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# Interaction of Korteweg–de Vries Solitons with External Sources

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### Abstract

We consider the problem of interaction of a solitary wave with a moving external source within the framework of Korteweg–de Vries (KdV) equation. We show that for certain profiles of external source the problem has exact solutions in the form of a stationary solitary waves coupled with the force. For the solitary waves which are not trapped by the external force of a small amplitude we obtain approximate solutions by means of the asymptotic method and analyse solutions with the arbitrary relationship between the widths of forcing function and solitary wave. Results obtained agree well with the results of previous works where only the limiting cases of very narrow or infinitely wide forcing as compared with the width of solitary wave were studied. Several new regimes of soliton interaction with width the forcing have been revealed. The theoretical results have been validated by the direct numerical modelling within the framework of forced KdV equation.

### Introduction

The forced Korteweg–de Vries (fKdV) equation is the canonical model for the description of resonant excitation of weakly nonlinear waves by moving perturbations. Such equation was derived by many authors for internal waves over a local topography in the atmosphere, in a water flow over bottom obstacles, surface and internal water waves generated by moving atmospheric perturbations, etc. The number of publications on these topic is so huge that it is impossible to mention all of them in this short article, therefore we only refer to the review [5] and relatively recent publication [6] where a reader can find more references on this topic.

In the papers [2, 3, 4] it was developed an effective method of asymptotic analysis of fKdV equation when the amplitude of external force acting on a KdV soliton is relatively small. Two limiting cases were analysed in those papers: (i) when the width of external force is very small in comparison with the width of a soliton and can be approximated by Dirac delta-function, and when (ii) the width of soliton is very small in comparison with the width of external perturbation. In the meantime, in the natural conditions the relationship between the widths of these entities can be arbitrary, therefore it is of interest to generalise the results of those papers for such cases. This is the main aim of the current study. In addition to that we show that for some special external forces exact solutions of fKdV equation can be obtained even when the amplitude of external force is not small.

### 1. The basic model equation and perturbation scheme

In this paper we follow the asymptotic method developed in [2, 3, 4] and apply it to the fKdV equation in the form:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \alpha u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = \varepsilon\frac{\partial f}{\partial x}, \tag{1}$$

where $c$, $\alpha$ and $\beta$ are constant coefficients, and $f(x,t)$ describes the external perturbation of amplitude $\varepsilon$ moving with the constant speed $V$.

By introduction of new variables $x' = x - Vt$, $t' = t$ we can transform Eq. (1) to the following form (the symbol $'$ is omitted):

$$\frac{\partial u}{\partial t} + (c-V)\frac{\partial u}{\partial x} + \alpha u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = \varepsilon\frac{\partial f}{\partial x}. \tag{2}$$

This form corresponds to the moving coordinate frame where the external force is stationary and depends only on the spatial coordinate $x$,

In the absence of external force ($f(x) \equiv 0$) Eq. (2) reduces to the well-known KdV equation which has stationary solutions in the form of periodic and solitary waves. We study here the dynamics of a solitary wave under the action of an external force of small amplitude $\varepsilon \ll 1$ assuming that in the zero approximation (when $\varepsilon = V = 0$) the solution is

$$u_0 = A_0\,\mathrm{sech}^2(\gamma_0\Phi), \tag{3}$$

where $\gamma_0 = \sqrt{\alpha A_0/12\beta}$ is the inverse half-width of a soliton, $\Phi = x - x_0 - \upsilon_0 t$ is the soliton "phase", $\upsilon_0 = c + \alpha A_0/3$ is the soliton speed, and $x_0$ is an arbitrary constant determining an initial position of the soliton at $t = 0$.

In the presence of external force of a small amplitude the solitary wave solution (3) is no longer valid, but one can assume that under the action of an external perturbation it will gradually vary so that its amplitude can be considered as a function of "slow time", $A(T)$, where $T = \varepsilon t$; correspondingly $\gamma(T)$ becomes also a function of time, and $\Phi = x - \Psi(T)$, where

$$\Psi(T) = x_0 + \frac{1}{\varepsilon}\int_0^T \upsilon(\tau)\,d\tau, \tag{4}$$

$$\upsilon = c - V + \alpha A(T)/3. \tag{5}$$

Now we need to define functions $A(T)$ and $\upsilon(T)$. This can be done by means of the asymptotic method developed, in particular, in Refs. [1, 2]. Following these papers, we seek for a solution of the perturbed KdV equation (2) in the form of the expansion series:

$$\begin{aligned} u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots \\ \upsilon &= \upsilon_0 + \varepsilon\upsilon_1 + \varepsilon^2\upsilon_2 + \ldots \end{aligned} \tag{6}$$

In the leading order of the perturbation method (in the zero approximation), when $\varepsilon = 0$, we obtain the solitary wave solution (3) for $u_0$ and $\upsilon_0$. In the next approximation we obtain the same solution but with slowly varying parameters in time. The dependence of soliton amplitude $A$ and phase $\Psi$ on $T$ can be found from the equation of energy balance [2]:

$$\frac{d}{dT}\int_{-\infty}^{\infty}\frac{u_0^2(\Phi)}{2}d\Phi = \int_{-\infty}^{\infty}u_0(\Phi)\frac{df(\Phi)}{d\Phi}d\Phi. \tag{7}$$

After substitution in Eq. (7) solution (3) we derive the set of equations for $A(T)$ and $\Psi(T)$:

$$\frac{dA}{dT} = \gamma\int_{-\infty}^{\infty}\mathrm{sech}^2(\gamma\Phi)\frac{df(\Phi+\Psi)}{d\Phi}d\Phi, \tag{8}$$

$$\frac{d\Psi}{dT} \quad = \quad \Delta V + \frac{\alpha A(T)}{3}, \tag{9}$$

where $\Delta V = c - V$. Note that in this approximation the rate of phase change is simply equal to the local speed of a soliton with the amplitude $A(T)$ which corresponds to the adiabatic theory.

In the second order of asymptotic theory the correction to the wave speed $\upsilon_1$ can be taken into account. Leaving aside the derivation of corrected equation (9) (the details can be found in Ref. [2]), we present the final equation:

$$\frac{d\Psi}{dT} = \Delta V + \frac{\alpha A(T)}{3} +$$

$$\frac{\varepsilon\alpha}{48\beta\gamma^2}\int_{-\infty}^{\infty}\frac{\sinh 2\gamma\Phi + 2\left(\gamma\Phi - 1\right)}{\cosh^2\gamma\Phi}\frac{\partial f(\Phi+\Psi)}{\partial\Phi}d\Phi. \tag{10}$$

Thus, the set of equations in the first approximation consists of Eqs. (8) and (9), whereas in the second approximation it consists of Eqs. (8) and (10). However, as has been shown in Ref. [2], the last term in Eq. (10) containing a small parameter $\varepsilon$ dramatically changes the qualitative character of solutions and makes the results realistic, whereas Eq. (9) provides unrealistic behaviour of a solitary wave in the vicinity of a forcing. This can be explained, apparently, by a structural instability of solutions with respect to small perturbations of the original set of equations. The difference between the solutions in the first and second approximations will be illustrated in the next Section, and then we will analyse only solutions corresponding to the second approximation as described by Eqs. (8) and (10) for different kinds of external force $f(x)$.

## 2. Forced KdV equation reducible to the KdV equation

Let us consider first the case when

$$f(x) = \mathrm{sech}^2\frac{x}{\Delta_f}, \quad V = c + \frac{4\beta}{\Delta_f^2} - \frac{\varepsilon\alpha\Delta_f^2}{12\beta}, \tag{11}$$

where $\Delta_f$ is a free parameter – the half-width of the external force, and such choice of velocity $V$ provides an exact solution to fKdV equation. In particular, with this external force one can find the exact solution to Eq. (2) in the form of a soliton (3) synchronously moving with the external perturbation with the same speed $\upsilon_s = V$ and having the amplitude $A_s = 12\beta/\alpha\Delta_f^2$ and $\gamma_s = 1/\Delta_f$. This solution represents a particular case of a family of exact solutions to the forced KdV equation constructed in Ref. [7]. Note that here the parameters $\varepsilon$ and $\Delta_f$ are arbitrary, and the amplitude $A_s$ of a soliton is determined only by the width of external force $\Delta_f$, whereas the soliton speed $V$ is determined both by the width $\Delta_f$ and amplitude $\varepsilon$ of external force.

Let us assume now that the parameter $\varepsilon$ is small, and we have the initial condition for Eq. (2) in the form of KdV soliton as per Eq. (3) with the initial amplitude $A_0 \neq A_s$. By substitution of function $f(x)$ from Eq. (11) in Eq. (8) we obtain for the parameter $\gamma$ the following equation:

$$\frac{d\gamma}{dT} = -\frac{2\alpha\varepsilon e^{2\theta}}{3\beta}\int_0^{\infty}\frac{q^K}{\left(e^{2\theta}+q^K\right)^2}\frac{q-1}{(q+1)^3}dq, \tag{12}$$

where $q = \exp\left(2\Phi/\Delta_f\right)$, $\theta = \gamma\Psi$, and $K = \gamma_0\Delta_f$ is the ratio of half-widths of external force and initial soliton. The parameter $K$ can be also presented in terms of the half-distance $D_f$ between the extrema of the force term: $K = 2\gamma_0 D_f/\ln\left(2+\sqrt{3}\right)$ (see the distance between the maximum and minimum of $f'_x$ in Fig. 1).

Equation (12) should be augmented by the equation for the wave speed, which follows from Eq. (9) in the first approximation (cf. [2]):

$$\frac{d\theta}{dT} = \Delta V\gamma + 4\beta\gamma^3, \tag{13}$$

where $\Delta V = c - V$. According to the asymptotic theory, soliton velocity should be approximately equal to the velocity of a forcing. If we assume that at the initial instant of time they are equal, then we obtain that the forcing amplitude $\varepsilon$ is linked with the initial soliton amplitude $A_0$ through the formula:

$$\varepsilon = \frac{\alpha A_0^2\left(1-K^2\right)}{3K^4}. \tag{14}$$

This formula shows that the polarity of the forcing depends on the sign of its amplitude $\varepsilon$ and is determined by the parameter $K$: it is positive if $K < 1$ and negative otherwise.

Dividing Eq. (12) by Eq. (13), we obtain:

$$\frac{d\gamma}{d\theta} = -\frac{2\alpha\varepsilon e^{2\theta}}{3\beta\left(\Delta V\gamma + 4\beta\gamma^3\right)}\int_0^{\infty}\frac{q^K}{\left(e^{2\theta}+q^K\right)^2}\frac{q-1}{(q+1)^3}dq. \tag{15}$$

This is the first-order separable equation whose general solution can be presented in the form:

$$\Gamma^2 + 2\Gamma = \frac{32\left(K^2-1\right)}{K^4}\iint_0^{\infty}\frac{q-1}{(q+1)^3}\frac{q^K dq}{\left(e^{2\theta}+q^K\right)^2}e^{2\theta}d\theta + C, \tag{16}$$

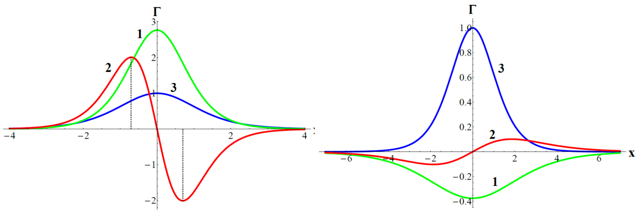
where $\Gamma = A/A_0$ is the dimensionless amplitude of a solitary wave, and $C$ is a constant of integration.

The integral in the right-hand side of Eq. (16) can be evaluated analytically for the particular values of $K$ (we do not present here the results of integration as they are very long and cumbersome), in other cases the integral can be calculated numerically. After evaluation of the integrals in Eq. (16), the phase portrait of the dynamical system (8)–(9) in terms of the dependence $\Gamma(\theta)$ can be plotted for any value of the parameter $K$ in the first approximation on the small parameter $\varepsilon$.

In the case when the width of initial solitary wave is the same as the width of external force, i.e., $K = 1$, we obtain $\Gamma = 1$ and $C = 3$. When $K$ varies in the range $0 < K < 1$, function $\varepsilon f(x) > 0$ (see Fig. 1a), and the right-hand side of Eq. (16) is positive, and the equilibrium state with $\Gamma = 1$ and $\theta = 0$ is of the centre-type in the phase plane. Therefore, if a solitary wave at the initial instant of time has the amplitude $A_0 \neq A_s$, then it will oscillate around the centre as shown in the phase plane of the system (8)–(9) in Fig. 2a). This formally corresponds to the trapping regime when a solitary wave is trapped in the neighbourhood of the centre of external force.

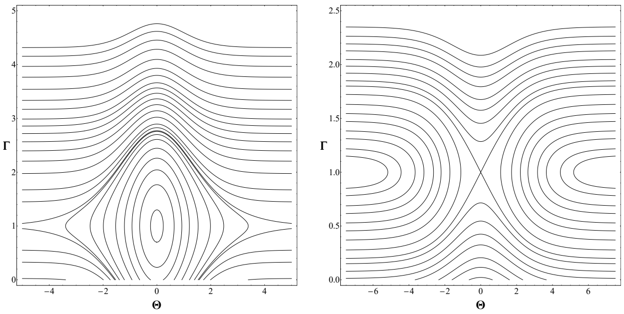
If the amplitude and speed of a soliton at the initial instant of time are big enough, then the soliton simply passes through the external perturbation and moves away. Such a regime of motion corresponds to the transient trajectories shown in the phase plane of Fig. 2a) above the separatrix (the line dividing trapped and transient trajectories).

There are also trajectories in the lower part of the phase plane which either bury into the horizontal axes with $A = 0$, or originate from this axis. Such trajectories correspond to the decay or birth of solitons from small perturbations, respectively. Some



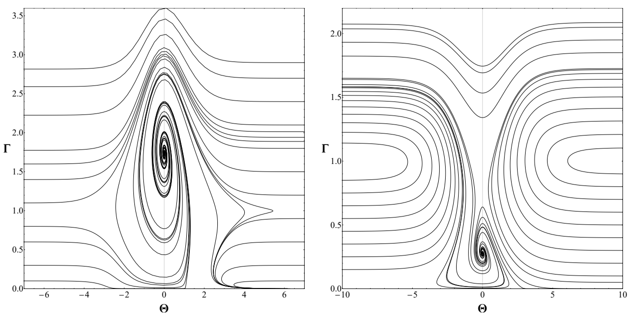

a) b)

Figure 1: The shape and polarity of the forcing $f(x)$ (green lines) as per Eqs. (11) and (14) for $K = 0.75$ (a) and $K = 2$ (b), red lines represent the derivatives $f_x'(x)$, and blue lines show the initial KdV solitons of unit amplitudes.




a) b)

Figure 2: The phase portraits of the dynamical system (8)–(9) as per Eq. (16) in the first approximation on the parameter ε for $K = 0.75$ (a) and $K = 2$ (b).




a) b)

Figure 3: The phase portraits of the dynamical system (12), (17) in the second approximation on the parameter ε for $K = 0.75$ (a) and $K = 2$ (b).

of these types of trajectories, which appear within the separatrix, correspond to the "virtual solitons"; they are generated in the neighbourhood of external perturbation, then increase, but after a while completely disappear.

When $K > 1$, then function $\varepsilon f(x) < 0$ (see Fig. 1b), the right-hand side of Eq. (16) is negative, and the equilibrium state with $\Gamma = 1$ and $\theta = 0$ is of the saddle-type as shown in Fig. 2b). In this case there are repulsive regimes, where solitary waves approach the external force and bounce back, and transient regimes, where solitons of big amplitudes and speeds simply pass through the external perturbation. There are also trajectories on phase plane which either bury into the horizontal axes with $A = 0$, or originate from this axis; this corresponds to the birth and decay of virtual solitons from small perturbations.

In this approximation our results are qualitatively similar to those obtained in Ref. [2], but in contrast to that paper and subsequent papers [3, 4], we do not use here the Dirac delta-function to approximate a soliton or an external force.

In the second approximation the dynamical system for Γ and θ becomes more realistic, but much more complex for the analysis, because Eq. (10) in terms of θ now becomes very long:

$$\frac{d\theta}{dT} = \Delta V\gamma + 4\beta\gamma^3 + \frac{\left(K^2-1\right)\Delta V^2}{2K^4\beta\gamma}\times$$

$$\int_0^\infty \frac{q^{2K} - 4e^{2\theta}q^K\left(1+\theta-K\ln q/2\right) - e^{4\theta}}{\left(e^{2\theta}+q^K\right)^2}\,\frac{q-1}{(1+q)^3}\,dq. \qquad (17)$$

The integral in the right-hand side of this equation can be calculated analytically and then, combining Eq. (17) with Eq. (12), we can plot the improved phase portrait of the system (see Fig. 3a). The phase portrait in the second approximation dramatically differs from the phase portrait of the first approximation. First of all, the equilibrium state of the centre-type in Fig. 2a) maps into the unstable focus (see Fig. 3a), as has been noticed earlier in Ref. [2]. Secondly, the equilibrium amplitude Γ in the second approximation is greater than in the first approximation. Thirdly, on transient trajectories of Fig. 3a) soliton amplitudes do not return back to their initial values (cf. asymptotics of transient trajectories above the focus when $\theta \to \pm\infty$). There are some other important features which were missed in Ref. [2]. In particular, when $K < 1$, there is a repulsive regime clearly visible in the right lower corner in Fig. 3a).

Similarly, there is a difference in the phase portraits of first and second approximation when $K > 1$. In particular, a new equilibrium state of a stable focus appears below the saddle as shown in Fig. 3b) (in Ref. [2] it was mistakenly identified with the centre-type equilibrium state). This equilibrium state corresponds to small-amplitude solitons generated by the external force with the negative potential shown in 1b). The external force corresponding to the positive potential as in Fig. 1a) cannot trap and confine a soliton. Therefore, we can conclude that in the case of wide forcing, when $K > 1$, a small amplitude KdV soliton can be trapped by external force in its minimum.

### 3. Forced KdV equation reducible to the KdVB equation

In this section we consider Eq. (2) with the different and non-symmetric potential function of the form:

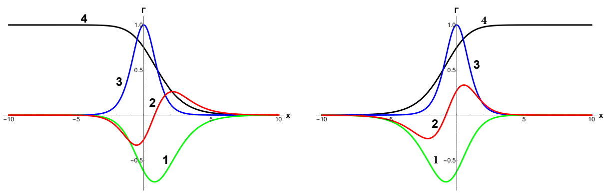
$$f(x) = \pm\left(1 \mp \tanh\frac{x}{\Delta_f}\right)\operatorname{sech}^2\frac{x}{\Delta_f}. \qquad (18)$$

Equation (1) with this potential function is reducible to the KdV–Burgers equation and has exact solution for any parameters ε and $\Delta_f$ in the form of a kink [8]:

$$u(x) = \varepsilon\Delta_f\left(1 \pm \tanh\frac{x}{\Delta_f} + \frac{1}{2}\operatorname{sech}^2\frac{x}{\Delta_f}\right) \qquad (19)$$

provided that the parameters ε and $\Delta_f$ are linked with the coefficients α and β by the formula $\varepsilon = 24\beta/\alpha\Delta_f^3$; then the speed of a stationary solution is $V = c + 24\beta^2/\Delta_f^2$.

In Fig. 4 we show the potential function (18) (lines 1) and its derivative $f_x'$ (lines 2), as well as the exact solutions (19) (lines 3). As follows from the exact solutions (19), a localised external force can produce a non-localised perturbation for $u(x)$ in the fKdV equation (1). Two different forcing functions corresponding to upper and lower signs in Eq. (18) are mirror symmetric with respect to the vertical axis, therefore we illustrate below the solutions generated by only one of them shown in Fig. 4a), but for the sake of generality, we present solutions for both signs in Eq. (18). Note that the forcing function (18)




a) b)

Figure 4: The potential function for the relatively wide external force with $K = 2$. Frame (a) pertains to upper sign of the potential functions in Eq. (18), and frame (b) – to the lower sign. Green lines (1) pertain to function $f(x)$, red lines (2) pertain to the derivatives $f'(x)$, blue lines 3 represent the KdV solitons at the initial instant of time, and black lines 4 are exact solutions of KdV-Burgers equation (19).

of any sign always represents only a negative potential shifted from the centre either to the right or to the left (see green lines 1 in the figure).

If the initial perturbation is chosen in the form of a KdV soliton (3) and the amplitude of external force is small, $\varepsilon \ll 1$, then we can apply the asymptotic theory presented in Sect. 1 to describe the evolution of a soliton under the influence of external force (18). Introducing the parameters $q = e^{2\Phi/\Delta_f}$ and $K = 2\gamma_0 D_f / \ln[(7+\sqrt{33})/4]$, where $D_f$ as above, is the half-distance between the extrema of function $f'_x$ (see Fig. 4) and skipping Eq. (9) of the first approximation, we present the set of equations (8) and (10) in the second approximation on the parameter $\varepsilon$:

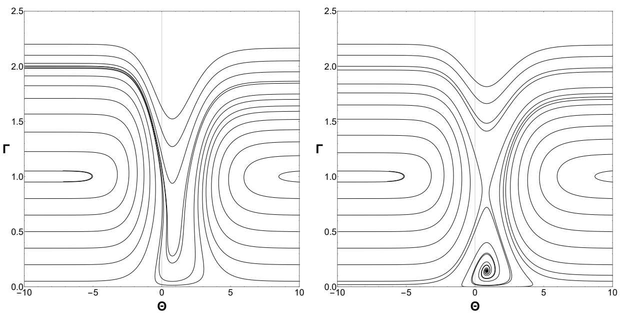
$$\frac{d\gamma}{dT} = \mp\frac{320\beta}{\Delta_f{}^4}\int\limits_0^\infty \frac{q^{K+1}e^{2\theta}}{\left(e^{2\theta}+q^K\right)^2}\,\frac{q^{\pm1}-2}{(q+1)^4}\,dq, \qquad (20)$$

$$\frac{d\theta}{dT} = \Delta V\gamma + 4\beta\gamma^3 \pm \frac{5\Delta V^2}{27\beta K^4\gamma}\times$$
$$\int\limits_0^\infty \frac{q^{2K} - 4e^{2\theta}q^K\left(1+\theta-K\ln q/2\right) - e^{4\theta}}{\left(e^{2\theta}+q^K\right)^2}\,\frac{q^{\pm1}-2}{(q+1)^4}\,q\,dq. \qquad (21)$$

The set of equations (20) and (21) does not have equilibrium states for the relatively narrow forcing with $K \leq 3$ as shown in Fig. 5a). In the phase plane there are either transient trajectories or bouncing trajectories in this case. If the forcing width increases, so that the parameter $K$ becomes greater than three, then the equilibrium state of a stable focus appears. This corresponds to the trapped KdV soliton of a small amplitude within the potential well as shown in Fig. 5b). But when the forcing width further increases, so that the parameter $K$ becomes greater than five, the equilibrium state disappears again, and the phase portrait of the system (20) and (21) becomes qualitatively similar to that shown in Fig. 2b).

## Conclusion

We have revised the asymptotic theory developed in the papers [2, 3, 4] to describe the interaction of a solitary wave with external sources. In those papers only limiting cases were studied, either when the forcing is infinitely narrow in comparison with the initial KdV soliton, or when the initial KdV soliton is very narrow in comparison with the forcing. Here we have considered an arbitrary relationship between the width of initial KdV soliton and external forcing. We have presented two examples of forced KdV equation which admit exact analytical solutions. In the case of small-amplitude forcing we have shown that in many cases solutions of approximate equations can be solved analytically, albeit the solutions look very cumbersome. In the limiting




a) b)

Figure 5: The phase portraits of the system (20) and (21) with $K = 2$ (frame a) and $K = 3.5$ (frame b).

cases of very narrow or very wide forcing our results converge to those obtained in the papers cited above. In the meantime, it has been shown in this paper that there are some physically interesting regimes which were missed in those papers. The most important among them represent trapped solitons of small amplitudes by external potentials. The detail paper with many examples including a periodic firing will be published in the journal Chaos.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