# On AIMD Congestion Control in Multiple Bottleneck Networks

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***Abstract*— We consider a linear algebraic model of the Additive-Increase Multiplicative-Decrease congestion control algorithm and present results on average fairness and convergence for multiple bottleneck networks. Results are presented for networks of both long-lived and short-lived flows.**

***Index Terms*— Additive-increase multiplicative-decrease (AIMD), network congestion control, multiple bottlenecks.**

## I. Introduction

TCP, in congestion avoidance mode, is based primarily on Chiu and Jain's [2] *Additive-Increase Multiplicative-Decrease* (AIMD) algorithm for decentralized allocation of a shared resource (e.g., bandwidth) among competing users. With some minor modifications, the AIMD algorithm has served the networking community well over the past two decades and it continues to provide the basic building block upon which today's internet communication is built.

The dynamics of communication networks employing the AIMD algorithm have been studied extensively; for example, see [1], [3], [4], [6], [7] and references therein. For networks where the resource constraint is a bound on the sum of the resource shares of the users, basic stability and convergence properties have been determined, both in a deterministic and in a stochastic setting. In particular, it has been shown that (with a fixed number of users) such networks possess unique stable equilibria to which the system converges geometrically from all starting points.

However, a common assumption is that all sources are limited by a single bottleneck link. Recently, a number of authors have reported that in such circumstances, AIMD dynamics can lead to network oscillations. Our interest here is to derive results that describe the behavior of AIMD networks in a quantifiable manner in the presence of multiple-bottleneck links, and to prove network convergence to well defined equilibria. The main contribution of this note is to present a variant of a recently proposed matrix model that allows us to derive results which predict a degree of fairness in resource allocation between flows that compete directly with each other, even in the presence of network oscillations.

## II. Mathematical Model

Similar to [6], we denote by $x_i(k) \in \mathbb{R}^+$ the flow rate of the $i^{th}$ source at the $k^{th}$ congestion event. Denote the additive increase and multiplicative decrease parameters of the $i^{th}$ source by $\alpha_i \in (0,\infty)$ and $\beta_i \in (0,1)$ respectively.

We assume that the network consists of $n_n$ nodes, labeled $j = 1, 2, \ldots, n_n$, and that each flow originates with a source, $i \in \{1, 2, \ldots, n_s\}$ and passes through a set of nodes $N_i \subset \{1, 2, \ldots, n_n\}$. Denote by $\Omega_j \subset \{1, 2, \ldots, n_s\}$ the set of flows that pass through node $j$. We assume that each node has a total capacity $B_j > 0$, so that the flow rates are constrained as:

$$B_j \geq \sum_{i\in\Omega_j} x_i(k); \forall j \in \{1, 2, \ldots, n_n\},\ k \in \mathbb{Z}_{\geq 0}. \quad (1)$$

We denote the stacked vector of source flows by $X(k)$ and write the constraints (1) in vector form as:

$$B_j \geq L_j^T X(k); \forall j \in \{1, 2, \ldots, n_n\}, k \in \mathbb{Z}_{\geq 0} \quad (2)$$

where $L_j$ is a vector with $i^{th}$ element unity if the $i^{th}$ flow includes node $j$, and zero otherwise. We assume that all flows include at least one node and thus all flow rates are bounded. At the $k^{th}$ congestion event, we assume that at least one node is congested. We denote by $J(k) \subset \{1, 2, \ldots, n_n\}$ the set of nodes congested at $k$; that is $J(k) = \{j : B_j = L_j^T X(k)\}$.

For synchronous traffic, the flow rates evolve according to:

$$X(k + 1) = A_{j(k)} X(k) + UT(k) \quad (3)$$

with $U$ a vector of the increase parameters, $T(k)$ the time between congestion events, and $A_j$ a diagonal matrix with $i^{th}$ element:

$$(A_j)_{ii} = \left\{ \begin{array}{c} \beta_i : i \in \Omega_j \\ 1 : otherwise \end{array} \right\}. \quad (4)$$

In other words, when node $j$ is congested, all flows that transit node $j$ (i.e., all flows in $\Omega_j$) reduce their rate to $\beta_i x_i(k)$, while flows not crossing node $j$ continue to increase their rates.

More generally, we can consider the case of asynchronous traffic where, at random, one or more flows will experience congestion rather than all flows being notified simultaneously. In this case the model of (3) becomes more complex, since the appropriate $A$ matrix is no longer a deterministic function of the congested node, $j(k)$. For simplicity, we follow [6] wherein the probability associated with whether or not source $i$ experiences congestion is independent[1] of other sources and is given by $\lambda_i$. The equivalent model to (3) becomes:

$$X(k + 1) = A_k X(k) + UT(k) \quad (5)$$

where $A_k$ is a diagonal random matrix with elements

$$A_k(ii) = \left\{ \begin{array}{ll} \beta_i & \text{w.p. } \lambda_i \text{ for } i \in \Omega_{j(k)} \\ 1 & otherwise \end{array} \right\}.$$

Under the assumption of independence of the probabilities in the elements of $A_k$, from (5) the expected value of $X(k)$ (denoted $\mathcal{E}\{X(k)\}$) satisfies the recursion:

$$\mathcal{E}\{X(k + 1)\} = A'_k \mathcal{E}\{X(k)\} + U\mathcal{E}\{T(k)\} \quad (6)$$

[1] Here the independence is both serially in time and between different flows.

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where $A'_k$ is a constant diagonal matrix with elements $\beta'_i = 1 - \lambda_i + \lambda_i \beta_i$ for $i \in \Omega_{j(k)}$.

The key to the following results is that we look at competition between flows that take a similar path through the network. We call a group of such flows *parallel flows*. From here on we consider $k$ to be the congestion epochs for the parallel flow grouping under consideration and denote the flow rates of the parallel flow group by $X_p(k)$ which evolve as

$$X_p(k+1) = A_p X_p(k) + U_p T(k), \qquad (7)$$

where $A_p$ is the $A_{j(k)}$ of (3) or $A'_k$ of (6) as appropriate.

## III. Results

### A. Synchronous Flows: Time Averages

*Claim 1:* Consider any set of parallel flows. Take any $U_p^\perp$ orthogonal to $U_p$ and suppose that either

$$\beta_{i_1} = \beta_{i_2} = \cdots = \beta_{i_p} =: \beta_p, \quad \text{or} \qquad (8)$$

$$\lim_{k\to\infty} T(k) = T_\infty \qquad (9)$$

then $\lim_{k\to\infty} \left(U_p^\perp (I - A_p) X_p(k)\right) = 0$.

*Proof:* Note that in either case, from (7) that:

$$X_p(k) = A_p^k X_p(0) + \sum_{\ell=0}^{k-1} A_p^{k-1-\ell} U_p T(\ell). \qquad (10)$$

The first term in (10) decays exponentially fast to zero, so it remains to evaluate properties of the remaining term.

First, suppose that (8) holds. Then it follows that $A_p = \beta_p I$, $A_p^\ell = \beta_p^\ell I$, and $U_p^\perp (I - A_p) = (1-\beta_p) U_p^\perp$. Using these facts along with (10) and ignoring initial conditions gives:

$$\left(U_p^\perp (I - A_p) X_p(k)\right) = (1-\beta_p) U_p^\perp U_p \sum_{\ell=0}^{k-1} \beta_p^{k-1-\ell} T(\ell)$$

and the result follows.

Alternatively, suppose that (9) holds. In this case we note that as $k \to \infty$, again ignoring initial conditions:

$$X_p(k) = \sum_{\ell=0}^{k-1} A_p^{k-1-\ell} U_p T_\infty + \sum_{\ell=0}^{k-1} A_p^{k-1-\ell} U_p (T(\ell) - T_\infty)$$

and the result follows. ∎

In other words, under the conditions stated in Claim 1, the states converge to a one dimensional subspace aligned with $(I - A_p)^{-1} U_p$ (the Perron eigenvector in [6, Theorem 2.1]) which has $i^{th}$ element $(1-\beta_i)^{-1}\alpha_i$. Note however, that this does not apply in general. However, the following is true:

*Claim 2:* Consider any set of parallel synchronized flows. Then, for any $k_0$,

$$\lim_{K\to\infty} \left( \frac{1}{K} \sum_{k=k_0}^{k_0+K} U_p^\perp (I - A_p) X_p(k) \right) = 0. \qquad (11)$$

*Proof:* Taking summations of (7) and rearranging yields

$$(I - A_p) \frac{1}{K} \sum_{k=k_0}^{k_0+K} X_p(k) = \frac{1}{K} \left( X_p(k_0) - X_p(k_0+K+1) \right) + U_p \frac{1}{K} \sum_{k=k_0}^{k_0+K} T(k).$$

Multiplying from the left by $U_p^\perp$ and taking the limit as $K \to \infty$ gives the result since $X_p(k)$ is bounded. ∎

*Remark 3:* Since Claim 2 holds for any set of synchronized flows, including any pair of flows, it represents a kind of average inter-flow fairness. The time average of the peak flows represented in $X_p(k)$ lies on a given ray from the origin. Moreover, for any flows $\ell$ and $m$ that are parallel, take $U_p^\perp$ as a vector with all elements zero, except the $\ell^{th}$ element $1/\alpha_\ell$ and the $m^{th}$ element $-1/\alpha_m$. We then have the long-term time average (which we denote with an overbar; i.e., $\overline{f_k} := \lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K} f_k$):

$$\overline{\left(\frac{1-\beta_\ell}{\alpha_\ell}\right) (X(k))_\ell - \left(\frac{1-\beta_m}{\alpha_m}\right) (X(k))_m} = 0 \qquad (12)$$

and therefore, provided the appropriate time averages exist,

$$\left(\frac{1-\beta_\ell}{\alpha_\ell}\right) \overline{(X(k))_\ell} = \left(\frac{1-\beta_m}{\alpha_m}\right) \overline{(X(k))_m}. \qquad (13)$$

In the case where the states do converge, Claim 2 immediately gives the following corollary.

*Corollary 4:* Consider any set of parallel synchronized flows and suppose that $X_p(k)$ converges to a limit denoted by $X_p(\infty)$. Then $U_p^\perp (I - A_p)^{-1} X_p(\infty) = 0$.

### B. Synchronous Parallel Flows: Ensemble Averages

Because the rest of the network can influence the detailed behavior of a set of parallel flows, even in the synchronous case, it is not possible to guarantee that the parallel flows converge. The results in Section III-A give time average results that apply in this case. Here we give some results for ensemble averages for the synchronous parallel flow case.

In the previous approach, the capacity constraint (1) can be thought of as a "router view" of congestion. Alternatively, we may consider the bandwidth a parallel flow group will obtain at congestion. We observe that this bandwidth will vary depending on which node is congested as well as how much capacity is taken by other flow groups. As such, the capacity constraint seen by any individual flow group will be random and time-varying. Using $\mathcal{I}_p$ to denote the $p^{th}$ parallel flow group, we can write the capacity constraint at congestion as

$$B_p(k) = \sum_{i\in\mathcal{I}_p} x_i(k); \ \forall k \in \mathbb{Z}_{\geq 0}. \qquad (14)$$

Note that $B_p(k)$ is necessarily bounded by the minimum capacity link traversed by the flow group $\mathcal{I}_p$.

Using the previous vector notation, at congestion we have

$$L_p^T X_p(k+1) = B_p(k+1), \qquad (15)$$

where $L_p$ is a vector of dimension $|\mathcal{I}_p|$ consisting of all ones. We assume that the process $B_p(k)$ is a stationary random process; i.e., there exists a finite real number $\bar{B}_p > 0$ such that $\mathcal{E}\{B_p(k)\} = \bar{B}_p$. Taking expectations on both sides of (15), and using the evolution equation (7) we obtain

$$L_p^T \left( A_p \mathcal{E}\{X_p(k)\} + U_p \mathcal{E}\{T(k)\} \right) = \bar{B}_p. \qquad (16)$$

The expected time between congestion events is then $\mathcal{E}\{T(k)\} = \bar{T}^* = \frac{\bar{B}_p}{L_p^T (I-A_p)^{-1} U_p}$ and it is not difficult to show that the expected flow rate converges exponentially to

$$\mathcal{E}\{X_p(k)\} = \bar{X}_p^* = (I - A_p)^{-1} U_p \bar{T}^*.$$

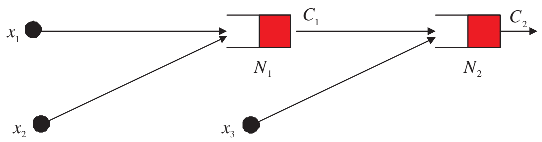


Fig. 1. Two node, three flow scenario

*Remark 5:* It is important to note here that we not only characterize the asymptote but also the dynamics of the process. Convergence to the equilibrium state is exponential and bounds on the rate of convergence can be derived. It is also important to note that the dynamics of the second moment can be expressed in a similar manner to the above analysis [5].

From a practical viewpoint, we may not know the expected value of the bandwidth, $\bar{B}_p$. However, the above analysis indicates how parallel flows will share available bandwidth within the parallel group. For example, if all flows in the group have the same increase and decrease parameters, the (unknown) bandwidth will be shared equally on average.

### C. Asynchronous Parallel Flows: Ensemble Averages

We now consider the more general model (5) that allows randomness in determining which flows experience lost packets at a congestion event. In this case, by the same arguments as in Claim 2, applied to (6), we obtain:

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=k_0}^{k_0+K} U_p^{\perp}(I - A_p)\mathcal{E}\left\{X_p(k)\right\} = 0. \qquad (17)$$

Furthermore, if the process is ergodic, then (17) simplifies to $U_p^{\perp}(I - A_p)\mathcal{E}\left\{X_p(k)\right\} = 0$.

## IV. Simulations

**Example 1:** Consider the network topology of Figure 1 with $C_1 = 2.5$ units and $C_2 = 5$ units. Drops are generated so that all nodes contributing to congestion are informed of congestion; namely, $x_1$ and $x_2$ are informed every time a node is congested, whereas $x_3$ is informed only when node $N_2$ is congested.

Here $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\beta_1 = 0.5$, $\beta_2 = 0.75$, and $\beta_3 = 0.9$. It follows from Claim 3 that $\bar{X}(k)_1 = 2\,\bar{X}(k)_2$. This is confirmed from the simulation results depicted in Figure 2.

**Example 2:** Consider again the network topology of Figure 1. Drops at congestion are generated at each node according to fixed probabilities (uniform for every source utilizing the constraint). Again, the flows $x_1(k)$ and $x_2(k)$ are parallel flows. In this simulation we observe $\lim_{K\to\infty} \frac{1}{K}\sum_{k=0}^{K} X_i(k)$, $i \in 1, 2$ over the first 1600 congestion events.

We see from Figure 3 that

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K} X_1(k) \approx 2\left(\lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K} X_2(k)\right),$$

which agrees with Claim 3 and the associated remarks.

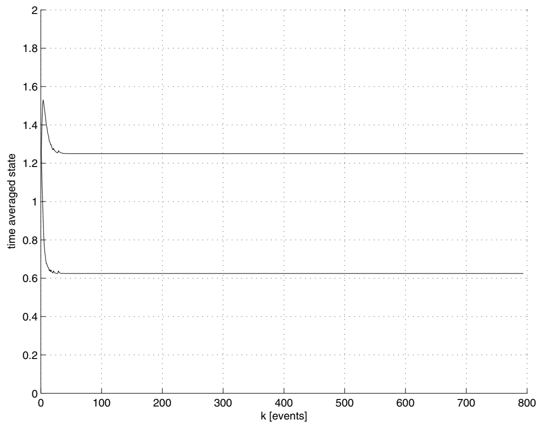


Fig. 2. Predictions of Claim 3: Synchronized network

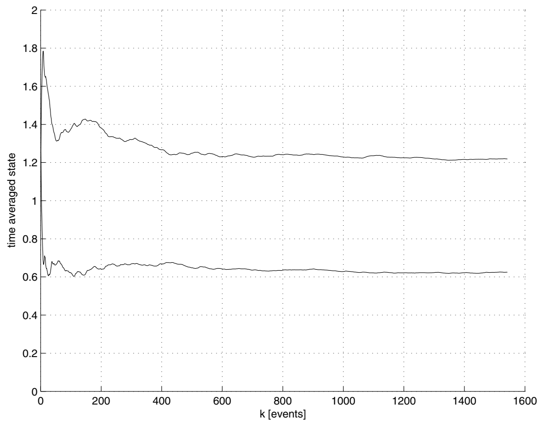


Fig. 3. Predictions of Claim 3: Stochastic network

## V. Conclusions

In this note we have demonstrated how network flows operating the AIMD congestion control protocol share bandwidth with other flows that have a similar path through the network. In particular, we have provided expressions in terms of the AIMD parameters for both the long-term time average and for ensemble averages, with the former statistic describing behavior of the network over long time scales, and the latter describing the network behavior over shorter time-scales.

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