Chaos, Solitons and Fractals 109 (2018) 31–35

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

A smooth-piecewise model to the Cord Attractor

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ARTICLE INFO

Article history:
Received 28 September 2017
Revised 2 February 2018
Accepted 3 February 2018
Available online 14 February 2018

Keywords:
Piecewise approximation
Cord attractor
Chaos
Nonlinear dynamics
Virtual fixed point

ABSTRACT

This paper reports a smooth-piecewise model to the Cord Attractor. The fact that the Cord Attractor has one real fixed point and two complex conjugate fixed points does not allow to use a technique based on the building of two affine subsystems, which requires at least two real fixed points [Chaos 16, 013115 (2006)]. In this work, we have presented a procedure to at least partially overcome this limitation using a virtual fixed point; the location of the fixed point is based on the topology of the original system. The switching function has been designed as a smooth function. The phase space and the local-finite largest Lyapunov exponent have been used to compare the resulting attractor with the original Cord Attractor.

1. Introduction

Piecewise linear representation has been considered as an important topic in the literature on circuits systems [1] and nonlinear dynamical systems [2,3] The research is concerned merely in mathematical modelling or in the implementation of circuits. It has been noticed that piecewise linear representation can reproduce the same dynamics of globally nonlinear dynamics. Some systems have already been represented, such as Rössler and Lorenz [2]. On the other hand, original piecewise linear systems, such as Chua circuit [4–6], can be represented by means of globally nonlinear models [7].

According to [1] there are two main motivations for studying piecewise approximation: *mathematical modelling* in a simpler way and *circuit modelling* to attend an implementation task. From the point of view of mathematical modelling, the author in [8] has developed an approximate method based on piecewise linearization for the determination of periodic orbits of nonlinear oscillators. Regarding the circuit modelling approach, the authors in [9] have recently developed a procedure to design chaotic systems by piecewise affine systems.

Research on the approximation of nonlinear dynamical system by a piecewise affine model is not new [3]. It has been proposed in [10] a set of high-level canonical piecewise linear (HL-CPWL) functions to form a representation basis for the set of piecewise linear functions. Although, this work represents a landmark on the research about piecewise approximation, it has been pointed out, from the point of view of mathematical modelling, that this technique demands a careful attention in the number of parameters [1]. Not differently, when the concern relies on circuit modelling, as the simplest circuit structure is generally desired. In such works, a black box system identification has usually been used.

A different approach may be found in [3,11] where the authors have developed a technique to build piecewise models by associating one affine subsystem to each real fixed point. The authors in [3] have also described some guidelines for constructing piecewise affine models based on feedback circuit analysis and on the identification of relevant terms of the differential equations. The main advantage of this technique is the possibility to build piecewise models from two affine systems, which is generally expected to be very simple. This is an important feature, as it is desired to have simple models obtained from the piecewise approximation [1]. However, this technique relies on the fact that the affine systems should be designed using real fixed points, as it has been done for Rössler and Lorenz [2,3]. In this approach, piecewise models are associated by one affine subsystem to each fixed point, in which the subsystems are linked by switching surfaces. One intrinsic requirement for this method is the existence of at least two real fixed points. Therefore, the approach developed in [2] is no longer efficient to systems that does not attend this requirement. This is the case of the Cord Attractor [12,13], which presents one real fixed point and two complex conjugate fixed points. To at least partially overcome this problem, this paper presents a procedure where a virtual fixed point is created based on the topology of the original system. It is presented an heuristic method to estimate the location of this new virtual fixed point. The

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https://doi.org/10.1016/j.chaos.2018.02.001
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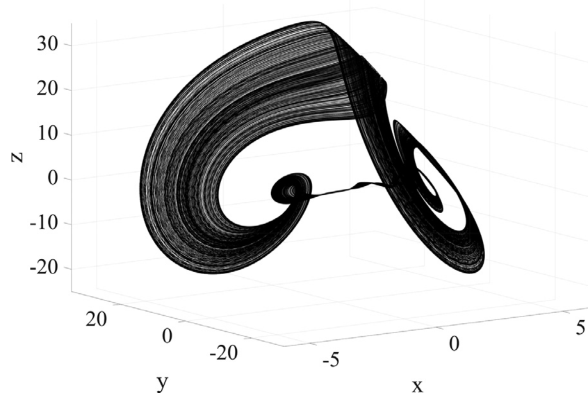

Fig. 1. The Cord Attractor. The parameters $(a, b, F, G) = (0.258, 4.033, 8, 1)$, with initial conditions $(x_0, y_0, z_0) = (0.1, 0.1, 0.1)$ according to [12].

switch between the two affine systems is based on a smooth-like procedure.

To proceed some validation of our proposed method, we have performed the visualisation of the original attractor and piecewise model. We have also presented a comparison between the local-finite largest Lyapunov exponent, calculated by the approaches described in [14] and in [15]. It is worth to state that recent results on the investigation over hidden attractor [16,17] bring some doubts on the reliability on computer simulation regarding the capacity to distinguish transient chaotic sets from an attractor, even for very long time computation. Similar concern is also presented from the perspective of the effects of discretization schemes. The authors in [18] show different qualitative outcomes by the simple fact of using different interval extension, that is, by the simple fact of applying basic mathematical properties in the equations, such as, associativity and distributivity. In this work, instead of trying to give a final word on the chaotic behaviour of Cord system, we are more interested in showing how to overwhelm the problem of using piecewise for the case of a system, wherein complex conjugate fixed points are presented. In such way, we use the same discretization scheme for the original and rebuilt system using piecewise technique to guarantee a fair comparison.

The remainder of this paper is organised as follows. Section 2 presents the general approach of piecewise affine model building. The Cord Attractor is shown in Section 3. Section 4 presents the methodology proposed for cases with no fixed points or with some complex fixed points. The results are presented in Sections 5 and 6 brings the final remarks.

2. Background

This section presents an overview of the method used for building models using piecewise affine as proposed in [11]. The Cord Attractor is also summarised.

2.1. Piecewise affine modelling

The number of affine subsystems is related to the number of fixed points. The definition of an affine systems is

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b} \tag{1}$$

where A and $\mathbf{b}$ are constants. Clearly, such an affine system with respect to the origin $\mathbf{x} = \mathbf{0}$ can be considered as a linear system centred at some point $\mathbf{p} = -A^{-1}\mathbf{b}$. Thus, the affine sub-model can be rewritten as

$$\dot{\mathbf{x}} = A(\mathbf{x} - \mathbf{p}) \tag{2}$$

A structure of piecewise affine model is described as

$$\dot{\mathbf{x}} = \sum_{i=1}^{m} f_i[s(\mathbf{x})]A_i(\mathbf{x} - \mathbf{p}_i), \tag{3}$$

where $\mathbf{x} \in \mathbb{R}$ is the state vector, m is the number of affine subsystems, and $\mathbf{p}_i \in \mathbb{R}^n$ is the fixed point to which the affine subsystem is associated. Constant matrices $A_i \in \mathbb{R}^{n\times n}$ defines the local linear dynamics of the affine subsystems and $s(\mathbf{x})$ is the switching surface between the domains where the subsystems are active and $f_i[\cdot]$ is a switching function. The switching function may be described by a smooth or Boolean function.

Nonlinear dynamical systems investigated in works, such as in [11], usually present at least two real fixed points which allows one to easily apply the Eq. (3). This is not the case for the Cord Attractor, as we see on the next section.

2.2. The Cord Attractor

The Cord Attractor [13] is defined by the following equations

$$\begin{aligned}\dot{x} &= -y - z - ax + aF\\ \dot{y} &= xy - bxz - y + G\\ \dot{z} &= bxy + xz - z.\end{aligned} \tag{4}$$

The authors in [12] explain that such system has been obtained by a replacement of the nonlinear terms $(-y^2 - z^2)$ of the system proposed in [19] by the linear terms $(-y - z)$. This modification has resulted in an increment on the observability of the systems, as well, in a generation of the a new attractor, called *Cord Attractor*. In this work, we use the same parameters used in [12], that is, $(a, b, F, G) = (0.258, 4.033, 8, 1)$, with initial conditions $(x_0, y_0, z_0) = (.1, .1, .1)$. The Cord Attractor can be seen in Fig. 1.

The fixed points of Eq. (4) with parameters $(a, b, F, G) = (0.258, 4.033, 8, 1)$ are given by

$$p_1 = \begin{vmatrix} x = 7.9091 \\ y = -0.0065 \\ z = 0.0299 \end{vmatrix} \tag{5}$$

$$p_{2,3} = \begin{vmatrix} x = 0.1034 \pm 0.1397i \\ y = 1.1856 \pm 0.6237i \\ z = 0.8517 \pm 0.5876i \end{vmatrix} \tag{6}$$

where p_1 is real and $p_{2,\,3}$ are complex conjugate.

3. Material and methods

This section presents the method used here to design a piecewise approximation to the Cord Attractor.

3.1. Virtual fixed point

The main contribution of this paper is the application of the concept of virtual fixed point to model a system with just one real fixed point. The creation of virtual fixed point allows to use the method developed in [2,3] and described in Eq. (4). Our inspiration comes from the fact that the complex fixed points are not touched by orbits of the system, but the global dynamics is influenced by its value. Keeping this in mind, we proposed the virtual fixed point must have the following features:

1. Being real.
2. Being located on the switching surface. This is a sufficient condition to guarantee that this fixed point in the smooth-piecewise system is virtual.

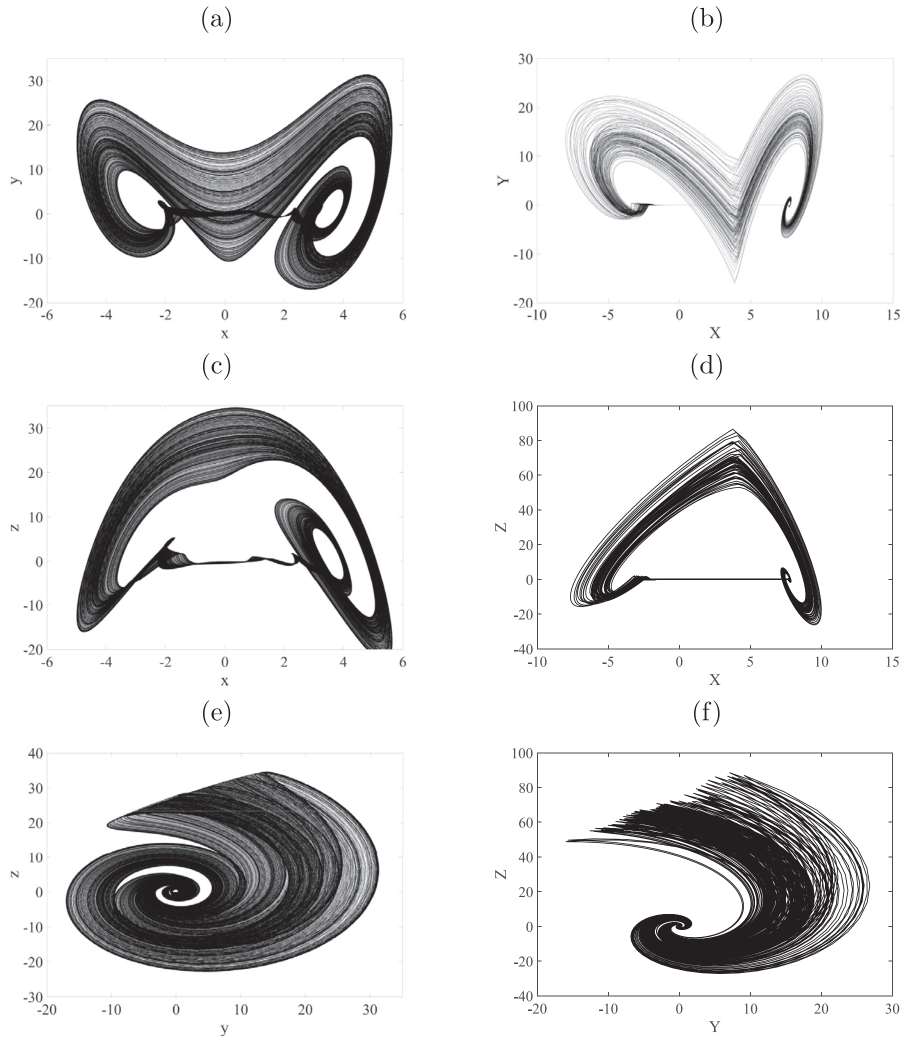

Fig. 2. Projections of the attractors. First column: original system with parameters chosen as $(a, b, F, G) = (0.258, 4.033, 8, 1)$, with initial conditions $(x_0, y_0, z_0) = (0.1, 0.1, 0.1)$. Second column: piecewise smooth system according to Eq. (9) for the same initial conditions.

3.2. Switching function

Similarly to what have been done in [11], the surface can be determined based on the topological properties of the attractor. Observing the attractor we have used a smooth function defined by

$$s(\mathbf{x}) = \frac{1}{1 + e^{-\tau(s(\mathbf{x}))}} \tag{7}$$

where the parameter τ is chosen empirically.

4. Results

We have chosen a switching surface as follows $s(\mathbf{x}) = x - 4.0$ and $\tau = 60$.

The virtual fixed point is given by

$$p_v = \begin{vmatrix} x = 4 \\ y = 0 \\ z = 0 \end{vmatrix} \tag{8}$$

The approximated system is thus given by

$$\dot{X} = f[s(X)]A_+(X - P_v) + (1 - f[s(X)])A_-(X - P_-) \tag{9}$$

where

$$p_v = P_-$$

$$A_+ = \begin{bmatrix} -0.258 & -1 & -1 \\ 0 & 6.9 & -7.9 \\ 0 & -31.8607 & 6.9 \end{bmatrix}$$

$$A_- = \begin{bmatrix} -0.258 & -1 & -1 \\ 0 & -4.5 & 3.5 \\ 0 & -14.1155 & -4.5 \end{bmatrix}$$

$$P_- = \begin{vmatrix} x = 7.9 \\ y = 0 \\ z = 0 \end{vmatrix} \tag{10}$$

It is already known that we may take the real fixed point from the original attractor slightly different without changing piecewise model. We have used this possibility to change the fixed point from Eqs. (5) to (10).

Fig. 2 presents the attractor projections of original system and the smooth piecewise. As procedure to check the similarity of the model, we have used the computation of the largest positive Lyapunov exponent (λ). In fact, we have carried out the computation of finite-time local Lyapunov exponent, in the sense described in [20], as we performed the computation over a finite set of points. As we are dealing with models, the technique proposed in [14] presents the robustness and simplicity for such task (see

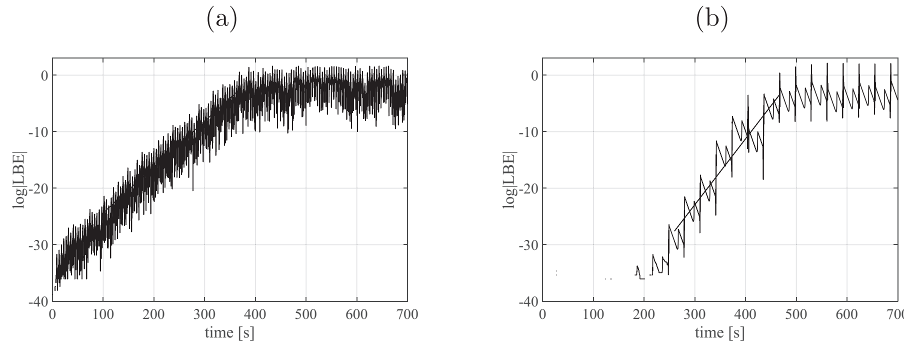

Fig. 3. Lower bound error from (a) original system (b) piecewise system. Largest Lyapunov exponent: (a) 0.08 (b) 0.11 are calculated as the slope of line in the first part of the graph.

Fig. 3, which are shown the computation of the largest positive Lyapunov exponent). The attractors of the original and piecewise are presented in Fig. 2. The original system has shown $\lambda = 0.08$, while $\lambda = 0.11$ for the piecewise. We also performed the computation of the λ using the package Lyapmax [15], which adopts the algorithm proposed by Wolf [21]. We have used 100.000 points and we have found $\lambda = 0.07$ and $\lambda = 0.06$ for the original and piecewise, respectively.

Although it seems obvious some similarity between the original system and smooth-piecewise approximation, we cannot state that there is a topological agreement between these two systems as these attractors did not align precisely. Future work should address the validation of this approach using topological analysis, as described in [22,23]. It is also possible that this topological analysis can reveal some insights about the location of virtual fixed point.

5. Conclusion

This paper presents a smooth-piecewise model to the Cord Attractor. The fixed points of the Cord Attractor are given by Eqs. (5) and (6), which means, that this system has one real fixed point and two complex conjugate fixed points. The methodology proposed in [2,3] requires two real fixed point to connect two affine subsystems. It has been proposed the creation of a virtual fixed point to at least partially overcome this problem. This virtual fixed point should be obviously real. The switching surface has been chosen as the location of the virtual fixed point.

After determining the virtual fixed point, a switching surface has been tested. The phase space and local-finite largest Lyapunov exponent have been applied to compare the smooth-piecewise model and the original Cord Attractor, which shows a satisfactory agreement. Although, we believed that creation of virtual fixed point is an important step towards a more comprehensive understanding of the piecewise approximation, not all answers have been considered. Usually one of the obstacle of this technique is the determination of the switching surface. This has been considered by means of topological analysis with some satisfactory success. Now, for this category of problems, that is, systems with less than two real fixed points, other problem comes out: the location of the fixed point. Here, we have placed the virtual fixed point on the switching surface. But there is no guarantee that it is going to work with systems presenting only complex fixed points. For instance, the authors in [24] have shown a collection of fifteen simple chaotic systems with no equilibria and their complex fixed-points. Five of them have no equilibria points, which requires a creation and location of two virtual fixed points. These systems are certainly challenging themes for future investigations. We also believe that a rigorous topological analysis, as detailed in [22,23], can also share light on this topic.

Acknowledgements

This work has been supported by CNPq/INERGE (465704/2014-0) and FAPEMIG (TEC- APQ- 00870-17). We also would like to thank CAPES for its support in the graduate program of Electrical Engineering at UFSJ.

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