



## Interval computing periodic orbits of maps using a piecewise approach



Erivelton G. Nepomuceno<sup>a</sup>, Heitor M. Rodrigues Junior<sup>a</sup>, Samir A.M. Martins<sup>a</sup>, Matjaž Perc<sup>b,c,d,\*</sup>, Mitja Slavinec<sup>b</sup>

<sup>a</sup>Control and Modelling Group (GCOM), Department of Electrical Engineering, Federal University of São João del-Rei, São João del-Rei, MG 36307-352, Brazil

<sup>b</sup>Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška cesta 160, Maribor SI-2000, Slovenia

<sup>c</sup>CAMTP – Center for Applied Mathematics and Theoretical Physics, University of Maribor, Mladinska 3, Maribor SI-2000, Slovenia

<sup>d</sup>School of Electronic and Information Engineering, Beihang University, Beijing 100191, P.R. China

### ARTICLE INFO

#### Keywords:

Interval arithmetic  
periodic orbit  
Natural interval extensions  
Discrete maps  
Dynamical systems

### ABSTRACT

Interval arithmetic applied to simulation of dynamical systems has attracted a great deal of interest in recent years. Much of this research has been carried out in the calculation of fixed points or low-period windows for nonlinear discrete maps. This study proposes a novel interval computation based on a piecewise method to calculate periodic orbits for the logistic map. Using the cobweb plot, three rounding situations have been applied to a correct outward rounding, as required by interval arithmetic. The proposed method is compared with results in the literature and with the results obtained by means of the Matlab toolbox Intlab. The comparison is accomplished for nine case studies using the logistic map. Numerical results explicitly indicate that the proposed method produces intervals that are substantially narrower than those obtained with the traditional techniques.

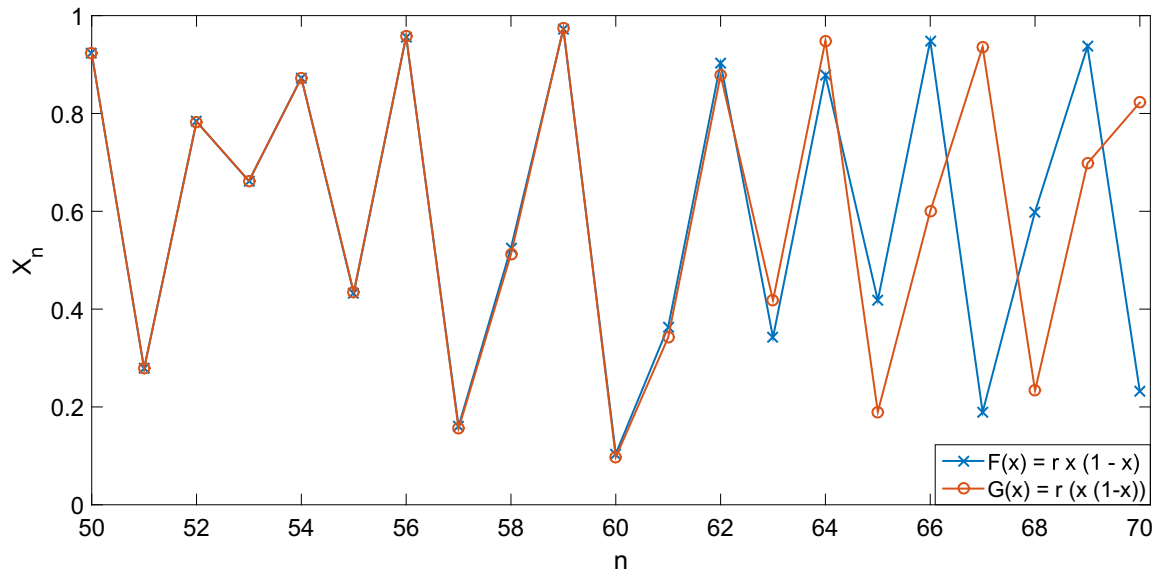
© 2018 Elsevier Inc. All rights reserved.

## 1. Introduction

Numerical computation is widely recognised to be of great importance in many fields of science. Many conclusions in nonlinear science and complex systems have been drawn upon simulation in digital computer [1,2]. The reliability of the results has been a subject of concern since the beginnings of digital computer in science. For instance, Hammel et al. [3] have applied one of the most powerful computer in late 1980s to present a computer-assisted proof of the reliability to simulate the logistic map along millions of iterates. The MIT scientist, E. N. Lorenz, well-known for its contributions on chaos theory, has also investigated computational instability on the simulation of nonlinear dynamics. He has pioneered to notice that chaos could be an artefact of finite precision in digital computer. On the other hand, Corless et al. [4] have demonstrated the opposite effect. They have shown that some numerical methods may produce discrete dynamical systems that are not chaotic, even when the original continuous dynamical system is believed to be chaotic. Although great advance has been observed in this topic, many works have been doing over the past few decades [1,5–8]. In fact, Lozi [1] has concluded his paper saying that “there is room for more study of the relationship between numerical computation and theoretical behaviour of chaotic solutions of dynamical systems”.

\* Corresponding author at: Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška cesta 160, SI-2000 Maribor, Slovenia.

E-mail addresses: [nepomuceno@ufsj.edu.br](mailto:nepomuceno@ufsj.edu.br) (E.G. Nepomuceno), [heitormrjunior@hotmail.com](mailto:heitormrjunior@hotmail.com) (H.M. Rodrigues Junior), [martins@ufsj.edu.br](mailto:martins@ufsj.edu.br) (S.A.M. Martins), [matjaz.perc@uni-mb.si](mailto:matjaz.perc@uni-mb.si) (M. Perc), [mitja.slavinec@uni-mb.si](mailto:mitja.slavinec@uni-mb.si) (M. Slavinec).



**Fig. 1.** Simulation in Matlab of Eq. (1) with  $r = 3.9$  e  $x_0 = 0.6$  for two natural interval extensions. The natural interval extensions  $F(x)$  and  $G(x)$  are mathematically equivalent. However, there is an extra parenthesis in  $G(x)$ , which changes the order of arithmetic operations. This slight difference is sufficient to produce an exponential divergence between the simulation of these two functions. This happens due to finite precision of floating point arithmetic. After  $n = 58$  obvious difference between these two pseudo-orbits is visible. The computations has been carried out in an Intel i7-5500 @ 2.41 GHz with Matlab.

A simple example of the influence of error propagation on numerical calculations of nonlinear dynamical systems is easily presented. Let the logistic map [9] given by Eq. (1)

$$x_{n+1} = rx_n(1 - x_n), \quad (1)$$

where  $r \in [0; 4]$  is the bifurcation parameter and  $x_n \in (0; 1)$ ,  $n \in \mathbb{N}$ . The study case is a simple example of computational error investigation of nonlinear dynamical systems, where two recursive functions representing the logistic map are given by  $F(x_n) = rx_n(1 - x_n)$  and  $G(x_n) = rx_n - rx_n^2$ . Fig. 1 shows the sequence of points obtained in a digital computer using floating point arithmetic and double precision. We have shown only iterations from 50 to 70 for  $F(x_n)$  and  $G(x_n)$  with  $r = 3.9$  and  $x_0 = 0.6$ . Although,  $F(\cdot)$  and  $G(\cdot)$  are mathematically equivalent, they have distinct sequences of arithmetic operations and a divergence between these two sequences becomes visually noticeable after approximately 60 iterations. We refer the reader to other similar works that deal with error in computer simulations [7,10–17].

Among many initiatives, interval arithmetic has been considered a systematic approach to increase the reliability of results in nonlinear dynamical systems [6,18–20]. The recognition of interval arithmetic has been strengthened after the approval of a specific standard about this topic by IEEE [21]. This document describes in details the main features of this approach and it shows maturity of scientific community to reach some consensus after almost five decades of intense research.

Although, it is undoubtedly true the significant advance of interval arithmetic applied to simulate dynamical systems, the interval width is still a leading concern, particularly in non-contracting maps. Many works have addressed the problem of interval width. For instance, Bruguera [22] proposed a innovative number representation to the interval that, instead of both endpoints, uses the lower endpoint and the width of the interval. According to his conclusion, this representation produces intervals that are substantially narrower than those obtained with the traditional representation. There also other variants for interval analysis, such as affine arithmetic that has been proposed in [23] to overcome the error explosion problem. These authors state that in many applications, the higher asymptotic accuracy of affine arithmetic is very attractive and compensates a higher computational cost. Constraint interval arithmetic and its variant, the single parameter level, are other exciting alternatives to arithmetic interval [24]. In these variants, the authors also deal with the problem of overestimations but also present some desirable properties, such as division inversion, not shared in general by standard interval arithmetic. Based on the interval Newton operator, Galias [25] has proposed a systematic method to find all low-period windows for the quadratic map. The author has shown how to calculate very accurate rigorous bounds of their widths for each low-period. Following this pursuit to improve the interval arithmetic, this paper proposes a novel interval computation based on a piecewise division of non-contracting map. Using the logistic map as our case study, we have carefully developed an algorithm that consider the range monotonicity of this function; this function has two clearly branches of positive and negative derivative. Using the cobweb plot of the logistic map, three rounding situations have been applied to a correct outward rounding, as required by interval arithmetic. The proposed technique has been compared with results with previous work [12] and outcomes yield using of Matlab toolbox Intlab. In nine numerical experiments, we have been able to produce intervals that are substantially narrower than those obtained by with the traditional approaches.

## 2. Background

In this section, some preliminary concepts on interval analysis and dynamical systems are briefly presented.

### 2.1. Interval analysis

According to Moore et al. [18], an interval  $X$  can be defined as a closed and limited set of real numbers  $x \in \mathbb{R}$ , such that

$$X = [\underline{X}, \overline{X}] = \{x : \underline{X} \leq x \leq \overline{X}\}.$$

It is called midpoint  $m = (\underline{X} + \overline{X})/2$  and interval width  $\omega = \overline{X} - \underline{X}$ .

Operations with intervals is like operations with sets. The basic interval operations are defined by:

$$X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}] \quad (2)$$

$$X - Y = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}] \quad (3)$$

$$X \cdot Y = [\min(S), \max(S)] \quad (4)$$

where  $S = \{XY, X\overline{Y}, \overline{X}Y, \overline{X}\overline{Y}\}$ . If 0 does not belongs to  $Y$ , then  $X/Y$  is given by

$$X/Y = X \cdot (1/Y) \quad (5)$$

where  $1/Y = [1/\overline{Y}, 1/\underline{Y}]$ .

Consider the following definition for a function  $f$  defined for a real variable  $x$  [18].

**Definition 2.1.** A natural interval extension of  $f$  is an interval function  $F$  defined to map an interval variable  $X$ , such that for real arguments:

$$F(x) = f(x). \quad (6)$$

Finally, we remind that intersection plays a key role in interval analysis. The size of the intersection between two intervals is at most the smallest of the intervals, as given by

$$w(X \cap Y) \leq \min\{w(X), w(Y)\}. \quad (7)$$

### 2.2. Discrete dynamical systems

Discrete dynamical systems are usually defined by a recursive function or a map given by Gilmore and Lefranc [26]:

$$x_{n+1} = f(x_n), \quad (8)$$

where  $n \in \mathbb{N}$ ,  $M \subset \mathbb{R}$  is a metric space and  $f: M \rightarrow M$  is the recursive function defined in state space and  $x_n$  denotes the state in discrete time  $n$  [1]. Eq. (8) can be written because of composite functions, given by Nepomuceno [12]:

$$x_n = f^1(x_{n-1}) = f^2(x_{n-2}) = \dots = f^n(x_0). \quad (9)$$

#### 2.2.1. Orbits and pseudo-orbits

The sequence defined by  $\{x_0, x_1, \dots, x_n\}$  obtained at each iteration from the map of Eq. (8) is called orbit of  $x_0$  [26], while the result of a computer-based computation of an orbit is called a pseudo-orbit. This is due to the fact of the obtained result approximates of the true orbit due to inherent properties of computers. A specific pseudo-orbit  $i \in \mathbb{N}$  is represented as  $\{\hat{x}_{i,0}, \hat{x}_{i,1}, \dots, \hat{x}_{i,n}\}$ , such as

$$|x_n - \hat{x}_{i,n}| \leq \delta_{i,n} \quad (10)$$

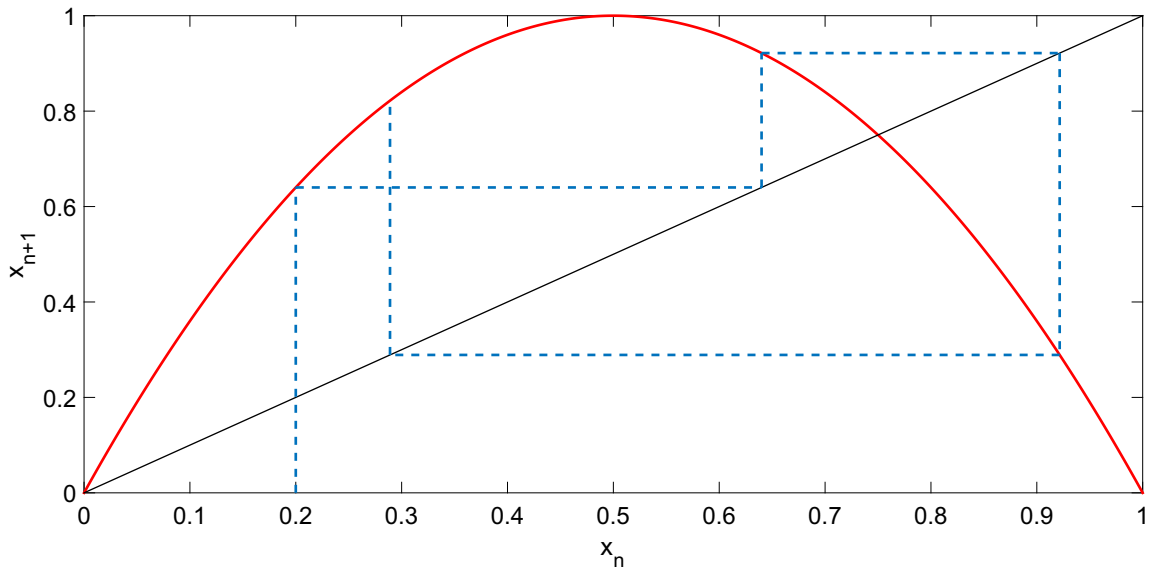
where  $\delta_{i,n} \geq 0$  and  $\delta_{i,n} \in \mathbb{R}$  [11].

#### 2.2.2. Fixed points and periodic orbits

In discrete dynamical systems, fixed points and periodic orbits are of great importance as elements for determining the equilibrium points of these systems [27]. The fixed point of an orbit is a point in which  $x_{n+1} = x_n = x^*$ , or

$$f(x^*) = x^*. \quad (11)$$

The periodic orbits or periodic points are a generalization of the fixed points. It is defined that if  $f^p(x^*) = x^*$ , then we have a periodic orbit of period  $p$ .



**Fig. 2.** Cobweb plot of logistic map with  $r = 4$  e  $x_0 = 0.2$ . This cobweb has been used to identify three regions with different rules to lead to a correct outward rounding. These three rules are summarized in Section 3.

2.2.3. Cobweb plot

The cobweb plot is a visual tool used in dynamical systems to investigate one-dimensional recursive functions [28]. It is a very useful instrument to study the behaviour of a dynamical system with the evolution of an initial condition under several iterations [29]. The cobweb plot, shown in Fig. 2, consists of a diagonal line  $y = x$  and a curve representing the map  $y = f(x)$ . To visualize the behaviour of the system from an initial condition  $x_0$  for a given parameter value, one must draw a vertical line from the point to the curve of the function and draw an horizontal line from the last coordinate to the diagonal line  $y = x$ , as shown in Fig. 2 [30].

3. Methodology

In this section, we present the method of interval computing periodic orbits using a piecewise approach. The key aspect is to guarantee outward rounding [31]. Using the cobweb plot of the logistic map (Fig. 2), it can be observed that there are three rounding situations for the results of calculations with the interval limits:

1.  $\bar{X}_n < 0.5$ : in this case, both limits of the interval are in the growing part of the curve. Thus, to ensure that the range covers all possible outcomes,  $X_{n+1}$  should be rounded down and  $\bar{X}_{n+1}$  should be rounded up.
2.  $X_n > 0.5$ : it indicates that the entire interval is in the decreasing part of cobweb plot. Thus, the solution found with lower bound  $X_n$  is the upper limit of the interval at the next iteration and the inverse occurs with the upper bound  $\bar{X}_{n+1}$ . Therefore,  $X_{n+1}$  should be rounded up and  $\bar{X}_{n+1}$  should be rounded down.
3.  $X_n < 0.5 < \bar{X}_n$ : in this case, the maximum point of the curve, given by  $f(0.5) = 0.25 \cdot r$ , belongs to the interval  $X_{n+1}$ . Therefore, the upper bound of the next iteration  $\bar{X}_{n+1}$  should be this point. The lower limit of the next iteration is given by  $X_{n+1} = \min\{f(X_n), f(\bar{X}_n)\}$ , which should be rounded down.

The following theorems are established to develop the proposed method. Let  $\delta \in \mathbb{R}$  be a positive number.

**Theorem 3.1.** If  $\bar{X}_n < 0.5$ , then  $f([X_n, \bar{X}_n]) = [f(X_n), f(\bar{X}_n)]$ .

**Proof.** If  $f(\bar{X}_n)$  is a upper bound of  $f([X_n, \bar{X}_n])$ , then  $f(\bar{X}_n) > f(\bar{X}_n - \delta)$ , for  $0 < \delta < \bar{X}_n$ . Thus,

$$f(\bar{X}_n - \delta) = r \cdot (\bar{X}_n - \delta) \cdot [1 - (\bar{X}_n - \delta)] = f(\bar{X}_n) + r \cdot \delta \cdot (2\bar{X}_n - 1 - \delta).$$

For  $f(\bar{X}_n) > f(\bar{X}_n - \delta)$ , as  $r \cdot \delta > 0$ , then,  $(2\bar{X}_n - 1 - \delta < 0)$ . As  $\bar{X}_n < 0.5$ , then  $2\bar{X}_n - 1 < 0$ . Thus, as  $\delta > 0$ , then  $(2\bar{X}_n - 1 - \delta) < 0$ . Consequently, it is confirmed that  $f(\bar{X}_n) > f(\bar{X}_n - \delta)$  for  $\bar{X}_n < 0.5$  and  $0 < \delta < \bar{X}_n$ .

For  $f(X_n)$  to be the lower bound of  $f([X_n, \bar{X}_n])$ , then  $f(X_n) < f(X_n + \delta)$ , for  $0 < \delta < 0.5 - X_n$ . Thus,

$$f(X_n + \delta) = r \cdot (X_n + \delta) \cdot [1 - (X_n + \delta)] = f(X_n) + r \cdot \delta \cdot (1 - 2X_n - \delta).$$

For  $f(X_n) < f(X_n + \delta)$ , as  $r \cdot \delta > 0$ , then,  $(1 - 2X_n - \delta) > 0$ . Assuming the highest possible value of  $\delta$ , a value very close to  $0.5 - X_n$  is reached. Thereby,

$$1 - 2X_n - \delta > 0,$$

$$1 - 2\underline{X}_n - (0.5 - \underline{X}_n) > 0,$$

$$0.5 - \underline{X}_n > 0.$$

As  $\underline{X}_n < 0.5$ , then  $0.5 - \underline{X}_n > 0$ . Consequently,  $f(\underline{X}_n) < f(\underline{X}_n + \delta)$ , for  $\overline{X}_n < 0.5$ , establishing the proof for the [Theorem 3.1](#). □

**Theorem 3.2.** *If  $\underline{X}_n > 0.5$ , then  $f([\underline{X}_n, \overline{X}_n]) = [f(\overline{X}_n), f(\underline{X}_n)]$ .*

**Proof.** If  $f(\overline{X}_n)$  is a lower bound of  $f([\underline{X}_n, \overline{X}_n])$ , then  $f(\overline{X}_n) < f(\overline{X}_n - \delta)$ , where  $0 < \delta < \overline{X}_n - 0.5$ . In a similar way to the proof presented in [Theorem 3.1](#), for  $f(\overline{X}_n) < f(\overline{X}_n - \delta)$ , then  $2\overline{X}_n - 1 - \delta > 0$ . Assuming the highest possible value of  $\delta$  as a value close to  $\overline{X}_n - 0.5$ , we have that:

$$2\overline{X}_n - 1 - \delta > 0,$$

$$\overline{X}_n - 0.5 > 0.$$

As  $\overline{X}_n > 0.5$ , then  $\overline{X}_n - 0.5 > 0$ . Consequently,  $f(\overline{X}_n) < f(\overline{X}_n - \delta)$ , for  $\underline{X}_n > 0.5$  and  $0 < \delta < \overline{X}_n - 0.5$ .

If  $f(\underline{X}_n)$  is an upper bound of  $f([\underline{X}_n, \overline{X}_n])$ , then  $f(\underline{X}_n) > f(\underline{X}_n + \delta)$ , for  $0 < \delta < 1 - \underline{X}_n$ . It has been shown that  $f(\underline{X}_n + \delta) = f(\underline{X}_n) + r\delta(1 - 2\underline{X}_n - \delta)$ . Therefore, as  $r\delta > 0$ , for  $f(\underline{X}_n) > f(\underline{X}_n + \delta)$ , then  $1 - 2\underline{X}_n - \delta < 0$ . As  $\underline{X}_n > 0.5$  and  $\delta > 0$ , then  $1 - 2\underline{X}_n - \delta < 0$ . Thereby,  $f(\underline{X}_n) > f(\underline{X}_n + \delta)$ , for  $\underline{X}_n > 0.5$  and  $0 < \delta < 1 - \underline{X}_n$ , establishing the proof for the [Theorem 3.2](#). □

**Theorem 3.3.** *If  $\underline{X}_n < 0.5 < \overline{X}_n$  and  $f(\underline{X}_n) < f(\overline{X}_n)$ , then  $f([\underline{X}_n, \overline{X}_n]) = [f(\underline{X}_n), f(0.5)]$ .*

**Proof.** If  $f(\underline{X}_n)$  is a lower bound of  $f([\underline{X}_n, \overline{X}_n])$ , then  $f(\underline{X}_n) < f(\underline{X}_n + \delta)$ , where  $0 < \delta < 1 - 2\underline{X}_n$ . As shown, for  $f(\underline{X}_n) < f(\underline{X}_n + \delta)$ , then, as  $r\delta > 0$ , we have  $1 - 2\underline{X}_n - \delta > 0$ . As  $\underline{X}_n < 0.5$ , then  $1 - 2\underline{X}_n > 0$ . And as  $\delta < 1 - 2\underline{X}_n$ , we have that the condition is satisfied, establishing the proof for the [Theorem 3.3](#). □

**Theorem 3.4.** *If  $\underline{X}_n < 0.5 < \overline{X}_n$  and  $f(\overline{X}_n) < f(\underline{X}_n)$ , then  $f([\underline{X}_n, \overline{X}_n]) = [f(\overline{X}_n), f(0.5)]$ .*

**Proof.** If  $f(\overline{X}_n)$  is a lower bound of  $f([\underline{X}_n, \overline{X}_n])$ , then  $f(\overline{X}_n) < f(\overline{X}_n - \delta)$ , where  $0 < \delta < 2\overline{X}_n - 1$ . As shown, for  $f(\overline{X}_n) < f(\overline{X}_n - \delta)$ , then, as  $r\delta > 0$ , we have that  $2\overline{X}_n - 1 - \delta > 0$ . As  $\overline{X}_n > 0.5$ , then  $2\overline{X}_n - 1 > 0$ . And as  $\delta < 2\overline{X}_n - 1$ , we have that the condition is satisfied, establishing the proof for the [Theorem 3.4](#). □

Now consider the following theorems related to fixed points.

**Theorem 3.5.** *Let the sequence  $\{\hat{X}_0, \hat{X}_1, \dots, \hat{X}_n\}$  be the interval pseudo-orbit of logistic map. If  $X_n \cap X_{n+1} \neq \emptyset$ , then  $x^* \in X_n$  is a fixed point.*

**Proof.** By definition, if  $x^*$  is a fixed point, then  $f(x^*) = x^*$ . If  $x^* \in X_n$ , then  $x^* \in f(X_n)$ , since the response obtained by the application of the function under the interval  $X_n$  contains the responses of all points belonging to the interval. Therefore, as  $f(X_n) = X_{n+1}$ , we have that  $X_n \cap X_{n+1} \neq \emptyset$  and the fixed point  $x^*$  belongs to the interval obtained by the intersection of these two intervals. □

**Theorem 3.6.** *Let the sequence  $\{\hat{X}_0, \hat{X}_1, \dots, \hat{X}_n\}$  be the interval pseudo-orbit of logistic map. If  $X_n \cap X_{n+p} \neq \emptyset$ , then  $x^* \in X_n$  characterizes one of the points of periodic orbit with period  $p$  of the map.*

**Proof.** By definition, if  $x^*$  is a point that belongs to a periodic orbit of period  $p$ , then  $f^p(x^*) = x^*$ . If  $x^* \in X_n$ , then  $x^* \in f^p(X_n)$ . Therefore, as  $f^p(X_n) = X_{n+p}$ , we have that  $X_n \cap X_{n+p} \neq \emptyset$  and  $x^*$  belongs to the interval obtained by the intersection of both intervals. □

Finally, a theorem on reduction interval width is presented. Let  $\{x_n\} = \{x_0, x_1, \dots, x_n\}$  be the true orbit of a one-dimensional discrete map  $f$  and  $\{\hat{X}_{i,n}\} = \{\hat{X}_{i,0}, \hat{X}_{i,1}, \dots, \hat{X}_{i,n}\}$  the pseudo-orbit computed by a natural interval extension of  $f$ , given by  $F_i$ , with  $i \in \mathbb{N}$ . It is clear that if  $x_0 \in \hat{X}_{i,0}$ , then

$$x_n \in \hat{X}_{i,n}. \tag{12}$$

From this, we can establish what follows for two natural interval extensions  $F_1$  e  $F_2$  of the same real function  $f$ .

**Theorem 3.7.** *If  $x_0 \in \hat{X}_{1,0}$  and  $x_0 \in \hat{X}_{2,0}$ , then  $x_n \in [\hat{X}_{1,n} \cap \hat{X}_{2,n}]$ ,  $n \in \mathbb{N}$ .*

**Proof.** Assuming that  $x_n \notin [\hat{X}_{1,n} \cap \hat{X}_{2,n}]$ , we have that  $x_n \notin \hat{X}_{1,n}$ , or  $x_n \notin \hat{X}_{2,n}$ , or both cases. According to [Eq. \(12\)](#), it means that  $\hat{X}_{1,n}$  or  $\hat{X}_{2,n}$  do not belong to the pseudo-orbit obtained by a natural interval extension of  $f$ , which is a contradiction. That completes the proof. □

The proposed method can be summarized in the following steps.

1. Obtain natural interval extensions  $F_i$  of the studied map function  $f$ .
2. Define the same parameters for all interval functions and a maximum number of iterations  $N$ .
3. Set an initial interval  $\hat{X}_0 = [\hat{X}_0, \hat{X}_0]$  in such a way that  $x_0 \in \hat{X}_0$ .

**Table 1**  
Main symbols and paramters used in the proposed method.

Symbol	Definition
$x_n$	variable at iteration $n \in \mathbb{N}$
$r$	bifurcation parameter of the logistic map
$F(\cdot), G(\cdot)$	natural interval extension
$f(\cdot)$	function upon real numbers
$f^n$	$n$ th composition of function $f$
$X, Y$	intervals
$\bar{X}$	superior limit of an interval
$\underline{X}$	inferior limit of an interval
$\omega$	width of an interval
$\omega(X \cap Y)$	size of intersection between two intervals
$\{x_0, x_1 \dots x_n\}$	sequence obtained from a map
$\delta_{i,n}$	error of a pseudo-orbit $i$ at discrete time $n$ .
$x^*$	fixed point
$f^p(x^*) = x^*$	periodic orbit of period $p$
$X_{i,n}^*$	an interval of a pseudo-orbit $i$ at discrete time $n$

**Table 2**  
Numerical examples investigated. The fourth column indicates the period for the fixed point.

Case	$r$	$x_0$	Period
1	$r = 327/110$	$x_0 = 100/327$	1
2	$r = 3.3$	$x_0 = 0.6$	2
3	$r = 3.47$	$x_0 = 0.6$	4
4	$r = 3.55$	$x_0 = 0.6$	8
5	$r = 3.566$	$x_0 = 0.6$	16
6	$r = 3.5689$	$x_0 = 0.6$	32
7	$r = 3.5699440$	$x_0 = 0.6$	512
8	$r = 3.5699453$	$x_0 = 0.6$	1024
9	$r = 3.5699456$	$x_0 = 0.6$	2048

4. Calculate the response of the current iteration for each natural interval extension with the interval obtained previously.
5. Calculate the intersection between the intervals obtained for each natural interval extension.
6. Check the occurrence of fixed point or periodic orbit according to the [Theorems 3.5](#) and [3.6](#). If the calculated pseudo-orbit reaches any region of stability, then the calculations must be stopped, and the intervals obtained must be analysed
7. Return to step 4 and repeat the calculations until the number of iterations is reached or the pseudo-orbit has converged.

To make clear the understanding of the proposed method the main parameters and symbols have been defined in [Table 1](#).

### 3.1. Case studies

Two natural interval extensions of the logistic are used as follows.

$$F_1(X_n) = rX_n(1 - X_n)$$

$$F_2(X_n) = r(X_n(1 - X_n))$$

These calculations were compared with the interval calculation performed by Intlab and by the method developed in [\[12\]](#). Nine investigated cases are summarized in [Table 2](#).

### 3.2. Hardware and software

All calculations were performed using the Matlab software on a computer with an Intel®i7-5500 @ 2.41 GHz processor, with a Windows operating system.

## 4. Results

[Tables 3](#) to [11](#) show the calculated period, the size of final interval and the number of iterations necessary to reach the fixed point or the periodic orbit. By the analysis of the [Table 3](#), all methods required the same number of iterations to converge to the fixed point. A small difference in the size of the intervals between the three approaches is noted. Within each methodology, the sizes of the intervals tend to be the same and small since the response converges already in the second iteration to the fixed point, presenting the same solution of the calculation done analytically in [Section 1](#).

**Table 3**  
Comparison of the calculations for Case 1 among the developed methods, *Intlab* and [12].

		Period	Width of interval	Iterations
Developed methodology	$F_1$	1	$3.3307 \times 10^{-16}$	2
	$F_2$	1	$3.3307 \times 10^{-16}$	2
	$F_1 \cap F_2$	1	$3.3307 \times 10^{-16}$	2
[12]	$F_1$	1	$1.6527 \times 10^{-15}$	2
	$F_2$	1	$1.0474 \times 10^{-15}$	2
<i>Intlab</i>	$F_1$	1	$2.4425 \times 10^{-15}$	2
	$F_2$	1	$2.4425 \times 10^{-15}$	2

**Table 4**  
Comparison of the calculations for Case 2 among the developed methods, *Intlab* and [12].

		Period	Width of interval	Iterations
Developed methodology	$F_1$	2	$6.6613 \times 10^{-16}$	55
	$F_2$	2	$4.4409 \times 10^{-16}$	55
	$F_1 \cap F_2$	2	$4.4409 \times 10^{-16}$	55
[12]	$F_1$	2	$1.4601 \times 10^{-8}$	27
	$F_2$	2	$2.4474 \times 10^{-8}$	28
<i>Intlab</i>	$F_1$	2	$3.6590 \times 10^{-6}$	20
	$F_2$	2	$3.6590 \times 10^{-6}$	20

**Table 5**  
Comparison of the calculations for Case 3 among the developed methods, *Intlab* and [12].

		Period	Width of interval	Iterations
Developed methodology	$F_1$	4	$1.1102 \times 10^{-15}$	207
	$F_2$	4	$8.8818 \times 10^{-16}$	207
	$F_1 \cap F_2$	4	$8.8818 \times 10^{-16}$	207
[12]	$F_1$	4	$7.4941 \times 10^{-5}$	33
	$F_2$	4	$3.2963 \times 10^{-5}$	33
<i>Intlab</i>	$F_1$	4	$6.4472 \times 10^{-4}$	23
	$F_2$	4	$6.4472 \times 10^{-4}$	23

**Table 6**  
Comparison of the calculations for Case 4 among the developed methods, *Intlab* and [12].

		Period	Width of interval	Iterations
Developed methodology	$F_1$	8	$1.6653 \times 10^{-15}$	299
	$F_2$	8	$6.6613 \times 10^{-16}$	303
	$F_1 \cap F_2$	8	$5.5511 \times 10^{-16}$	303
[12]	$F_1$	8	$2.1521 \times 10^{-3}$	35
	$F_2$	8	$9.4794 \times 10^{-4}$	35
<i>Intlab</i>	$F_1$	8	$9.1500 \times 10^{-4}$	23
	$F_2$	8	$9.1500 \times 10^{-4}$	23

For the second case shown in Table 4, the proposed methodology presents intervals with considerably greater precision. For cases 3 and 4, whose orbit should converge for periods 4 and 8, respectively, the responses, shown in Tables 5 and 6, present similar behaviour to the previous case: all methods converge to orbits with the expected period, although there is a considerable difference in the final size of the intervals and in the number of iterations of the methodology developed in relation to the other methods. The responses obtained with *Intlab* for Case 5, instead of reaching period 16, as expected, converged to orbits with period 8. It can be noted that the size of the intervals obtained for this method is much larger in relation to the others, showing that the obtained answers have lower accuracy. All other methods converged to orbits with the expected period for both extensions, as can be seen in the Table 7.

For the sixth case, presented in Table 8, only our method has presented results according to the literature. The orbit found by the method using the *Intlab* tool continued with period 8 and the method of [12] presented period 16, rather than converging to period 32. Again, it is noted that the amount of Iterations of the developed methodology is much larger in relation to the other methods, however the size of the final interval remains small.

The remain cases are represented in Tables 9 to 11. Only our method has converged to orbits with the expected periods according to the literature. An interesting fact that had not occurred for the previous cases is that, from Case 7, the final interval width obtained for the method that uses the intersection of the responses between the natural interval extensions is not narrower in relation to the size presented for one or both extensions when applied in the developed method. This shows that, despite obtaining very small intervals, the method  $F_1 \cap F_2$  does not reduce the limits of the true solution for all cases.

**Table 7**Comparison of the calculations for Case 5 among the developed methods, *Intlab* and [12].

		Period	Width of interval	Iterations
Developed methodology	$F_1$	16	$6.6613 \times 10^{-16}$	415
	$F_2$	16	$2.1094 \times 10^{-15}$	417
	$F_1 \cap F_2$	16	$5.5511 \times 10^{-16}$	415
[12]	$F_1$	16	$3.9800 \times 10^{-4}$	32
	$F_2$	16	$3.6663 \times 10^{-4}$	33
<i>Intlab</i>	$F_1$	8	0.0012	23
	$F_2$	8	0.0012	23

**Table 8**Comparison of the calculations for Case 6 among the developed methods, *Intlab* and [12].

		Period	Width of interval	Iterations
Developed methodology	$F_1$	32	$1.1546 \times 10^{-14}$	1991
	$F_2$	32	$6.4393 \times 10^{-15}$	1999
	$F_1 \cap F_2$	32	$3.1086 \times 10^{-15}$	2015
[12]	$F_1$	16	0.0028	34
	$F_2$	16	0.0018	35
<i>Intlab</i>	$F_1$	8	0.0110	25
	$F_2$	8	0.0110	25

**Table 9**Comparison of the calculations for Case 7 among the developed methods, *Intlab* and [12].

		Period	Width of interval	Iterations
Developed methodology	$F_1$	512	$1.2212 \times 10^{-15}$	3999
	$F_2$	512	$3.9968 \times 10^{-15}$	4001
	$F_1 \cap F_2$	512	$4.1078 \times 10^{-15}$	4003
[12]	$F_1$	16	$4.4852 \times 10^{-4}$	32
	$F_2$	16	$4.1711 \times 10^{-4}$	33
<i>Intlab</i>	$F_1$	8	0.01558	25
	$F_2$	8	0.01558	25

**Table 10**Comparison of the calculations for Case 8 among the developed methods, *Intlab* and [12].

		Period	Width of interval	Iterations
Developed methodology	$F_1$	1024	$6.3616 \times 10^{-14}$	10,711
	$F_2$	1024	$9.4480 \times 10^{-14}$	10,739
	$F_1 \cap F_2$	1024	$2.3948 \times 10^{-13}$	10,833
[12]	$F_1$	16	$4.4856 \times 10^{-4}$	32
	$F_2$	16	$4.1715 \times 10^{-4}$	33
<i>Intlab</i>	$F_1$	8	0.0116	25
	$F_2$	8	0.0116	25

**Table 11**Comparison of the calculations for Case 9 among the developed methods, *Intlab* and [12].

		Period	Width of interval	Iterations
Developed methodology	$F_1$	2048	$8.9040 \times 10^{-14}$	21,551
	$F_2$	2048	$3.9524 \times 10^{-14}$	21,695
	$F_1 \cap F_2$	2048	$2.2249 \times 10^{-13}$	21,887
[12]	$F_1$	16	$4.4857 \times 10^{-4}$	32
	$F_2$	16	$4.1715 \times 10^{-4}$	33
<i>Intlab</i>	$F_1$	8	0.0134	25
	$F_2$	8	0.0134	25

## 5. Conclusions

In this work, an interval computing periodic orbits of maps has been developed. We have used a piecewise approach to consider different modes of rounding according to the monotonicity of the map. The logistic map has been used as a case study. Nine numerical examples have been investigated. In general, the proposed method has produced intervals that are significant narrower than those obtained by *Intlab* approach and in [12].



For the first four studied cases, all methods were effective in reaching the expected periodic orbits according to the literature. For the fifth case, *Intlab* was not able to obtain the expected response. In cases 6, 7, 8 and 9, only our method has been able to reach orbits with the periods specified in the literature. It should be noticed that the interval width obtained by the methodology developed in this work are relative small when it is compared with range of values in which the map variable  $x$  could assume. However, it is also clear that our method requires a significant larger number of iterates.

Finally, we emphasize the effectiveness of the developed methodology, since it was the only one to achieve and guarantee the results established in the literature. In addition, the high precision of the results can be confirmed by the width of the obtained intervals. Future work should investigate the behaviour of the method in chaotic regime and to examine how its use can be applied in the construction of bifurcation diagrams. Other nonlinear discrete maps should also be investigated.

## Acknowledgements

The authors are thankful to the Brazilian Research Agencies CNPq/INERGE (465704/2014-0) and Fapemig (TEC-APQ 00870-17), and to the Slovenian Research Agency (grants P5-0029 and J1-7009). We also would like to thank CAPES for its support in the joint graduate program of Electrical Engineering of UFSJ and CEFET-MG.

## References

- [1] R. Lozi, Can we trust in numerical computations of chaotic solutions of dynamical systems?, in: C. Letellier, R. Gilmore (Eds.) *Topology and Dynamics of Chaos: In Celebration of Robert Gilmore's 70th Birthday*, World Scientific, London, 2013, pp. 63–98.
- [2] E. Ott, *Chaos in Dynamical Systems* (Second edition), Cambridge University Press, Cambridge, 2002.
- [3] S.M. Hammel, J.A. Yorke, C. Grebogi, Do numerical orbits of chaotic dynamical processes represent true orbits? *J Complex* 3 (2) (1987) 136–145.
- [4] R.M. Corless, C. Essex, M.A.H. Nerenberg, Numerical methods can suppress chaos, *Phys. Lett. A* 157 (1) (1991) 27–36.
- [5] T. Sauer, C. Grebogi, J.A. Yorke, How long do numerical chaotic solutions remain valid? *Phys. Rev. Lett.* 79 (1) (1997) 59–62.
- [6] Z. Galias, The dangers of rounding errors for simulations and analysis of nonlinear circuits and systems - and how to avoid them, *IEEE Circuits Syst. Mag.* 13 (3) (2013) 35–52.
- [7] E.J. Rothwell, M.J. Cloud, Automatic error analysis using intervals, *IEEE Trans. Educ.* 55 (1) (2012) 9–15.
- [8] E. Tlelo-Cuautle, L.G. de la Fraga, V.T. Pham, C. Volos, S. Jafari, A. de Jesus Quintas-Valles, Dynamics, FPGA realization and application of a chaotic system with an infinite number of equilibrium points, *Nonlinear Dyn.* 89 (2) (2017) 1129–1139.
- [9] R.M. May, Simple mathematical models with very complicated dynamics, *Nature* 261 (1976) 459–467.
- [10] E.G. Nepomuceno, S.A. Martins, B.C. Silva, G.F. Amaral, M. Perc, Detecting unreliable computer simulations of recursive functions with interval extensions, *Appl. Math. Comput.* 329 (2018) 408–419.
- [11] E.G. Nepomuceno, S.A.M. Martins, A lower bound error for free-run simulation of the polynomial NARMAX, *Syst. Sci. Control Eng.* 4 (1) (2016) 50–58.
- [12] E.G. Nepomuceno, Convergence of recursive functions on computers, *J. Eng.* 2014 (10) (2014) 1–3.
- [13] D. Faranda, M.F. Mestre, G. Turchetti, Analysis of round off errors with reversibility test as a dynamical indicator, *Int. J. Bifurc. Chaos* 22 (9) (2012) 1250215, doi:10.1142/S021812741250215X. <http://www.worldscientific.com/doi/abs/10.1142/S021812741250215X>
- [14] L.S. Yao, Computed chaos or numerical errors, *Nonlinear Anal.* 15 (1) (2005) 109–126. <http://arxiv.org/abs/0506045>
- [15] C. Spandl, Computational complexity of iterated maps on the interval, *Math. Comput. Simul.* 82 (8) (2012) 1459–1477. <http://arxiv.org/abs/1006.0404>arXiv:1006.0404
- [16] A. Hasan, E.C. Kerrigan, G.A. Constantinides, Control-theoretic forward error analysis of iterative numerical algorithms, *IEEE Trans. Autom. Control* 58 (6) (2013) 1524–1529, doi:10.1109/TAC.2012.2225513. <http://ieeexplore.ieee.org/document/6332482/>
- [17] C. Adler, R. Kneusel, W. Younger, Chaos, number theory, and computers, *J. Comput. Phys.* 166 (1) (2001) 165–172.
- [18] R.E. Moore, R.B. Kearfott, M.J. Cloud, *Introduction to interval analysis*, in: Society for Industrial and Applied Mathematics, Philadelphia, 2009.
- [19] G.I. Hargreaves, *Interval Analysis in Matlab*, University of Manchester, 2002 Master's thesis.
- [20] Z. Galias, Proving the existence of long periodic orbits in 1D maps using interval Newton method and backward shooting, *Topol. Appl.* 124 (1) (2002) 25–37.
- [21] Institute of electrical and electronics engineers (IEEE), IEEE standard for interval arithmetic, IEEE std 1788–2015, 2015, 1–97, doi:10.1109/IEEESTD.2015.7140721
- [22] J.D. Bruguera, Optimizing the representation of intervals, *Sci. Comput. Program.* 90 (2014) 21–33. (Part A)
- [23] J. Stolfi, L.H. Figueiredo, An introduction to affine arithmetic, *Tema (São Carlos)* 4 (3) (2003) 297–312.
- [24] Y. Chalco-Cano, W.A. Lodwick, B. Bede, Single level constraint interval arithmetic, *Fuzzy Sets Syst.* 257 (16) (2014) 146–168.
- [25] Z. Galias, Rigorous numerical study of low-period windows for the quadratic map, *Int. J. Bifurc. Chaos* 25 (10) (2015) 1550139.
- [26] R. Gilmore, M. Lefranc, *The Topology of Chaos: Alice in Stretch and Squeezeland*, John Wiley & Sons, 2012.
- [27] R.L. Devaney, *A First Course in Chaotic Dynamical Systems : Theory and Experiment*, Westview Press, 1992.
- [28] A. Paul, N. Das, A.K. Prusty, An advanced gray image encryption scheme by using discrete logarithm with logistic and HEH64 chaotic functions, in: Proceedings of the IEEE Third International Advance Computing Conference (IACC), IEEE, 2013, pp. 1114–1120.
- [29] O. Alpar, Analysis of a new simple one dimensional chaotic map, *Nonlinear Dyn.* 78 (2) (2014) 771–778.
- [30] A.G. Tomida, Matlab toolbox and GUI for analyzing one-dimensional chaotic maps, in: Proceedings of the International Conference on Computational Sciences and its Applications, ICCSA 2008, 2008, pp. 321–330.
- [31] J.H.M. Lee, M.V. Emden, Interval computation as deduction in chip, *J. Logic Progr.* 16 (3–4) (1993) 255–276.