}{2}}(\mathbb{R})\,\hat{\otimes}(\mathbb{H}\oplus\mathbb{H}v^-)$.
  - If $\epsilon(\chi)=-1$ and $\epsilon(\vartheta)=-1$, then $(e_\chi+e_{\mathrm{sgn}(\chi)})\mathbb{R}[G,N]\cong M_{\frac{\vartheta(1)}{2}}(\mathbb{R})\,\hat{\otimes}(\mathbb{H}\oplus\mathbb{H}v^+)$.
  - If $\epsilon(\chi)=0$ and $\epsilon(\vartheta)=1$, then $(e_\chi+e_{\mathrm{sgn}(\chi)})\mathbb{R}[G,N]\cong M_{\vartheta(1)}(\mathbb{R})\,\hat{\otimes}(\mathbb{R}\oplus\mathbb{R}v^-)$.

  If ϑ is complex, then the superalgebra $(e_\chi+e_{\overline{\chi}}+e_{\mathrm{sgn}(\chi)}+e_{\overline{\mathrm{sgn}(\chi)}})\mathbb{R}[G,N]$ is $(e_\chi+e_{\overline{\chi}}+e_{\mathrm{sgn}(\chi)}+e_{\overline{\mathrm{sgn}(\chi)}})\mathbb{R}G$ as an algebra, but graded in the following way:

  - Its 0-component is $(e_\chi+e_{\overline{\chi}}+e_{\mathrm{sgn}(\chi)}+e_{\overline{\mathrm{sgn}(\chi)}})\mathbb{R}N$.
  - Its 1-component is $(e_\chi+e_{\overline{\chi}}+e_{\mathrm{sgn}(\chi)}+e_{\overline{\mathrm{sgn}(\chi)}})\mathbb{R}[G\backslash N]$.

  As a superalgebra, $(e_\chi+e_{\overline{\chi}}+e_{\mathrm{sgn}(\chi)}+e_{\overline{\mathrm{sgn}(\chi)}})\mathbb{R}[G,N]\cong M_{\chi(1)}(\mathbb{R})\,\hat{\otimes}(\mathbb{C}\oplus\mathbb{C}v)$

The table below describes the super simple subsuperalgebras of $\mathbb{R}[G,N]$. The first column of the table describes the irreducible $\mathbb{C}$-characters of G involved in each stage of the direct sum decomposition of $\mathbb{R}[G,N]$.

$\mathbb{C}$-Characters of G involved in the decomposition	Super simple subsuperalgebra of $\mathbb{R}[G,N]$	Subsuperalgebra as a subalgebra of $\mathbb{R}G$	0-component as a subalgebra of $\mathbb{R}N$
$\{\chi,\chi'\}\in\mathcal{P}_2(\mathrm{Irr}_{\mathbb{C}}(G))$: $\chi'=\overline{\chi},\ \epsilon(\chi)=\epsilon(\chi')=0$, $\chi_{\downarrow N}=\vartheta+\vartheta^g,\ \vartheta,\vartheta^g\in\mathrm{Irr}_{\mathbb{C}}(N)$, $\epsilon(\vartheta)=\epsilon(\vartheta^g)=0$	$M_{(\frac{\chi(1)}{2},\frac{\chi(1)}{2})}(\mathbb{C})$	$M_{\chi(1)}(\mathbb{C})$	$M_{\vartheta(1)}(\mathbb{C})$ $\times M_{\vartheta^g(1)}(\mathbb{C})$
$\{\chi,\chi',\psi,\psi'\}\in\mathcal{P}_4(\mathrm{Irr}_{\mathbb{C}}(G))$: $\chi'=\overline{\chi},\ \psi=\mathrm{sgn}(\chi),\ \psi'=\overline{\psi}$ $\epsilon(\chi)=\epsilon(\chi')=\epsilon(\psi)=\epsilon(\psi)=0$, $\chi_{\downarrow N}=\psi_{\downarrow N}=\vartheta\in\mathrm{Irr}_{\mathbb{C}}(N)$, $\epsilon(\vartheta)=0$	$M_{\chi(1)}(\mathbb{R})\,\hat{\otimes}(\mathbb{C}\oplus\mathbb{C}v)$	$M_{\chi(1)}(\mathbb{C})$ $\times M_{\psi(1)}(\mathbb{C})$	$M_{\vartheta(1)}(\mathbb{C})$
$\chi\in\mathrm{Irr}_{\mathbb{C}}(G)$: $\epsilon(\chi)=1$, $\chi_{\downarrow N}=\vartheta+\vartheta^g,\ \vartheta,\vartheta^g\in\mathrm{Irr}_{\mathbb{C}}(N)$, $\epsilon(\vartheta)=\epsilon(\vartheta^g)=1$	$M_{(\frac{\chi(1)}{2},\frac{\chi(1)}{2})}(\mathbb{R})$	$M_{\chi(1)}(\mathbb{R})$	$M_{\vartheta(1)}(\mathbb{R})$ $\times M_{\vartheta^g(1)}(\mathbb{R})$
$\{\chi,\chi'\}\in\mathcal{P}_2(\mathrm{Irr}_{\mathbb{C}}(G))$: $\chi'=\mathrm{sgn}(\chi),\ \epsilon(\chi)=\epsilon(\chi')=1$ $\chi_{\downarrow N}=\chi'_{\downarrow N}=\vartheta\in\mathrm{Irr}_{\mathbb{C}}(N)$ $\epsilon(\vartheta)=1$	$M_{\chi(1)}(\mathbb{R})\,\hat{\otimes}(\mathbb{R}\oplus\mathbb{R}v^+)$	$M_{\chi(1)}(\mathbb{R})$ $\times M_{\chi'(1)}(\mathbb{R})$	$M_{\vartheta(1)}(\mathbb{R})$
$\chi\in\mathrm{Irr}_{\mathbb{C}}(G)$: $\epsilon(\chi)=1$, $\chi_{\downarrow N}=\vartheta+\vartheta^g,\ \vartheta,\vartheta^g\in\mathrm{Irr}_{\mathbb{C}}(N)$, $\epsilon(\vartheta)=\epsilon(\vartheta^g)=0$	$M_{\frac{\chi(1)}{2}}(\mathbb{R})\,\hat{\otimes}(\mathbb{C}\oplus\mathbb{C}v^+)$	$M_{\chi(1)}(\mathbb{R})$	$M_{\vartheta(1)}(\mathbb{C})$
$\{\chi,\chi'\}\in\mathcal{P}_2(\mathrm{Irr}_{\mathbb{C}}(G))$: $\chi'=\mathrm{sgn}(\chi)=\overline{\chi},\ \epsilon(\chi)=\epsilon(\chi')=0$ $\chi_{\downarrow N}=\chi'_{\downarrow N}=\vartheta\in\mathrm{Irr}_{\mathbb{C}}(N)$ $\epsilon(\vartheta)=-1$	$M_{\frac{\chi(1)}{2}}(\mathbb{R})\,\hat{\otimes}(\mathbb{H}\oplus\mathbb{H}v^-)$	$M_{\chi(1)}(\mathbb{C})$	$M_{\frac{\vartheta(1)}{2}}(\mathbb{H})$
$\chi\in\mathrm{Irr}_{\mathbb{C}}(G)$: $\epsilon(\chi)=-1$, $\chi_{\downarrow N}=\vartheta+\vartheta^g,\ \vartheta,\vartheta^g\in\mathrm{Irr}_{\mathbb{C}}(N)$, $\epsilon(\vartheta)=\epsilon(\vartheta^g)=-1$	$M_{(\frac{\chi(1)}{4},\frac{\chi(1)}{4})}(\mathbb{H})$	$M_{\frac{\chi(1)}{2}}(\mathbb{H})$	$M_{\frac{\vartheta(1)}{2}}(\mathbb{H})$ $\times M_{\frac{\vartheta^g(1)}{2}}(\mathbb{H})$
$\{\chi,\chi'\}\in\mathcal{P}_2(\mathrm{Irr}_{\mathbb{C}}(G))$: $\chi'=\mathrm{sgn}(\chi),\ \epsilon(\chi)=\epsilon(\chi')=-1$ $\chi_{\downarrow N}=\chi'_{\downarrow N}=\vartheta\in\mathrm{Irr}_{\mathbb{C}}(N)$ $\epsilon(\vartheta)=-1$	$M_{\frac{\chi(1)}{2}}(\mathbb{R})\,\hat{\otimes}(\mathbb{H}\oplus\mathbb{H}v^+)$	$M_{\frac{\chi(1)}{2}}(\mathbb{H})$ $\times M_{\frac{\chi'(1)}{2}}(\mathbb{H})$	$M_{\frac{\vartheta(1)}{2}}(\mathbb{H})$
$\chi\in\mathrm{Irr}_{\mathbb{C}}(G)$: $\epsilon(\chi)=-1$, $\chi_{\downarrow N}=\vartheta+\vartheta^g,\ \vartheta,\vartheta^g\in\mathrm{Irr}_{\mathbb{C}}(N)$, $\epsilon(\vartheta)=\epsilon(\vartheta^g)=0$	$M_{\frac{\chi(1)}{2}}(\mathbb{R})\,\hat{\otimes}(\mathbb{C}\oplus\mathbb{C}v^-)$	$M_{\frac{\chi(1)}{2}}(\mathbb{H})$	$M_{\vartheta(1)}(\mathbb{C})$
$\{\chi,\chi'\}\in\mathcal{P}_2(\mathrm{Irr}_{\mathbb{C}}(G))$: $\chi'=\mathrm{sgn}(\chi)=\overline{\chi},\ \epsilon(\chi)=\epsilon(\chi')=0$ $\chi_{\downarrow N}=\chi'_{\downarrow N}=\vartheta\in\mathrm{Irr}_{\mathbb{C}}(N)$ $\epsilon(\vartheta)=1$	$M_{\chi(1)}(\mathbb{R})\,\hat{\otimes}(\mathbb{R}\oplus\mathbb{R}v^-)$	$M_{\chi(1)}(\mathbb{C})$	$M_{\vartheta(1)}(\mathbb{R})$

4.4 Super irreducible $\mathbb{C}[G, N]$-supermodules

We know that any super irreducible $\mathbb{C}[G, N]$-supermodule can be constructed by inducing an irreducible ungraded $\mathbb{C}N$-module. A super irreducible $\mathbb{C}[G, N]$-supermodule V corresponds to an element of $BW(\mathbb{C}) = C_2$, which is determined by the super division algebra $\mathrm{End}^{\mathfrak{s}}_{\mathbb{C}[G,N]}(V)$. Let us go through the two possibilities for a super irreducible $\mathbb{C}[G, N]$-supermodule. To do this, we will explore how super irreducible $\mathbb{C}[G, N]$-supermodules can be constructed from irreducible $\mathbb{C}G$-modules.

Let V_χ be an irreducible $\mathbb{C}G$-module with character χ.

- If $(V_\chi)_{\downarrow N}$ is not an irreducible $\mathbb{C}N$-module, then:
  - $(V_\chi)_{\downarrow N} = V_\vartheta \oplus V_{\vartheta^g}$, where V_ϑ and V_{ϑ^g} are irreducible $\mathbb{C}N$-modules with characters ϑ and ϑ^g respectively, and $V_\vartheta \not\cong V_{\vartheta^g}$.
  - ▶ V_χ is itself a super irreducible $\mathbb{C}[G, N]$-supermodule with components V_ϑ and V_{ϑ^g}.
  - $V_\vartheta^{\uparrow G} = V_{\vartheta^g}^{\uparrow G} = V_\chi$.
  - The corresponding super simple two-sided superideal summand, $\mathbb{C}[G, N]^{\mathfrak{s}}(V_\chi)$, is isomorphic to $M_{(\vartheta(1),\vartheta(1))}(\mathbb{C})$. The 0-component is isomorphic to $M_{\vartheta(1)}(\mathbb{C}) \times M_{\vartheta(1)}(\mathbb{C})$.
  - $\mathrm{End}^{\mathfrak{s}}_{\mathbb{C}[G,N]}(V_\chi) \cong \mathbb{C} \oplus 0$.
  - χ would be a super irreducible $\mathbb{C}$-super character of (G, N).

  When $(V_\chi)_{\downarrow N}$ is not an irreducible $\mathbb{C}N$-module, we will say that it has poset type $\wedge$, as its corresponding poset diagram resembles the "$\wedge$" symbol.

$$\begin{tikzcd} G & & \chi \arrow[dl, dash] \arrow[dr, dash] & \\ N & \vartheta & & \vartheta^g \end{tikzcd}$$

- If $(V_\chi)_{\downarrow N}$ is an irreducible $\mathbb{C}N$-module, then:
  - $(V_\chi)_{\downarrow N} = V_\vartheta$, where V_ϑ is an irreducible $\mathbb{C}N$-modules with character ϑ.
  - $V_\vartheta^{\uparrow G} = V_\chi \oplus V_{\mathrm{sgn}(\chi)}$.
  - ▶ $V_\chi \oplus V_{\mathrm{sgn}(\chi)}$ is a super irreducible $\mathbb{C}[G, N]$-supermodule with components constructible in the following way:
    - ∗ Since $(V_\chi)_{\downarrow N} = V_\vartheta \simeq (V_{\mathrm{sgn}(\chi)})_{\downarrow N}$, there is a $\mathbb{C}N$-module isomorphism $\varphi : V_\chi \to V_{\mathrm{sgn}(\chi)}$.
    - ∗ The components of $V_\chi \oplus V_{\mathrm{sgn}(\chi)}$ are: $V_0 = \{a + \varphi(a) : a \in V_\chi\}$ and $V_1 = \{a - \varphi(a) : a \in V_\chi\}$. Both of these components are isomorphic to V_ϑ as $\mathbb{C}N$-modules.
    - ▷ As $\mathbb{C}G$-modules, $V_\chi \not\cong V_{\mathrm{sgn}(\chi)}$. As $\mathbb{C}N$-modules, $V_\chi \cong V_{\mathrm{sgn}(\chi)}$.
  - The corresponding super simple two-sided superideal summand, $\mathbb{C}[G, N]^{\mathfrak{s}}(V_\chi \oplus V_{\mathrm{sgn}(\chi)})$, is isomorphic to $M_{\vartheta(1)}(\mathbb{C}) \times M_{\vartheta(1)}(\mathbb{C})$. The 0-component is $\{(A, A) : A \in M_{\vartheta(1)}(\mathbb{C})\}$, which is isomorphic to $M_{\vartheta(1)}(\mathbb{C})$. The 1-component is $\{(A, -A) : A \in M_{\vartheta(1)}(\mathbb{C})\}$.

- $\mathrm{End}^{\mathfrak{s}}_{\mathbb{C}[G,N]}(V_\chi \oplus V_{\mathrm{sgn}(\chi)}) \cong \mathbb{C} \oplus \mathbb{C}v$.
- $\chi + \mathrm{sgn}(\chi)$ would be a super irreducible $\mathbb{C}$-super character of (G,N).

When $(V_\chi)_{\downarrow N}$ is an irreducible $\mathbb{C}N$-module, we will say that it has poset type $\vee$, as its corresponding poset diagram resembles the "$\vee$" symbol.

$$\begin{tikzcd}[column sep=small] G & \chi \arrow[dr, dash] & & \mathrm{sgn}(\chi) \arrow[dl, dash] \\ N & & \vartheta & \end{tikzcd}$$

The table below summarises our analysis. The super division algebra specified in each row is the super division algebra part of $\mathbb{C}[G,N]^{\mathfrak{s}}(V)$ for each possibility of a $\mathbb{C}[G,N]$-supermodule $V = V_0 \oplus V_1$.

SDA / $\mathbb{C}$	Poset type	$\mathbb{C}G$-mod. decomp.	SDA a simple alg.?	$V_0 \cong V_1$?	Irred. $\mathbb{C}G$-module?
$\mathbb{C} \oplus 0$	$\wedge$	V_χ	Yes	No	Yes
$\mathbb{C} \oplus \mathbb{C}v$	$\vee$	$V_\chi + V_{\mathrm{sgn}(\chi)}$	No	Yes	No

4.5 Super irreducible $\mathbb{R}[G,N]$-supermodules

Any super irreducible $\mathbb{R}[G,N]$-supermodule can be constructed by inducing an irreducible ungraded $\mathbb{R}N$-module. The diagram below shows the different restrictions we will be performing when finding super irreducible $\mathbb{R}[G,N]$-supermodules.

$$\begin{tikzcd} \mathbb{C}[G,N] \arrow[r] \arrow[d] & \mathbb{R}[G,N] \arrow[d] \\ \mathbb{C}N \arrow[r] & \mathbb{R}N \end{tikzcd}$$

Let V_χ be an irreducible $\mathbb{C}G$-module with character χ.

- If V_χ is not an irreducible $\mathbb{C}N$-module, we know that $(V_\chi)_{\downarrow N} = V_\vartheta \oplus V_{\vartheta^g}$, where V_ϑ and V_{ϑ^g} are irreducible $\mathbb{C}N$-modules with characters ϑ and ϑ^g respectively, and $V_\vartheta \not\cong V_{\vartheta^g}$.
  - If $\epsilon(\chi) = 0$, and $\epsilon(\vartheta) = \epsilon(\vartheta^g) = 0$, then:
    * $\vartheta^g \neq \overline{\vartheta}$.
    * $(V_\chi)_{\downarrow\mathbb{R}} = \tilde{V}_\chi$ where $\tilde{V}_\chi$ is an irreducible $\mathbb{R}G$-module of type $\mathbb{C}$.
    * $(V_\chi)_{\downarrow\mathbb{R}N} = ((V_\chi)_{\downarrow N})_{\downarrow\mathbb{R}} = (V_\vartheta \oplus V_{\vartheta^g})_{\downarrow\mathbb{R}} = (V_\vartheta)_{\downarrow\mathbb{R}} \oplus (V_{\vartheta^g})_{\downarrow\mathbb{R}} = \tilde{V}_\vartheta \oplus \tilde{V}_{\vartheta^g}$ where $\tilde{V}_\vartheta$ and $\tilde{V}_{\vartheta^g}$ are irreducible $\mathbb{R}N$-modules of type $\mathbb{C}$. Note that $(\tilde{V}_\chi)_{\downarrow N} = \tilde{V}_\vartheta \oplus \tilde{V}_{\vartheta^g}$.
    ► $\tilde{V}_\chi$ is itself a super irreducible $\mathbb{R}[G,N]$-supermodule with components $\tilde{V}_\vartheta$ and $\tilde{V}_{\vartheta^g}$.
    * $\tilde{V}_\vartheta^{\uparrow G} = \tilde{V}_{\vartheta^g}^{\uparrow G} = \tilde{V}_\chi$.
    * The corresponding super simple two-sided superideal summand, $\mathbb{R}[G,N]^{\mathfrak{s}}(\tilde{V}_\chi)$, is isomorphic to $M_{(\vartheta(1),\vartheta(1))}(\mathbb{C})$. The 0-component is isomorphic to $M_{\vartheta(1)}(\mathbb{C}) \times M_{\vartheta(1)}(\mathbb{C})$.

  * $\operatorname{End}^{\mathfrak{s}}_{\mathbb{R}[G,N]}(\tilde{V}_\chi) \cong \mathbb{C} \oplus 0$.
  * When induced up to $\mathbb{C}$:
    - ▷ $\tilde{V}_\chi^{\uparrow\mathbb{C}G} = \tilde{V}_\chi \otimes_{\mathbb{R}} \mathbb{C} \cong V_\chi \oplus V_{\overline{\chi}}$.
    - ▷ $\tilde{V}_\vartheta^{\uparrow\mathbb{C}N} = \tilde{V}_\vartheta \otimes_{\mathbb{R}} \mathbb{C} \cong V_\vartheta \oplus V_{\overline{\vartheta}}$ and $\tilde{V}_{\vartheta^g}^{\uparrow\mathbb{C}N} = \tilde{V}_{\vartheta^g} \otimes_{\mathbb{R}} \mathbb{C} \cong V_{\vartheta^g} \oplus V_{\overline{\vartheta^g}}$.
  * $\chi + \overline{\chi}$ would be a super irreducible $\mathbb{R}$-super character of (G,N).

- If $\epsilon(\chi) = 1$, and $\epsilon(\vartheta) = \epsilon(\vartheta^g) = 1$, then:
  * $\vartheta^g \neq \overline{\vartheta}$.
  * $(V_\chi)_{\downarrow\mathbb{R}} = \tilde{V}_\chi \oplus \tilde{V}_\chi$ where $\tilde{V}_\chi$ is an irreducible $\mathbb{R}G$-module of type $\mathbb{R}$.
  * $(V_\chi)_{\downarrow\mathbb{R}N} = ((V_\chi)_{\downarrow N})_{\downarrow\mathbb{R}} = (V_\vartheta \oplus V_{\vartheta^g})_{\downarrow\mathbb{R}} = (V_\vartheta)_{\downarrow\mathbb{R}} \oplus (V_{\vartheta^g})_{\downarrow\mathbb{R}} = \tilde{V}_\vartheta \oplus \tilde{V}_\vartheta \oplus \tilde{V}_{\vartheta^g} \oplus \tilde{V}_{\vartheta^g}$ where $\tilde{V}_\vartheta$ and $\tilde{V}_{\vartheta^g}$ are irreducible $\mathbb{R}N$-modules of type $\mathbb{R}$.
  * Given V_χ, we can construct $\tilde{V}_\chi$ by setting a $\mathbb{C}$-basis $\{x_1, \ldots, x_{\chi(1)}\}$ on V_χ such that its $\mathbb{R}$-span is an irreducible $\mathbb{R}G$-module. Similarly, we can construct $\tilde{V}_\vartheta$ and $\tilde{V}_{\vartheta^g}$ by setting a $\mathbb{C}$-basis $\{y_1, \ldots, y_{\vartheta(1)}\}$ and $\{z_1, \ldots, z_{\vartheta(1)}\}$ on V_ϑ and V_{ϑ^g} such that their $\mathbb{R}$-spans are irreducible $\mathbb{R}N$-modules. Note that $(\tilde{V}_\chi)_{\downarrow N} = \tilde{V}_\vartheta \oplus \tilde{V}_{\vartheta^g}$.
  ▶ $\tilde{V}_\chi$ is itself a super irreducible $\mathbb{R}[G,N]$-supermodule with components $\tilde{V}_\vartheta$ and $\tilde{V}_{\vartheta^g}$.
  * $\tilde{V}_\vartheta^{\uparrow G} = \tilde{V}_{\vartheta^g}^{\uparrow G} = \tilde{V}_\chi$.
  * The corresponding super simple two-sided superideal summand, $\mathbb{R}[G,N]^{\mathfrak{s}}(\tilde{V}_\chi)$, is isomorphic to $M_{(\vartheta(1),\vartheta(1))}(\mathbb{R})$. The 0-component is isomorphic to $M_{\vartheta(1)}(\mathbb{R}) \times M_{\vartheta(1)}(\mathbb{R})$.
  * $\operatorname{End}^{\mathfrak{s}}_{\mathbb{R}[G,N]}(\tilde{V}_\chi) \cong \mathbb{R} \oplus 0$.
  * When induced up to $\mathbb{C}$:
    - ▷ $\tilde{V}_\chi^{\uparrow\mathbb{C}G} = \tilde{V}_\chi \otimes_{\mathbb{R}} \mathbb{C} \cong V_\chi$.
    - ▷ $\tilde{V}_\vartheta^{\uparrow\mathbb{C}N} = \tilde{V}_\vartheta \otimes_{\mathbb{R}} \mathbb{C} \cong V_\vartheta$ and $\tilde{V}_{\vartheta^g}^{\uparrow\mathbb{C}N} = \tilde{V}_{\vartheta^g} \otimes_{\mathbb{R}} \mathbb{C} \cong V_{\vartheta^g}$.
  * χ would be a super irreducible $\mathbb{R}$-super character of (G,N).

- If $\epsilon(\chi) = 1$, and $\epsilon(\vartheta) = \epsilon(\vartheta^g) = 0$, then:
  * $\vartheta^g = \overline{\vartheta}$.
  * $(V_\chi)_{\downarrow\mathbb{R}} = \tilde{V}_\chi \oplus \tilde{V}_\chi$ where $\tilde{V}_\chi$ is an irreducible $\mathbb{R}G$-module of type $\mathbb{R}$.
  * $(V_\chi)_{\downarrow\mathbb{R}N} = ((V_\chi)_{\downarrow N})_{\downarrow\mathbb{R}} = (V_\vartheta \oplus V_{\vartheta^g})_{\downarrow\mathbb{R}} = (V_\vartheta)_{\downarrow\mathbb{R}} \oplus (V_{\vartheta^g})_{\downarrow\mathbb{R}} = \tilde{V}_\vartheta \oplus \tilde{V}_{\vartheta^g}$ where $\tilde{V}_\vartheta$ and $\tilde{V}_{\vartheta^g}$ are irreducible $\mathbb{R}N$-modules of type $\mathbb{C}$. Note that $(\tilde{V}_\chi \oplus \tilde{V}_\chi)_{\downarrow N} = \tilde{V}_\vartheta \oplus \tilde{V}_{\vartheta^g}$.
  ▶ $\tilde{V}_\chi \oplus \tilde{V}_\chi$ is itself a super irreducible $\mathbb{R}[G,N]$-supermodule with components $\tilde{V}_\vartheta$ and $\tilde{V}_{\vartheta^g}$.
    - ▷ As $\mathbb{R}G$-modules, $\tilde{V}_\chi \oplus 0 \cong 0 \oplus \tilde{V}_\chi$.
  * $\tilde{V}_\vartheta^{\uparrow G} = \tilde{V}_{\vartheta^g}^{\uparrow G} = \tilde{V}_\chi \oplus \tilde{V}_\chi$.
  * The corresponding super simple two-sided superideal summand, $\mathbb{R}[G,N]^{\mathfrak{s}}(\tilde{V}_\chi \oplus \tilde{V}_\chi)$, is isomorphic to $M_{\vartheta(1)}(\mathbb{R}) \hat{\otimes} (\mathbb{C} \oplus \mathbb{C}v^+)$, which, as an algebra, is isomorphic to $M_{\chi(1)}(\mathbb{R})$. The 0-component is isomorphic to $M_{\vartheta(1)}(\mathbb{C})$.
  * $\operatorname{End}^{\mathfrak{s}}_{\mathbb{R}[G,N]}(\tilde{V}_\chi \oplus \tilde{V}_\chi) \cong \mathbb{C} \oplus \mathbb{C}v^-$.
  * When induced up to $\mathbb{C}$:

- ▷ $(\tilde{V}_\chi \oplus \tilde{V}_\chi)^{\uparrow \mathbb{C}G} = (\tilde{V}_\chi \oplus \tilde{V}_\chi) \otimes_{\mathbb{R}} \mathbb{C} \cong V_\chi \oplus V_\chi$.
- ▷ $\tilde{V}_\vartheta^{\uparrow \mathbb{C}N} \cong \tilde{V}_{\vartheta^g}^{\uparrow \mathbb{C}N} = \tilde{V}_\vartheta \otimes_{\mathbb{R}} \mathbb{C} \cong V_\vartheta \oplus V_{\overline{\vartheta}}$.

∗ 2χ would be a super irreducible $\mathbb{R}$-super character of (G,N).

– If $\epsilon(\chi) = -1$, and $\epsilon(\vartheta) = \epsilon(\vartheta^g) = -1$, then:

∗ $\vartheta^g \neq \overline{\vartheta}$.

∗ $(V_\chi)_{\downarrow \mathbb{R}} = \tilde{V}_\chi$ where $\tilde{V}_\chi$ is an irreducible $\mathbb{R}G$-module of type $\mathbb{H}$.

∗ $(V_\chi)_{\downarrow \mathbb{R}N} = ((V_\chi)_{\downarrow N})_{\downarrow \mathbb{R}} = (V_\vartheta \oplus V_{\vartheta^g})_{\downarrow \mathbb{R}} = (V_\vartheta)_{\downarrow \mathbb{R}} \oplus (V_{\vartheta^g})_{\downarrow \mathbb{R}} = \tilde{V}_\vartheta \oplus \tilde{V}_{\vartheta^g}$ where $\tilde{V}_\vartheta$ and $\tilde{V}_{\vartheta^g}$ are irreducible $\mathbb{R}N$-modules of type $\mathbb{H}$. Note that $(\tilde{V}_\chi)_{\downarrow N} = \tilde{V}_\vartheta \oplus \tilde{V}_{\vartheta^g}$.

► $\tilde{V}_\chi$ is itself a super irreducible $\mathbb{R}[G,N]$-supermodule with components $\tilde{V}_\vartheta$ and $\tilde{V}_{\vartheta^g}$.

∗ $\tilde{V}_\vartheta^{\uparrow G} = \tilde{V}_{\vartheta^g}^{\uparrow G} = \tilde{V}_\chi$.

∗ The corresponding super simple two-sided superideal summand, $\mathbb{R}[G,N]^{\mathfrak{s}}(\tilde{V}_\chi)$, is isomorphic to $M_{\left(\frac{\vartheta(1)}{2},\frac{\vartheta(1)}{2}\right)}(\mathbb{H})$. The 0-component is isomorphic to $M_{\frac{\vartheta(1)}{2}}(\mathbb{H}) \times M_{\frac{\vartheta(1)}{2}}(\mathbb{H})$.

∗ $\mathrm{End}^{\mathfrak{s}}_{\mathbb{R}[G,N]}(\tilde{V}_\chi) \cong \mathbb{H} \oplus 0$.

∗ When induced up to $\mathbb{C}$:
- ▷ $\tilde{V}_\chi^{\uparrow \mathbb{C}G} = \tilde{V}_\chi \otimes_{\mathbb{R}} \mathbb{C} \cong V_\chi \oplus V_\chi$.
- ▷ $\tilde{V}_\vartheta^{\uparrow \mathbb{C}N} = \tilde{V}_\vartheta \otimes_{\mathbb{R}} \mathbb{C} \cong V_\vartheta \oplus V_\vartheta$ and $\tilde{V}_{\vartheta^g}^{\uparrow \mathbb{C}N} = \tilde{V}_{\vartheta^g} \otimes_{\mathbb{R}} \mathbb{C} \cong V_{\vartheta^g} \oplus V_{\vartheta^g}$.

∗ 2χ would be a super irreducible $\mathbb{R}$-super character of (G,N).

– If $\epsilon(\chi) = -1$, and $\epsilon(\vartheta) = \epsilon(\vartheta^g) = 0$, then:

∗ $\vartheta^g = \overline{\vartheta}$.

∗ $(V_\chi)_{\downarrow \mathbb{R}} = \tilde{V}_\chi$ where $\tilde{V}_\chi$ is an irreducible $\mathbb{R}G$-module of type $\mathbb{H}$.

∗ $(V_\chi)_{\downarrow \mathbb{R}N} = ((V_\chi)_{\downarrow N})_{\downarrow \mathbb{R}} = (V_\vartheta \oplus V_{\vartheta^g})_{\downarrow \mathbb{R}} = (V_\vartheta)_{\downarrow \mathbb{R}} \oplus (V_{\vartheta^g})_{\downarrow \mathbb{R}} = \tilde{V}_\vartheta \oplus \tilde{V}_{\vartheta^g}$ where $\tilde{V}_\vartheta$ and $\tilde{V}_{\vartheta^g}$ are irreducible $\mathbb{R}N$-modules of type $\mathbb{C}$. Note that $(\tilde{V}_\chi)_{\downarrow N} = \tilde{V}_\vartheta \oplus \tilde{V}_{\vartheta^g}$.

► $\tilde{V}_\chi$ is itself a super irreducible $\mathbb{R}[G,N]$-supermodule with components $\tilde{V}_\vartheta$ and $\tilde{V}_{\vartheta^g}$.

∗ $\tilde{V}_\vartheta^{\uparrow G} = \tilde{V}_{\vartheta^g}^{\uparrow G} = \tilde{V}_\chi$.

∗ The corresponding super simple two-sided superideal summand, $\mathbb{R}[G,N]^{\mathfrak{s}}(\tilde{V}_\chi)$, is isomorphic to $M_{\vartheta(1)}(\mathbb{R}) \hat{\otimes} (\mathbb{C} \oplus \mathbb{C}v^-)$, which, as an algebra, is isomorphic to $M_{\vartheta(1)}(\mathbb{H})$. The 0-component is isomorphic to $M_{\vartheta(1)}(\mathbb{C})$.

∗ When induced up to $\mathbb{C}$:
- ▷ $\tilde{V}_\chi^{\uparrow \mathbb{C}G} = \tilde{V}_\chi \otimes_{\mathbb{R}} \mathbb{C} \cong V_\chi \oplus V_\chi$.
- ▷ $\tilde{V}_\vartheta^{\uparrow \mathbb{C}N} \cong \tilde{V}_{\vartheta^g}^{\uparrow \mathbb{C}N} = \tilde{V}_\vartheta \otimes_{\mathbb{R}} \mathbb{C} \cong V_\vartheta \oplus V_{\overline{\vartheta}}$.

∗ $\mathrm{End}^{\mathfrak{s}}_{\mathbb{R}[G,N]}(\tilde{V}_\chi) \cong \mathbb{C} \oplus \mathbb{C}v^+$.

∗ 2χ would be a super irreducible $\mathbb{R}$-super character of (G,N).

- If V_χ is an irreducible $\mathbb{C}N$-module, we know that $(V_\chi)_{\downarrow N} = V_\vartheta$, where V_ϑ is an irreducible $\mathbb{C}N$-module with character ϑ, $V_\vartheta^{\uparrow G} = V_\chi \oplus V_{\mathrm{sgn}(\chi)}$, and the components V_0 and V_1 of $V_\vartheta^{\uparrow G} = V_\chi \oplus V_{\mathrm{sgn}(\chi)}$ are isomorphic to V_ϑ, and are constructed in the following way:

* Since $(V_\chi)_{\downarrow N} = V_\vartheta \simeq (V_{\text{sgn}(\chi)})_{\downarrow N}$, there is a $\mathbb{C}N$-module isomorphism $\varphi : V_\chi \to V_{\text{sgn}(\chi)}$.
* The components of $V_\chi \oplus V_{\text{sgn}(\chi)}$ are: $V_0 = \{a + \varphi(a) : a \in V_\chi\}$ and $V_1 = \{a - \varphi(a) : a \in V_\chi\}$. Both of these components are isomorphic to V_ϑ as $\mathbb{C}N$-modules.

– If $\epsilon(\chi) = \epsilon(\text{sgn}(\chi)) = 0$ and $\epsilon(\vartheta) = 0$, then:

∗ $\text{sgn}(\chi) \neq \overline{\chi}$.

∗ $(V_\chi \oplus V_{\text{sgn}(\chi)})_{\downarrow\mathbb{R}} = (V_\chi)_{\downarrow\mathbb{R}} \oplus (V_{\text{sgn}(\chi)})_{\downarrow\mathbb{R}} = \tilde{V}_\chi \oplus \tilde{V}_{\text{sgn}(\chi)}$, where $\tilde{V}_\chi$ and $\tilde{V}_{\text{sgn}(\chi)}$ are irreducible $\mathbb{R}G$-modules of type $\mathbb{C}$.

∗ $(V_\chi \oplus V_{\text{sgn}(\chi)})_{\downarrow\mathbb{R}N} = ((V_\chi \oplus V_{\text{sgn}(\chi)})_{\downarrow N})_{\downarrow\mathbb{R}} \cong (V_\vartheta \oplus V_\vartheta)_{\downarrow\mathbb{R}} = (V_\vartheta)_{\downarrow\mathbb{R}} \oplus (V_\vartheta)_{\downarrow\mathbb{R}} = \tilde{V}_\vartheta \oplus \tilde{V}_\vartheta$ where $\tilde{V}_\vartheta$ is an irreducible $\mathbb{R}N$-module of type $\mathbb{C}$. Note that $(\tilde{V}_\chi)_{\downarrow N} \cong (\tilde{V}_{\text{sgn}(\chi)})_{\downarrow N} \cong \tilde{V}_\vartheta$.

► $\tilde{V}_\chi \oplus \tilde{V}_{\text{sgn}(\chi)}$ is itself a super irreducible $\mathbb{R}[G,N]$-supermodule with components $(V_0)_{\downarrow\mathbb{R}}$ and $(V_1)_{\downarrow\mathbb{R}}$. Here, $(V_0)_{\downarrow\mathbb{R}} \cong (V_1)_{\downarrow\mathbb{R}} \cong \tilde{V}_\vartheta$.

▷ As $\mathbb{R}G$-modules, $\tilde{V}_\chi \ncong \tilde{V}_{\text{sgn}(\chi)}$. As $\mathbb{R}N$-modules, $\tilde{V}_\chi \cong \tilde{V}_{\text{sgn}(\chi)} \cong \tilde{V}_\vartheta$.

∗ The corresponding super simple two-sided superideal summand, $\mathbb{R}[G,N]^{\mathfrak{s}}(\tilde{V}_\chi \oplus \tilde{V}_{\text{sgn}(\chi)})$, is isomorphic to $M_{\vartheta(1)}(\mathbb{C}) \times M_{\vartheta(1)}(\mathbb{C})$. The 0-component is $\{(A,A) : A \in M_{\vartheta(1)}(\mathbb{C})\}$, which is isomorphic to $M_{\vartheta(1)}(\mathbb{C})$. The 1-component is $\{(A,-A) : A \in M_{\vartheta(1)}(\mathbb{C})\}$.

∗ $\text{End}^{\mathfrak{s}}_{\mathbb{R}[G,N]}(\tilde{V}_\chi \oplus \tilde{V}_{\text{sgn}(\chi)}) \cong \mathbb{C} \oplus \mathbb{C}v$.

∗ When induced up to $\mathbb{C}$:

▷ $(\tilde{V}_\chi \oplus \tilde{V}_{\text{sgn}(\chi)})^{\uparrow\mathbb{C}G} = (\tilde{V}_\chi \oplus \tilde{V}_{\text{sgn}(\chi)}) \otimes_{\mathbb{R}} \mathbb{C} \cong V_\chi \oplus V_{\overline{\chi}} \oplus V_{\text{sgn}(\chi)} \oplus V_{\overline{\text{sgn}(\chi)}}$.

▷ $\tilde{V}_0^{\uparrow\mathbb{C}N} \cong \tilde{V}_1^{\uparrow\mathbb{C}N} = \tilde{V}_\vartheta \otimes_{\mathbb{R}} \mathbb{C} \cong V_\vartheta \oplus V_{\overline{\vartheta}}$.

∗ $\chi + \overline{\chi} + \text{sgn}(\chi) + \overline{\text{sgn}(\chi)}$ would be a super irreducible $\mathbb{R}$-super character of (G,N).

– If $\epsilon(\chi) = \epsilon(\text{sgn}(\chi)) = 1$ and $\epsilon(\vartheta) = 1$, then:

∗ $\text{sgn}(\chi) \neq \overline{\chi}$.

∗ $(V_\chi \oplus V_{\text{sgn}(\chi)})_{\downarrow\mathbb{R}} = (V_\chi)_{\downarrow\mathbb{R}} \oplus (V_{\text{sgn}(\chi)})_{\downarrow\mathbb{R}} = \tilde{V}_\chi \oplus \tilde{V}_\chi \oplus \tilde{V}_{\text{sgn}(\chi)} \oplus \tilde{V}_{\text{sgn}(\chi)}$, where $\tilde{V}_\chi$ and $\tilde{V}_{\text{sgn}(\chi)}$ are irreducible $\mathbb{R}G$-modules of type $\mathbb{R}$.

∗ $(V_\chi \oplus V_{\text{sgn}(\chi)})_{\downarrow\mathbb{R}N} = ((V_\chi \oplus V_{\text{sgn}(\chi)})_{\downarrow N})_{\downarrow\mathbb{R}} \cong (V_\vartheta \oplus V_\vartheta)_{\downarrow\mathbb{R}} = (V_\vartheta)_{\downarrow\mathbb{R}} \oplus (V_\vartheta)_{\downarrow\mathbb{R}} = \tilde{V}_\vartheta \oplus \tilde{V}_\vartheta \oplus \tilde{V}_\vartheta \oplus \tilde{V}_\vartheta$ where $\tilde{V}_\vartheta$ is an irreducible $\mathbb{R}N$-module of type $\mathbb{R}$.

∗ Given V_χ and $V_{\text{sgn}(\chi)}$, we can construct $\tilde{V}_\chi$ and $\tilde{V}_{\text{sgn}(\chi)}$ by setting a $\mathbb{C}$-basis $\{x_1, \ldots, x_{\chi(1)}\}$ and $\{y_1, \ldots, y_{\chi(1)}\}$ for V_χ and $V_{\text{sgn}(\chi)}$ such that their $\mathbb{R}$-spans are irreducible $\mathbb{R}G$-modules.

► $\tilde{V}_\chi \oplus \tilde{V}_{\text{sgn}(\chi)}$ is itself a super irreducible $\mathbb{R}[G,N]$-supermodule with components $\tilde{V}_0$ and $\tilde{V}_1$, where $\tilde{V}_0$ and $\tilde{V}_1$ are the $\mathbb{R}$-spans of $\{x_1 + \varphi(x_1), \ldots, x_{\chi(1)} + \varphi(x_{\chi(1)})\}$ and $\{x_1 - \varphi(x_1), \ldots, x_{\chi(1)} - \varphi(x_{\chi(1)})\}$. Here, $\tilde{V}_0 \cong \tilde{V}_1 \cong \tilde{V}_\vartheta$.

▷ As $\mathbb{R}G$-modules, $\tilde{V}_\chi \ncong \tilde{V}_{\text{sgn}(\chi)}$. As $\mathbb{R}N$-modules, $\tilde{V}_\chi \cong \tilde{V}_{\text{sgn}(\chi)} \cong \tilde{V}_\vartheta$.

∗ The corresponding super simple two-sided superideal summand, $\mathbb{R}[G,N]^{\mathfrak{s}}(\tilde{V}_\chi \oplus \tilde{V}_{\text{sgn}(\chi)})$, is isomorphic to $M_{\vartheta(1)}(\mathbb{R}) \times M_{\vartheta(1)}(\mathbb{R})$. The 0-component is $\{(A,A) : A \in M_{\vartheta(1)}(\mathbb{R})\}$, which is isomorphic to $M_{\vartheta(1)}(\mathbb{R})$. The 1-component is $\{(A,-A) : A \in M_{\vartheta(1)}(\mathbb{R})\}$.

* $\mathrm{End}^{\mathfrak{s}}_{\mathbb{R}[G,N]}(\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}) \cong \mathbb{R} \oplus \mathbb{R}v^-$.
* When induced up to $\mathbb{C}$:
  ▷ $(\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)})^{\uparrow \mathbb{C}G} = (\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}) \otimes_{\mathbb{R}} \mathbb{C} \cong V_\chi \oplus V_{\mathrm{sgn}(\chi)}$.
  ▷ $\tilde{V}_0^{\uparrow \mathbb{C}N} \cong \tilde{V}_1^{\uparrow \mathbb{C}N} = \tilde{V}_\vartheta \otimes_{\mathbb{R}} \mathbb{C} \cong V_\vartheta$.
* $\chi + \mathrm{sgn}(\chi)$ would be a super irreducible $\mathbb{R}$-super character of (G,N).

– If $\epsilon(\chi) = \epsilon(\mathrm{sgn}(\chi)) = 0$ and $\epsilon(\vartheta) = -1$, then:
* $\mathrm{sgn}(\chi) = \overline{\chi}$.
* $(V_\chi \oplus V_{\mathrm{sgn}(\chi)})_{\downarrow \mathbb{R}} = (V_\chi)_{\downarrow \mathbb{R}} \oplus (V_{\mathrm{sgn}(\chi)})_{\downarrow \mathbb{R}} = \tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}$, where $\tilde{V}_\chi$ and $\tilde{V}_{\mathrm{sgn}(\chi)}$ are irreducible $\mathbb{R}G$-modules of type $\mathbb{C}$.
* $(V_\chi \oplus V_{\mathrm{sgn}(\chi)})_{\downarrow \mathbb{R}N} = ((V_\chi \oplus V_{\mathrm{sgn}(\chi)})_{\downarrow N})_{\downarrow \mathbb{R}} \cong (V_\vartheta \oplus V_\vartheta)_{\downarrow \mathbb{R}} = (V_\vartheta)_{\downarrow \mathbb{R}} \oplus (V_\vartheta)_{\downarrow \mathbb{R}} = \tilde{V}_\vartheta \oplus \tilde{V}_\vartheta$ where $\tilde{V}_\vartheta$ is an irreducible $\mathbb{R}N$-module of type $\mathbb{H}$. Note that $(\tilde{V}_\chi)_{\downarrow N} \cong (\tilde{V}_{\mathrm{sgn}(\chi)})_{\downarrow N} \cong \tilde{V}_\vartheta$.
▶ $\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}$ is itself a super irreducible $\mathbb{R}[G,N]$-supermodule with components $(V_0)_{\downarrow \mathbb{R}}$ and $(V_1)_{\downarrow \mathbb{R}}$. Here, $(V_0)_{\downarrow \mathbb{R}} \cong (V_1)_{\downarrow \mathbb{R}} \cong \tilde{V}_\vartheta$.
  ▷ As $\mathbb{R}G$-modules, $\tilde{V}_\chi \cong \tilde{V}_{\mathrm{sgn}(\chi)}$.
* The corresponding super simple two-sided superideal summand, $\mathbb{R}[G,N]^{\mathfrak{s}}(\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)})$, is isomorphic to $M_{\frac{\vartheta(1)}{2}}(\mathbb{R}) \hat{\otimes} (\mathbb{H} \oplus \mathbb{H}v^-)$ which, as an algebra, is isomorphic to $M_{\vartheta(1)}(\mathbb{C})$. Its 0-component is isomorphic to $M_{\frac{\vartheta(1)}{2}}(\mathbb{H})$.
* $\mathrm{End}^{\mathfrak{s}}_{\mathbb{R}[G,N]}(\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}) \cong \mathbb{H} \oplus \mathbb{H}v^+$.
* When induced up to $\mathbb{C}$:
  ▷ $(\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)})^{\uparrow \mathbb{C}G} = (\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}) \otimes_{\mathbb{R}} \mathbb{C} \cong V_\chi \oplus V_{\overline{\chi}} \oplus V_{\mathrm{sgn}(\chi)} \oplus V_{\overline{\mathrm{sgn}(\chi)}} \cong V_\chi \oplus V_\chi \oplus V_{\overline{\chi}} \oplus V_{\overline{\chi}}$.
  ▷ $\tilde{V}_0^{\uparrow \mathbb{C}N} \cong \tilde{V}_1^{\uparrow \mathbb{C}N} = \tilde{V}_\vartheta \otimes_{\mathbb{R}} \mathbb{C} \cong V_\vartheta \oplus V_\vartheta$.
* $2(\chi + \overline{\chi}) = 2(\chi + \mathrm{sgn}(\chi))$ would be a super irreducible $\mathbb{R}$-super character of (G,N).

– If $\epsilon(\chi) = \epsilon(\mathrm{sgn}(\chi)) = -1$ and $\epsilon(\vartheta) = -1$, then:
* $\mathrm{sgn}(\chi) \neq \overline{\chi}$.
* $(V_\chi \oplus V_{\mathrm{sgn}(\chi)})_{\downarrow \mathbb{R}} = (V_\chi)_{\downarrow \mathbb{R}} \oplus (V_{\mathrm{sgn}(\chi)})_{\downarrow \mathbb{R}} = \tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}$, where $\tilde{V}_\chi$ and $\tilde{V}_{\mathrm{sgn}(\chi)}$ are irreducible $\mathbb{R}G$-modules of type $\mathbb{H}$.
* $(V_\chi \oplus V_{\mathrm{sgn}(\chi)})_{\downarrow \mathbb{R}N} = ((V_\chi \oplus V_{\mathrm{sgn}(\chi)})_{\downarrow N})_{\downarrow \mathbb{R}} \cong (V_\vartheta \oplus V_\vartheta)_{\downarrow \mathbb{R}} = (V_\vartheta)_{\downarrow \mathbb{R}} \oplus (V_\vartheta)_{\downarrow \mathbb{R}} = \tilde{V}_\vartheta \oplus \tilde{V}_\vartheta$ where $\tilde{V}_\vartheta$ is an irreducible $\mathbb{R}N$-module of type $\mathbb{H}$. Note that $(\tilde{V}_\chi)_{\downarrow N} \cong (\tilde{V}_{\mathrm{sgn}(\chi)})_{\downarrow N} \cong \tilde{V}_\vartheta$.
▶ $\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}$ is itself a super irreducible $\mathbb{R}[G,N]$-supermodule with components $(V_0)_{\downarrow \mathbb{R}}$ and $(V_1)_{\downarrow \mathbb{R}}$. Here, $(V_0)_{\downarrow \mathbb{R}} \cong (V_1)_{\downarrow \mathbb{R}} \cong \tilde{V}_\vartheta$.
  ▷ As $\mathbb{R}G$-modules, $\tilde{V}_\chi \ncong \tilde{V}_{\mathrm{sgn}(\chi)}$. As $\mathbb{R}N$-modules, $\tilde{V}_\chi \cong \tilde{V}_{\mathrm{sgn}(\chi)} \cong \tilde{V}_\vartheta$.
* The corresponding super simple two-sided superideal summand, $\mathbb{R}[G,N]^{\mathfrak{s}}(\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)})$, is isomorphic to $M_{\frac{\vartheta(1)}{2}}(\mathbb{H}) \times M_{\frac{\vartheta(1)}{2}}(\mathbb{H})$. The 0-component is $\{(A,A) : A \in M_{\frac{\vartheta(1)}{2}}(\mathbb{H})\}$, which is isomorphic to $M_{\frac{\vartheta(1)}{2}}(\mathbb{H})$. The 1-component is $\{(A,-A) : A \in M_{\vartheta(1)}(\mathbb{H})\}$.
* $\mathrm{End}^{\mathfrak{s}}_{\mathbb{R}[G,N]}(\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}) \cong \mathbb{H} \oplus \mathbb{H}v^-$.

* When induced up to $\mathbb{C}$:
  - $(\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)})^{\uparrow \mathbb{C}G} = (\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}) \otimes_{\mathbb{R}} \mathbb{C} \cong V_\chi \oplus V_\chi \oplus V_{\mathrm{sgn}(\chi)} \oplus V_{\mathrm{sgn}(\chi)}$.
  - $\tilde{V}_0^{\uparrow \mathbb{C}N} \cong \tilde{V}_1^{\uparrow \mathbb{C}N} = \tilde{V}_\vartheta \otimes_{\mathbb{R}} \mathbb{C} \cong V_\vartheta \oplus V_\vartheta$.
* $2\chi + 2\mathrm{sgn}(\chi)$ would be a super irreducible $\mathbb{R}$-super character of (G,N).

- If $\epsilon(\chi) = \epsilon(\mathrm{sgn}(\chi)) = 0$ and $\epsilon(\vartheta) = 1$, then:
  * $\mathrm{sgn}(\chi) = \overline{\chi}$.
  * $(V_\chi)_{\downarrow \mathbb{R}} = \tilde{V}_\chi$, where $\tilde{V}_\chi$ is an irreducible $\mathbb{R}G$-modules of type $\mathbb{C}$.
  * $(V_\chi)_{\downarrow \mathbb{R}N} = ((V_\chi)_{\downarrow N})_{\downarrow \mathbb{R}} \cong (V_\vartheta)_{\downarrow \mathbb{R}} = \tilde{V}_\vartheta \oplus \tilde{V}_\vartheta$ where $\tilde{V}_\vartheta$ is an irreducible $\mathbb{R}N$-module of type $\mathbb{R}$. Note that $(\tilde{V}_\chi)_{\downarrow N} \cong \tilde{V}_\vartheta \oplus \tilde{V}_\vartheta$.
  ► $\tilde{V}_\chi$ is itself a super irreducible $\mathbb{R}[G,N]$-supermodule. To construct its components $\tilde{V}_0$ and $\tilde{V}_1$, we first note that $(V_\chi)_{\downarrow N} = V_\vartheta$ is an irreducible $\mathbb{C}N$-module such that $\epsilon(\vartheta) = 1$. Hence $(V_\vartheta)_{\downarrow \mathbb{R}} = \tilde{V}_\vartheta \oplus \tilde{V}_\vartheta$. We can construct $\tilde{V}_0$ by taking a $\mathbb{C}$-basis $\{x_1, \ldots, x_{\chi(1)}\}$ of V_ϑ whose $\mathbb{R}$-span is an irreducible $\mathbb{R}N$-module. $\tilde{V}_1$ will be the $\mathbb{R}$-span of $\{ix_1, \ldots, ix_{\chi(1)}\}$. We note that $\tilde{V}_0 \cong \tilde{V}_1 \cong \tilde{V}_\vartheta$.
  * The corresponding super simple two-sided superideal summand, $\mathbb{R}[G,N]^{\mathfrak{s}}(\tilde{V}_\chi)$, is isomorphic to $M_{\vartheta(1)}(\mathbb{R}) \hat{\otimes} (\mathbb{R} \oplus \mathbb{R}v^-)$, which, as an algebra, is isomorphic to $M_{\vartheta(1)}(\mathbb{C})$. The 0-component is isomorphic to $M_{\vartheta(1)}(\mathbb{R})$.
  * When induced up to $\mathbb{C}$:
    - $\tilde{V}_\chi^{\uparrow \mathbb{C}G} = \tilde{V}_\chi \otimes_{\mathbb{R}} \mathbb{C} \cong V_\chi \oplus V_{\overline{\chi}}$.
    - $\tilde{V}_0^{\uparrow \mathbb{C}N} \cong \tilde{V}_1^{\uparrow \mathbb{C}N} = \tilde{V}_\vartheta \otimes_{\mathbb{R}} \mathbb{C} \cong V_\vartheta$.
  * $\mathrm{End}^{\mathfrak{s}}_{\mathbb{R}[G,N]}(\tilde{V}_\chi) \cong \mathbb{R} \oplus \mathbb{R}v^+$.
  * $\chi + \overline{\chi}$ would be a super irreducible $\mathbb{R}$-super character of (G,N).

The table below summarises our analysis. The super division algebra specified in each row is the super division algebra part of $\mathbb{R}[G,N]^{\mathfrak{s}}(V)$ for each possibility of an $\mathbb{R}[G,N]$-supermodule $V = V_0 \oplus V_1$.

SDA / $\mathbb{R}$	Poset type	$\mathbb{R}G$-mod. decomp.	$\mathbb{R}G$-mod. type	SDA a simple alg.?	$V_0 \cong V_1$?	$\mathbb{R}N$-mod. type	Irred. $\mathbb{R}G$-module?
$\mathbb{C} \oplus 0$	$\wedge$	$\tilde{V}_\chi$	$\mathbb{C}$	Yes	No	$\mathbb{C}$	Yes
$\mathbb{C} \oplus \mathbb{C}v$	$\vee$	$\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}$	$\mathbb{C}$	No	Yes	$\mathbb{C}$	No
$\mathbb{R} \oplus 0$	$\wedge$	$\tilde{V}_\chi$	$\mathbb{R}$	Yes	No	$\mathbb{R}$	Yes
$\mathbb{R} \oplus \mathbb{R}v^+$	$\vee$	$\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}$	$\mathbb{R}$	No	Yes	$\mathbb{R}$	No
$\mathbb{C} \oplus \mathbb{C}v^+$	$\wedge$	$\tilde{V}_\chi \oplus \tilde{V}_\chi$	$\mathbb{R}$	Yes	Yes	$\mathbb{C}$	No
$\mathbb{H} \oplus \mathbb{H}v^-$	$\vee$	$\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}$	$\mathbb{C}$	Yes	Yes	$\mathbb{H}$	No
$\mathbb{H} \oplus 0$	$\wedge$	$\tilde{V}_\chi$	$\mathbb{H}$	Yes	No	$\mathbb{H}$	Yes
$\mathbb{H} \oplus \mathbb{H}v^+$	$\vee$	$\tilde{V}_\chi \oplus \tilde{V}_{\mathrm{sgn}(\chi)}$	$\mathbb{H}$	No	Yes	$\mathbb{H}$	No
$\mathbb{C} \oplus \mathbb{C}v^-$	$\wedge$	$\tilde{V}_\chi$	$\mathbb{H}$	Yes	Yes	$\mathbb{C}$	Yes
$\mathbb{R} \oplus \mathbb{R}v^-$	$\vee$	$\tilde{V}_\chi$	$\mathbb{C}$	Yes	Yes	$\mathbb{R}$	Yes

We end this section with an interpretation of the Gow indicator. Since the Gow indicator $\eta(\vartheta)$ of an irreducible $\mathbb{C}$-character ϑ of N is equal to either 1, 0 and -1, it is tempting to think that the Gow indicator is acting like the Frobenius-Schur indicator on some kind of module. This is in fact true, and we will outline this in the upcoming subsection.

4.6 Interpretation of the Gow indicator

Ideas in this subsection are inspired from [9]. Again, let G be a group and let $N \triangleleft G$ be a normal subgroup of index 2. Then we can construct the group superalgebra $\mathbb{R}[G, N]$. From this, we construct a different superalgebra $\mathcal{A} = (\mathbb{R} \oplus \mathbb{R}v^-) \hat{\otimes} \mathbb{R}[G, N]$. This has homogeneous components $\mathcal{A}_0 = (\mathbb{R} \otimes \mathbb{R}N) + (\mathbb{R}v^- \otimes \mathbb{R}[G\backslash N]) \cong \mathbb{R}G$ and $\mathcal{A}_1 = (\mathbb{R} \otimes \mathbb{R}[G\backslash N]) + (\mathbb{R}v^- \otimes \mathbb{R}N)$.

Now let V be an irreducible $\mathcal{A}$-module. Our task is to relate $\mathrm{End}_{\mathcal{A}}(V)$ with the Gow indicator. To this end, we note that V is itself an $\mathbb{R}N$-module. Let $W \subset V$ be an irreducible $\mathbb{R}N$-submodule of V. Let $\tilde{\vartheta}$ be the character of the $\mathbb{C}N$-module $\mathbb{C} \otimes_{\mathbb{R}} W$. This is a character of N, and is either irreducible, or a sum of two irreducible characters of N. Let ϑ be an irreducible constituent of $\tilde{\vartheta}$. We can now state the following theorem:

Theorem 4.2 (Gow indicator Theorem)

$$\eta(\vartheta) := \frac{1}{|N|} \sum_{g \in G\backslash N} \vartheta(g^2) = \begin{cases} 1 & \text{if } \mathrm{End}_{\mathcal{A}}(V) = \mathbb{R}, \\ 0 & \text{if } \mathrm{End}_{\mathcal{A}}(V) = \mathbb{C}, \\ -1 & \text{if } \mathrm{End}_{\mathcal{A}}(V) = \mathbb{H}. \end{cases}$$

A proof of this theorem can be sketched by comparing the Super Frobenius-Schur indicator table with the $\mathbb{R}[G, N]$ decomposition table on page 53.

$\mathcal{S}(\vartheta)$	$q(\vartheta)$	$\epsilon(\chi)$	$\epsilon(\vartheta)$	$\eta(\vartheta)$
0	0	0	0	0
0	1	0	0	0
1	0	1	1	0
ω	1	1	1	1
$\omega^2 = i$	0	1	0	1
ω^3	1	0	-1	1
$\omega^4 = -1$	0	-1	-1	0
ω^5	1	-1	-1	-1
$\omega^6 = -i$	0	-1	0	-1
ω^7	1	0	1	-1

Super Frobenius-Schur indicator table

We note that in the above table, the values in the $\epsilon(\chi)$ and $\eta(\vartheta)$ columns follow the same cyclic sequence in the rows with non-zero $\mathcal{S}(\vartheta)$ value. In the last 8 rows, the sequence in the $\epsilon(\chi)$ column is the same as the sequence in the $\eta(\vartheta)$ column, but shifted up by one space. So given any $\vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$ and its corresponding super simple subsuperalgebra

$B = B_0 \oplus B_1$, the superalgebra $(\mathbb{R} \oplus \mathbb{R}v^-) \hat{\otimes} B$ will be, as an algebra, isomorphic to either $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, $M_n(\mathbb{H})$, $M_n(\mathbb{R}) \times M_n(\mathbb{R})$, $M_n(\mathbb{C}) \times M_n(\mathbb{C})$ or $M_n(\mathbb{H}) \times M_n(\mathbb{H})$ for some $n \geq 1$, and we can see that:

- $\eta(\vartheta) = 1$ if $(\mathbb{R} \oplus \mathbb{R}v^-) \hat{\otimes} B$ is isomorphic to $M_n(\mathbb{R})$ or $M_n(\mathbb{R}) \times M_n(\mathbb{R})$ for some $n \geq 1$.
- $\eta(\vartheta) = 0$ if $(\mathbb{R} \oplus \mathbb{R}v^-) \hat{\otimes} B$ is isomorphic to $M_n(\mathbb{C})$ or $M_n(\mathbb{C}) \times M_n(\mathbb{C})$ for some $n \geq 1$.
- $\eta(\vartheta) = -1$ if $(\mathbb{R} \oplus \mathbb{R}v^-) \hat{\otimes} B$ is isomorphic to $M_n(\mathbb{H})$ or $M_n(\mathbb{H}) \times M_n(\mathbb{H})$ for some $n \geq 1$.

From this observation, we get the Gow indicator theorem. This result was interpreted in a similar way in [9, Theorem 4.2].

5 Clifford Algebras

In this section, we will give one final family of examples of superalgebras, namely Clifford Algebras (named after William K. Clifford). We will construct Clifford Algebras over $\mathbb{R}$ and $\mathbb{C}$, and see how we can relate them to an element of $BW(\mathbb{R})$ and $BW(\mathbb{C})$. The theory in section is based on material from [2, Chapter 9, Section 2].

5.1 Introduction to Clifford Algebras

Let V be a finite dimensional F-vector space. Let us introduce some notation. The n-fold tensor product of V with itself,

$$\underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}},$$

will be written as $\bigotimes^{n} V$. By convention, $\bigotimes^{0} V := F$.

Definition 5.1 The ***tensor algebra*** of V is the vector space

$$T(V) := \sum_{n=0}^{\infty} \left(\bigotimes^{n} V \right) = F \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

So each element of $T(V)$ is a sequence of tensor products, only finitely many of which are non-zero. Clearly $T(V)$ is infinite dimensional over F. The multiplication in $T(V)$ is induced by

$$\left(\bigotimes^{n} V \right) \times \left(\bigotimes^{m} V \right) \to \bigotimes^{n+m} V,$$

where $(v_1 \otimes v_2 \otimes \cdots \otimes v_n, w_1 \otimes w_2 \otimes \cdots \otimes w_m) \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes w_1 \otimes w_2 \otimes \cdots \otimes w_m$. This makes $T(V)$ into a ring, and indeed an F-algebra.

For example, if V is a 1-dimensional F-vector space, we may write $V = Fx$ for any non-zero x. Then $T(V) \cong F[x]$, which is the algebra of F-polynomials in the variable x. Under this isomorphism, $x \otimes x$ is mapped to x^2, etc. So $T(V)$ is a polynomial algebra.

Now, given a finite dimensional F-vector space, let $q : V \to F$ be a quadratic form (q is not necessarily non-singular). Let $I(q)$ be the two-sided ideal in $T(V)$ generated by

$$\left\{ v \otimes v - q(v) \cdot 1_{T(V)} : v \in V \right\}.$$

Note that $1_{T(V)} = 1_F$.

Definition 5.2 Given a finite dimensional F-vector space V and quadratic form q on V, the Clifford Algebra $C(V, q)$ is defined as

$$C(V, q) = T(V)/I(q).$$

We note that the composition $V \to T(V) \to C(V, q)$ gives us a natural injection from V to $C(V, q)$, since $V \cap I(q) = 0_V$.

Example. Let $V = Fx$ (where x is non-zero), and let $q = \langle \alpha \rangle$ be a 1-dimensional form on V (where $\alpha \in F^{\times}$). Suppose $q(x) = \alpha$. Then we have $T(V) = F[x]$ and $I(q) = (x^2 - \alpha)$, which is the ideal generated by $x^2 - \alpha$. That means $C(V,q) \cong F[x]/(x^2 - \alpha)$. If α is not a square in F, then $C(V,q) \cong F(\sqrt{\alpha})$, which is a quadratic extension field.

Now that we have defined $C(V,q)$, we give it a superalgebra structure as follows: Write $C(V,q) = A_0 \oplus A_1$, where A_0 is the image of $\sum\limits_{n \text{ even}} \left(\overset{n}{\otimes} V \right)$ in the natural map $T(V) \to C(V,q)$, and A_1 is the image of $\sum\limits_{n \text{ odd}} \left(\overset{n}{\otimes} V \right)$.

Essentially, $C(V,q)$ inherits the natural grading of $T(V)$. We note that $I(q)$ is in the even part of $T(V)$. We shall denote the even part of $C(V,q)$ as $C_0(V,q)$ and the odd part as $C_1(V,q)$.

Let us outline some properties of Clifford Algebras.

Let V be an F-vector space, and let q be a quadratic form on V. If $\{x_1, \dots, x_n\}$ is an orthogonal basis for V (with respect to the associated bilinear form $\langle a, b \rangle = q(a+b) - q(a) - q(b)$), then, in $C(V,q)$, $x_i x_j = -x_j x_i$ whenever $i \neq j$. In other words, distinct x_i's anti-commute in the Clifford Algebra.

This can be quickly verified: x_i, x_j being orthogonal with respect to q means $q(x_i + x_j) - q(x_i) - q(x_j) = 0$. Hence, in $C(V,q)$, $(x_i + x_j)^2 - x_i^2 - x_j^2 = x_i^2 + x_i x_j + x_j x_i + x_j^2 - x_i^2 - x_j^2 = x_i x_j + x_j x_i = 0 \implies x_i x_j = -x_j x_i$.

Proposition 5.3 *Let (V,q) be a quadratic vector space, and let $\{x_1, \dots, x_n\}$ be an orthogonal basis for V with respect to q. Set $\alpha_i = q(x_i)$, $i = 1, \dots, n$.*

Then $\{1, x_1, x_2, \dots, x_n, x_1x_2, x_1x_3, \dots, x_{n-1}x_n, x_1x_2x_3, \dots, x_1x_2 \cdots x_n\}$ is a basis for $C(V,q)$. In particular, $\dim_F(C(V,q)) = 2^n$, independent of the choice of basis for V.

We remark that not all α_i's are necessarily non-zero.

Proposition 5.4 *Let (V,q) be a quadratic F-vector space. If V is even dimensional, then $C(V,q)$ is a central simple algebra over F. If V is of odd dimension, then $C_0(V,q)$ is central simple over F.*

Before we look at the super tensor product of two Clifford Algebras, let us outline a universal property of $C(V,q)$.

Proposition 5.5 *Let F be a field, (V,q) a quadratic F-vector space, and A an F-algebra. Suppose we have a linear mapping $f : V \to A$ which is compatible with q in the sense that $f(v)^2 = q(v) \cdot 1_A$. Then there exists a unique F-algebra homomorphism $g : C(V,q) \to A$ such that $f = g \circ i$, where $i : V \to C(V,q)$ is the natural injective map we outlined earlier. We have the following picture:*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "i"'] & A \\ C(V,q) \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Proof. We have our natural map $V \xrightarrow{i'} T(V) \xrightarrow{p} C(V,q)$. That means $i = p \circ i'$. Let us define the linear map $h : T(V) \to A$. Given a sequence $(\lambda_0, (v_{1,1}), (v_{2,1} \otimes v_{2,2}), (v_{3,1} \otimes v_{3,2} \otimes v_{3,3}), \dots)$ in $T(V)$, h would map this to $\lambda_0 + f(v_{1,1}) + f(v_{2,1})f(v_{2,2}) + f(v_{3,1})f(v_{3,2})f(v_{3,3}) + \cdots$. We now have the following picture:

$$\begin{tikzcd} & & A & \\ V \arrow[urr, "f"] \arrow[r, "i'"'] & T(V) \arrow[ur, dashed, "h"] \arrow[rr, "p"'] & & C(V,q) \arrow[ul, dashed, "g"'] \end{tikzcd}$$

By the universal property of tensor product, f factors via $h : T(V) \to A$. Let us show that h vanishes on $I(q)$ (this would yield a well-defined map $g : C(V, q) \to A$). We have $h(v\otimes v - q(v)\cdot 1_A) = h(v\otimes v) - h(q(v)\cdot 1_A) = f(v)^2 - q(v)\cdot 1_A = 0$ by compatibility. ■

Let us now recall that, given a quadratic form q on V, there is a symmetric bilinear form $\phi : V \times V \to F$ associated with q. ϕ would be defined by $\phi(v, w) = \frac{1}{2}(q(v + w) - q(v) - q(w))$. This works when $\mathrm{char} F \neq 2$. When our underlying field F does not have characteristic 2, we have a one-to-one correspondence between the set of quadratic forms on V and the set of symmetric bilinear forms on V. The reverse correspondence is given by $q(v) = \phi(v, v)$.

⋆ From now on, we will assume the characteristic of F is not 2.

Let V_1, V_2 be F-vector spaces with respective quadratic forms q_1 and q_2. Denote the symmetric bilinear forms associated with q_1 and q_2 as ϕ_1 and ϕ_2 respectively. Let $V_1 \oplus V_2$ be the direct sum of the vector spaces. We define a symmetric form on $V_1 \oplus V_2$ as follows:

$$(V_1 \oplus V_2) \times (V_1 \oplus V_2) \to F$$

$$(v_1 \oplus v_2, v_1' \oplus v_2') \mapsto \phi_1(v_1, v_1') + \phi_2(v_2, v_2').$$

It is straightforward to check that this is a symmetric bilinear form. It is called the orthogonal sum of ϕ_1 and ϕ_2. It is denoted as $\phi_1 \perp \phi_2$. We note that, for any $v_1 \in V_1$, $v_2 \in V_2$, $(\phi_1 \perp \phi_2)(v_1, v_2) = \phi_1(v_1, 0) + \phi_1(0, v_2) = 0 + 0 = 0$. So, under this symmetric bilinear form, every vector in V_1 is orthogonal to every vector in V_2. We may denote the quadratic form associated with $\phi_1 \perp \phi_2$ as $q_1 \perp q_2$.

We can now explore the super tensor product of two Clifford Algebras.

Proposition 5.6 *Let (V_1, q_1), (V_2, q_2) be quadratic F-vector spaces. Then $C(V_1 \oplus V_2, q_1 \perp q_2) \cong C(V_1, q_1) \hat{\otimes} C(V_2, q_2)$ as superalgebras.*

Recall that, given two superalgebras A, B, $f : A \to B$ is a superalgebra homomorphism if f is an algebra homomorphism that satisfies $f(A_i) \subset B_i$ for $i = 0, 1$.

Proof. Let us construct the linear map $f_0 : V_1 \oplus V_2 \to C(V_1, q_1) \hat{\otimes} C(V_2, q_2)$, $v_1 \oplus v_2 \mapsto (v_1 \otimes 1) + (1 \otimes v_2)$. This is compatible with $q_1 \perp q_2$ since, using multiplication in $C(V_1, q_1) \hat{\otimes} C(V_2, q_2)$,

$$f_0(v_1 \oplus v_2)^2 = (v_1 \otimes 1 + 1 \otimes v_2)(v_1 \otimes 1 + 1 \otimes v_2) = v_1^2 \otimes 1 + 1 \otimes v_2^2$$

$$+ v_1 \otimes v_2 - v_1 \otimes v_2 = v_1^2 \otimes 1 + 1 \otimes v_2^2 = q_1(v_1) \otimes 1 + 1 \otimes q_2(v_2) =$$

$$q_1(v_1) \cdot 1 + q_2(v_2) \cdot 1 = (q_1(v_1) + q_2(v_2)) \cdot 1 = (q_1 \perp q_2)(v_1 \oplus v_2) \cdot 1.$$

Hence, by the universal property of $C(V_1 \oplus V_2, q_1 \perp q_2)$, f_0 factors through $f : C(V_1 \oplus V_2, q_1 \perp q_2) \to C(V_1, q_1) \hat{\otimes} C(V_2, q_2)$. We have the following picture:

$$\begin{tikzcd} & C(V_1,q_1)\hat{\otimes} C(V_2,q_2) & \\ V_1 \oplus V_2 \arrow[ur, "f_0"] \arrow[r] & T(V_1 \oplus V_2) \arrow[u, dashed] \arrow[r] & C(V_1 \oplus V_2, q_1 \perp q_2) \arrow[ul, dashed, "f"'] \end{tikzcd}$$

Note that f is surjective as the image of f_0 generates all of $C(V_1, q_1) \hat{\otimes} C(V_2, q_2)$.

To show that f is a superalgebra isomorphism, we can construct the inverse of f. To this end, we note that the map

$$V_1 \to V_1 \oplus V_2 \xrightarrow{\text{natural injection}} C(V_1 \oplus V_2, q_1 \perp q_2)$$

is compatible with q_1. Hence, by universality, there exists a map $\alpha_1 : C(V_1, q_1) \to C(V_1 \oplus V_2, q_1 \perp q_2)$. Similarly, the natural map

$$V_2 \to V_1 \oplus V_2 \xrightarrow{\text{natural injection}} C(V_1 \oplus V_2, q_1 \perp q_2)$$

gives us $\alpha_2 : C(V_2, q_2) \to C(V_1 \oplus V_2, q_1 \perp q_2)$. Each α_i is a superalgebra homomorphism, and the images of α_1 and α_2 "anti-commute" in the sense described earlier. This is a consequence of the earlier remark that elements of an orthogonal basis anti-commute in the Clifford Algebra. Hence, we can create a well-defined superalgebra homomorphism:

$$\alpha_1 \hat{\otimes} \alpha_2 : C(V_1, q_1) \hat{\otimes} C(V_2, q_2) \to C(V_1 \oplus V_2, q_1 \perp q_2).$$

We can check that $(\alpha_1 \hat{\otimes} \alpha_2) \circ f$ is the identity map (on $C(V_1 \oplus V_2, q_1 \perp q_2)$) by verifying that it is the identity on elements of an orthogonal basis for $(V_1 \oplus V_2, q_1 \perp q_2)$. We conclude that f is a superalgebra isomorphism. ■

5.2 Clifford Algebras over $\mathbb{C}$

Let us now give an example of a $\mathbb{C}$-Clifford Algebra $C(V, q)$ for every class in $BW(\mathbb{C})$, the Brauer-Wall group of $\mathbb{C}$. We need to specify both the real vector space and the non-singular quadratic form.

(i) $x^0 = [\mathbb{C} \oplus 0] = 1_{BW(\mathbb{C})}$:
$V = \mathbb{R}^2$, q is represented as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The basis of $C(V, q)$ is $\{1, x, y, xy\}$. So $C_0(V, q) = \text{span}(1, xy)$ and $C_1(V, q) = \text{span}(x, y)$. We can see that, in this case $C(V, q) \cong M_{(1,1)}(\mathbb{C})$.

(ii) $x^1 = [\mathbb{C} \oplus \mathbb{C}v]$:
$V = \mathbb{R}x$, q is represented as $\left(1\right)$. The basis of $C(V, q)$ is $\{1, x\}$. So $C_0(V, q) = \mathbb{R} \cdot 1$ and $C_1(V, q) = \mathbb{R}x$. We can see that, in this case $C(V, q) \cong \mathbb{C} \oplus \mathbb{C}v$.

5.3 Clifford Algebras over $\mathbb{R}$

Let us now give an example of an $\mathbb{R}$-Clifford Algebra $C(V, q)$ for every class in $BW(\mathbb{R})$, the Brauer-Wall group of $\mathbb{R}$. We need to specify both the real vector space and the non-singular quadratic form.

(i) $x^0 = [\mathbb{R} \oplus 0] = 1_{BW(\mathbb{R})}$:
$V = \mathbb{R}^2$, q is represented as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The basis of $C(V, q)$ is $\{1, x, y, xy\}$. So $C_0(V, q) = \text{span}(1, xy)$ and $C_1(V, q) = \text{span}(x, y)$. We can see that, in this case $C(V, q) \cong M_{(1,1)}(\mathbb{R})$.

(ii) $x^1 = [\mathbb{R} \oplus \mathbb{R}v^+]$:
$V = \mathbb{R}x$, q is represented as $\left(1\right)$. The basis of $C(V, q)$ is $\{1, x\}$. So $C_0(V, q) = \mathbb{R} \cdot 1$ and $C_1(V, q) = \mathbb{R}x$. We can see that, in this case $C(V, q) \cong \mathbb{R} \oplus \mathbb{R}v^+$.

(*iii*) $x^2 = [\mathbb{C} \oplus \mathbb{C}v^+]$:
$V = \mathbb{R}^2$, q is represented as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The basis of $C(V,q)$ is $\{1, x, y, xy\}$. So $C_0(V,q) = \mathrm{span}(1, xy)$ and $C_1(V,q) = \mathrm{span}(x,y)$. We can see that, in this case $C(V,q) \cong \mathbb{C} \oplus \mathbb{C}v^+$.

(*iv*) $x^3 = [\mathbb{H} \oplus \mathbb{H}v^-]$:
$V = \mathbb{R}^3$, q is represented as $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The basis of $C(V,q)$ is $\{1, x, y, z, xy, xz, yz, xyz\}$. So $C_0(V,q) = \mathrm{span}(1, xy, xz, yz)$ and $C_1(V,q) = \mathrm{span}(x, y, z, \ xyz)$. In this case, $C(V,q) \cong \mathbb{H} \oplus \mathbb{H}v^-$.

(*v*) $x^4 = [\mathbb{H} \oplus 0]$:
$V = \mathbb{R}^3$, q is represented as $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The basis of $C(V,q)$ is $\{1, w, x, y, z, wx, wy, wz, xy, xz, yz, wxy, wxz, wyz, xyz, wxyz\}$. So $C_0(V,q) = \mathrm{span}(1, \ wx, wy, wz, xy, xz, yz, wxyz)$ and $C_1(V,q) = \mathrm{span}(w, x, y, z, wxy, wxz, wyz, xyz)$. In this case, $C(V,q) \cong M_{1,1}(\mathbb{H})$.

(*vi*) $x^5 = [\mathbb{H} \oplus \mathbb{H}v^+]$:
$V = \mathbb{R}^3$, q is represented as $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. The basis of $C(V,q)$ is $\{1, x, y, z, xy, xz, yz, xyz\}$. So $C_0(V,q) = \mathrm{span}(1, xy, xz, yz)$ and $C_1(V,q) = \mathrm{span}(x, y, z, xyz)$. In this case, $C(V,q) \cong \mathbb{H} \oplus \mathbb{H}v^+$.

(*vii*) $x^6 = [\mathbb{C} \oplus \mathbb{C}v^-]$:
$V = \mathbb{R}^2$, q is represented as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The basis of $C(V,q)$ is $\{1, x, y, xy\}$. So $C_0(V,q) = \mathrm{span}(1, xy)$ and $C_1(V,q) = \mathrm{span}(x,y)$. In this case, $C(V,q) \cong \mathbb{C} \oplus \mathbb{C}v^-$.

(*viii*) $x^7 = [\mathbb{R} \oplus \mathbb{R}v^-]$:
$V = \mathbb{R}x$, q is represented as $\begin{pmatrix} -1 \end{pmatrix}$. The basis of $C(V,q)$ is $\{1, x\}$. So $C_0(V,q) = \mathbb{R} \cdot 1$ and $C_1(V,q) = \mathbb{R}x$. In this case, $C(V,q) \cong \mathbb{R} \oplus \mathbb{R}v^-$.

We note that when $\mathrm{char}(F) \neq 2$, a Clifford Algebra $C(V,q)$ over F is ordinary central simple if V is of even dimension, and $C(V,q)$ is not ordinary central simple if V is of odd dimension.

6 Conclusion and Future work

Throughout the thesis, we have seen that many results regarding algebras and modules have their equivalents in Superalgebra Theory. One important result we have proven is the Super Wedderburn's Theorem, which states that any super semisimple superalgebra is a direct sum of super simple superalgebras of the form $M_{(r,s)} \hat{\otimes} D$ where D is a super division algebra.

Given a field F, we have also defined a group of equivalence classes of super central simple F-superalgebras called the Brauer-Wall group of F (denoted as $BW(F)$), which is the superalgebra equivalent of the Brauer group. One important result is that, if $\mathrm{char}(F) \neq 2$, then we have $BW(F)/P(F) \cong C_2$, and $P(F)/Br(F) \cong F^\times/F^{\times 2}$.

We have also introduced the Super Frobenius-Schur indicator of a super irreducible $\mathbb{C}$-super character of (G, N). It associates a $\mathbb{C}[G, N]$-supermodule with a super division algebra over $\mathbb{R}$. We also studied in detail the supermodules of the superalgebras $\mathbb{C}[G, N]$ and $\mathbb{R}[G, N]$, and how we can construct them from ordinary modules.

There are a couple of topics in the theory of superalgebras that can be explored in future work. For example, one could examine the theory of super semisimple superalgebras over a field of characteristic 2. Further reading on this topic can be found in [5].

One may also study super representations and super characters of specific pairs (G, N), where $N \triangleleft G$ and $|G : N| = 2$. A potentially interesting set of examples would be the pairs (S_n, A_n), $(2 \cdot S_n^+, 2 \cdot A_n)$, and $(2 \cdot S_n^-, 2 \cdot A_n)$.

One could also explore the theory of graded algebras when the grading is over a different group. Given a group G, one can explore if there are complications to the theory of G-graded algebras if the characteristic of F divides the order of G.

To generalise the idea of a group superalgebra, let G be a group and let $N \triangleleft G$ be normal subgroup of index p. Let $g \in G \backslash N$. Then we could give $\mathbb{C}G$ a C_p-grading by letting the 0-component be $\mathbb{C}N$, and the i-component be $\mathbb{C}[Ng^i]$. One can introduce a generalised version of the Frobenius-Schur indicator that applies to characters of $\mathbb{C}G$. The C_p-graded Frobenius-Schur indicator would relate characters of $\mathbb{C}G$ with C_p-graded division algebras.

Another further topic one could explore is the Category Theory of superalgebras, supermodules and super homomorphisms. Studying this topic can explain why much of the structure theory of algebras and modules can be carried over to superalgebras and supermodules.

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