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SUPERPOSITION OPERATORS ON DIRICHLET TYPE SPACES

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Abstract. We characterize the entire functions φ for which the induced nonlinear superposition operator $f \mapsto \varphi \circ f$ maps one Besov space B^p into another B^q, where B^∞ can be taken to be any of the following natural spaces: $VMOA$, $BMOA$, $\mathcal{B}_0$, and $\mathcal{B}$. We do the same for the superpositions from one unweighted Dirichlet-type space D^p into another, and from B^p into the weighted space D^q_α. The admissible functions are typically polynomials whose degree depends on p and q, or entire functions whose order and type are determined by those exponents. We prove some new Trudinger-type inequalities for analytic functions along the way.

0. Introduction

Let X, Y be spaces of analytic functions on the unit disk $\mathbf{D} \subset \mathbf{C}$ which contain the polynomials. The *nonlinear superposition operator* S_φ on X (with *symbol* φ) is defined by

$$S_\varphi(f) = \varphi \circ f, \qquad f \in X.$$

It is immediate that if $S_\varphi(X) \subset Y$ then the symbol φ must be an entire function. The graph of S_φ is usually closed but, since the operator is nonlinear, this is not enough to assure its boundedness. Nonetheless, for a number of important spaces X, Y, such as Hardy, Bergman, Dirichlet, Bloch, etc., the mere action $S_\varphi : X \to Y$ implies that φ must belong to a very special class of entire functions, which in turn implies boundedness. Our goal is to study the following questions:

(a) Which entire functions can transform one space into another?
(b) Are there spaces (of the type considered) which are transformed one into another by specified classes of entire functions (polynomials of certain degree, entire functions of given type and order, etc.)?

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Such questions have been extensively studied for real valued functions (*cf.* [AZ], for example). In the context of analytic functions, the question was investigated for the Hardy and Bergman spaces and the Nevanlinna class by Cámera and Giménez [C], [CG]. Our work is motivated by [CG] where, among other results, the authors characterize the superposition operators between Bergman spaces. The Bergman space A^p is the space of all L^p functions (with respect to Lebesgue area measure) which are analytic in the unit disk. Cámera and Giménez prove that $S_\varphi(A^p) \subset A^q$ if and only if φ is a polynomial of degree at most p/q; note that our notation is different from theirs. Next, they show that such operators are necessarily continuous, bounded and locally Lipschitz. They also consider similar problems for superposition operators acting from Bergman spaces into the Nevanlinna area class, etc. Their method is based on choosing certain A^p "test functions" with the largest possible range and applying suitable Cauchy estimates. Naturally, we employ the same idea where possible, but the technical difficulties with the "derivative spaces" studied here are greater, and various additional tools are needed.

The material is organized as follows. In Section 1 we review the preliminary material, including the basic definitions and facts about function spaces and conformal geometry. In Section 2, by means of test functions which are Riemann maps onto special domains, we characterize the symbols of superposition operators acting between different Besov-type spaces, including the "endpoint spaces" $VMOA$, $BMOA$, little Bloch $\mathcal{B}_0$, and Bloch $\mathcal{B}$. The operator S_φ acts from any one of these spaces into another of them if and only if φ is either a linear function or a constant, depending on the specific case in question.

The Dirichlet-type spaces D^p consist of functions whose derivatives are in A^p. With the exception of one delicate case, we find the operators between such spaces in Section 3. The admissible functions φ are either polynomials, or all entire functions, depending on the relationship between the exponents of the spaces.

The most interesting and delicate case, that of the superposition operators acting from the Dirichlet space into D^q, $q < 2$, requires a more sophisticated embedding theorem, namely a Trudinger-Moser type inequality for Dirichlet functions. This analysis is carried out in Section 4. Here the answer assumes the form of a sharp dichotomy: all functions of order less than two, or of order two and finite type, are "good" for every $q < 2$, while all the remaining entire functions are "bad". The proof of sufficiency follows essentially from the Trudinger-Moser type inequality, while necessity requires certain estimates for conformal mappings and the Poincaré metric.

The results of Section 4 raise the question of characterizing the entire functions of up to arbitrary order by their transformation properties as symbols of superposition operators. For this, one needs a different scale of spaces. To this end, we consider (Section 6) maps from the Besov spaces B^p, $p > 2$, into weighted spaces D^q_β, for certain range of indices. This analysis requires a preliminary detour in Section 5 to prove some new (non-standard exponent) Trudinger inequalities for analytic functions, similar to inequalities for Sobolev functions found by the first author and O'Shea in [BO].

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1. Background

We shall write $dA(z)$ to denote the Lebesgue area measure: $dA(z) = r\,dr\,dt = dx\,dy$. Given a positive finite p and a real number $\alpha > -1$, the *weighted Bergman space* A^p_α is defined as the space of analytic functions f in $\mathbf{D}$ such that

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbf{D}} |f(z)|^p\,(1-|z|^2)^\alpha\,dA \right)^{1/p} < \infty\,.$$

From now on, all upper indices denoted by p, q will be assumed positive. A lot of information on weighted Bergman spaces can be found in [DS], [HKZ], or [Z]. We denote the μ-average of a function g on a set S as

$$⨍_S g\,d\mu \equiv \frac{1}{\mu(S)} \int_S g\,d\mu.$$

If $d\mu = dA$, we also write this average as g_S, and we write $|S|$ for the area of S. We denote by $D(a,r)$ the disk with center a and radius r, and by $Q(z,r)$ the open square of center z and side-length $2r$. We sometimes write $A \lesssim B$ if $A \le CB$ for some constant C dependent only on allowed parameters, and we write $A \approx B$ if $A \lesssim B \lesssim A$.

We shall frequently use the following elementary estimate.

Lemma 1.

(a) *If $h \in A^p_\alpha$, then $|h(z)| = o\left((1-|z|)^{-(2+\alpha)/p}\right)$ when $|z| \to 1$.*

(b) *For $1 < p < q < \infty$, the inclusion $A^p_{p-2} \subset A^q_{q-2}$ holds.*

Proof. We first prove (a). For z sufficiently close to the unit circle, integrate the subharmonic function $|h|^p$ over the annulus $\{w : 3|z| - 1 \le 2|w| \le |z| + 1\}$, use the area version of the submean value property for the disk $D(z, (1-|z|)/2)$, and apply Lebesgue's Dominated Convergence Theorem.

As for (b), the assumption $f \in A^p_{p-2}$ and part (a) imply that $(1-|z|^2)\,|f(z)| \le M$ for some $M > 0$ and for all $z \in \mathbf{D}$. Therefore,

$$\int_{\mathbf{D}} |f(z)|^q\,(1-|z|^2)^{q-2}\,dA(z) \le M^{q-p} \int_{\mathbf{D}} |f(z)|^p\,(1-|z|^2)^{p-2}\,dA(z) < \infty\,. \qquad \square$$

We also study the *weighted Dirichlet-type spaces* D^p_α, $0 < p < \infty$, $-1 < \alpha < \infty$, of analytic functions f in the disk such that $f' \in A^p_\alpha$, equipped with the "norm"

$$\|f\|_{D^p_\alpha} = |f(0)| + \left(\int_{\mathbf{D}} |f'|^p\,(1-|z|^2)^\alpha\,dA \right)^{1/p} = |f(0)| + \|f'\|_{A^p_\alpha}\,.$$

For $p < 1$ this is not a true norm, but it satisfies $\|f+g\|^p_{D^p_\alpha} \le C_p\,(\|f\|^p_{D^p_\alpha} + \|g\|^p_{D^p_\alpha})$, where the constant C_p depends only on p. We write $D^p = D^p_0$. The space $\mathcal{D} = D^2$ is the classical *Dirichlet space* of analytic functions whose image Riemann surface has finite area. Clearly, the inclusion $D^p \subset D^q$ holds for $q < p < \infty$.

The spaces D^p_α include as special cases the *analytic Besov spaces* $B^p = D^p_{p-2}$, $1 < p < \infty$; we also define $\|f\|_{B^p} = \|f\|_{D^p_{p-2}}$. These spaces are *conformally invariant*: if $f \in B^p$, then $f \circ \varphi \in B^p$, for every disk automorphism $\varphi : \mathbf{D} \to \mathbf{D}$. Lemma 1(b) shows that $B^p \subset B^q$ whenever $1 < p < q < \infty$. The space B^1 is separately defined as the space of analytic functions for which

$$\int_{\mathbf{D}} |f''(z)|\, dA(z) < \infty\,.$$

Although this semi-norm is not conformally invariant, the space B^1 is. Equivalently, B^1 is the set of all functions

$$f = a_0 + \sum_{n=1}^{\infty} a_n \varphi_{\lambda_n}\,,$$

for some absolutely summable sequence $(a_n)_{n=0}^{\infty}$ and a sequence of points $\lambda_n \in \mathbf{D}$. Here

$$\varphi_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda} z}, \qquad z \in \mathbf{D},$$

is an *involutive disk automorphism* for every $\lambda \in \mathbf{D}$. The norm of f can then be defined as the infimum of the l^1 norms of (a_n) taken over all such representations. Thus it is clear that $B^1 \subset H^\infty$. It was shown in [AFP] that the two definitions of B^1 are equivalent. Lemma 2(b) below shows that $B^1 \subset B^p$ for all $1 < p < \infty$.

The other extreme among spaces of Besov type is the *Bloch space* $\mathcal{B}$ of all analytic functions f in $\mathbf{D}$ for which

$$\sup_{z \in \mathbf{D}} (1 - |z|)\, |f'(z)| < \infty\,,$$

while the *little Bloch space* $\mathcal{B}_0 \subset \mathcal{B}$ consists of all functions analytic in $\mathbf{D}$ for which

$$\lim_{|z| \to 1} (1 - |z|)\, |f'(z)| = 0\,.$$

A classical source for Bloch spaces is [ACP]. More about analytic Besov spaces can be found in [AFP] and [Z]; see also [W] and [DGV].

We denote by $BMOA$ the space of analytic functions of *bounded mean oscillation*, which consists of all functions f in the Hardy space H^2 whose boundary values lie in BMO of the unit circle $\mathbf{T}$. Equivalently, f belongs to $BMOA$ if and only if

$$\|f\|_{BMOA} = |f(0)| + \sup_{\lambda \in \mathbf{D}} \|f \circ \varphi_\lambda - f(\lambda)\|_{H^2} < \infty\,.$$

The space $VMOA$ of analytic functions of *vanishing mean oscillation* is the set of all $f \in BMOA$ such that

$$\lim_{|\lambda| \to 1} \|f \circ \varphi_\lambda - f(\lambda)\|_{H^2} = 0\,.$$

The reader is invited to consult [G] or [Ba] for more on these spaces. As a consequence of Lemma 1, we have $B^p \subset \mathcal{B}_0$ for all $1 < p < \infty$ (hence also for $p = 1$). In fact, more is true:

$$B^p \subset VMOA \subset \left\{ \begin{matrix} \mathcal{B}_0 \\ BMOA \end{matrix} \right\} \subset \mathcal{B}.$$

All of the above inclusions are strict, while neither of the spaces $\mathcal{B}_0$, $BMOA$ is contained in the other. Thus, in a sense, the four rightmost spaces above play the role of B^∞. The inclusion $B^p \subset VMOA$ is more delicate, but is known among the experts; see [DGV] for an indication of a proof.

We shall need the following results on integration of D^p functions.

Lemma 2.

(a) *$D^p \subset A^{2p/(2-p)}$ for all $p < 2$. Moreover, $\|f\|_{A^{2p/(2-p)}} \le C_p \|f\|_{D^p}$.*

(b) *If $f' \in A^1$, then $f \in A^p_{p-2}$ for every $p > 1$.*

Proof. First (a) follows from a variant of the Sobolev Imbedding Theorem [GT, Theorem 7.26] applied to analytic functions. A proof of the inclusion $D^p \subset A^{2p/(2-p)}$ involving only holomorphic functions was given by Flett [F, Theorem 5]. The associated norm dependence follows from versions of the classical Closed Graph Theorem. The Banach space version suffices when $p \ge 1$, while for $p < 1$ we use the version given by [DS, Theorem II.2.4].

As for (b), we assume that $f' \in A^1$. Since $f(z) - f(0) = \int_0^z f'(\zeta)\, d\zeta$, we see that

$$\int_0^{2\pi} |f(re^{it})|\, dt \le 2\pi|f(0)| + \int_0^{2\pi}\int_0^r |f'(\rho e^{it})|\, d\rho\, dt, \qquad 0 < r < 1,$$

and so $f \in H^1$. Thus f satisfies the estimate $|f(z)| \le C\,(1-|z|)^{-1}$ (see [D, p. 36]), where C depends on $f(0)$ but not on z. It follows that for $p > 1$,

$$\begin{aligned}\int_{\mathbf{D}} |f(z)|^p\,(1-|z|)^{p-2}\, dA(z) &\le \int_0^1 \left(\int_0^{2\pi} |f(re^{it})|(1-|z|)^{2p-3} dt\right) r\, dr \\ &\le C\int_0^1 (1-r)^{2p-3}\, dr < \infty. \qquad \square\end{aligned}$$

The one-parameter family of functions $\{f_\alpha(z) = (1-z)^{-\alpha} : \alpha > 0\}$ provides a collection of typical examples of unbounded functions in H^p, A^p, D^p, etc. We record as a lemma the easily verified conditions concerning the membership of these functions in Bergman and Dirichlet spaces.

Lemma 3. *Let f_α be as above, $\alpha > 0$. Then*

(a) *$f_\alpha \in A^p$, $p < \infty$, if and only if $\alpha < 2/p$.*

(b) *$f_\alpha \in D^p$, $p < 2$, if and only if $\alpha < (2-p)/p$.*

We now review some basic facts about conformal mappings and the Poincaré metric; for more details, we refer the reader to [Ah] and [P]. A mapping of one complex domain onto another is said to be *univalent* or *conformal* if it is one-to-one. By a *Riemann map* associated with a simply connected domain $\Omega \neq \mathbf{C}$ in the plane we shall always mean a conformal mapping F from $\mathbf{D}$ onto Ω (traditionally, the mapping goes the other way around).

Given a domain Ω in the plane and a point $w \in \Omega$, we write $d_\Omega(w)$ to denote the (Euclidean) distance from w to the boundary $\partial\Omega$. The following useful result is well-known [P, Corollary 1.4].

Lemma 4. *If f is a univalent map of $\mathbf{D}$ onto a simply connected domain Ω then*

$$d_\Omega(f(z)) \le |f'(z)|\,(1-|z|^2) \le 4\,d_\Omega(f(z))\,.$$

The *Poincaré metric* on the unit disk $\mathbf{D}$ is defined by

$$\lambda_{\mathbf{D}}(z_1,z_2) = \min_\gamma \int_\gamma \frac{2|dz|}{1-|z|^2} = \log\frac{1+|z_1-z_2|/|1-\overline{z_1}z_2|}{1-|z_1-z_2|/|1-\overline{z_1}z_2|}\,,$$

the minimum being taken over all rectifiable curves γ from z_1 to z_2 in $\mathbf{D}$. On a simply connected domain $\Omega \subset \mathbf{C}$ ($\Omega \neq \mathbf{C}$), the Poincaré metric is defined in terms of a Riemann map $F : \mathbf{D} \to \Omega$ via the pull-back:

$$\lambda_\Omega(w_1,w_2) = \lambda_{\mathbf{D}}(F^{-1}(w_1),F^{-1}(w_2))\,.$$

The minimum is actually attained, and the metric is independent of the Riemann map chosen [P]. Observe that for $w = F(z)$, $F(0) = 0$ we have the following useful estimate (with universal constants of comparison):

$$\lambda_\Omega(0,F(z)) \approx \log\frac{1}{1-|z|}\,, \quad |z| \to 1.$$

2. Operators acting between Besov, Bloch, and related spaces

In this section, we use B^∞ (which has not yet been defined) to denote any one of the four spaces $\mathcal{B}$, $\mathcal{B}_0$, $BMOA$, and $VMOA$. Statements about B^∞ should be interpreted as a statements that apply to all of these spaces.

We start off with an easy result based on our earlier observations.

Proposition 5. *For any entire function φ, we have $S_\varphi(B^1) \subset B^1$, whence $S_\varphi(B^1) \subset B^p$ for all $1 \le p \le \infty$.*

Proof. Recall that $f \in B^1$ implies that f is bounded in $\mathbf{D}$, and so is $g \circ f$ for any entire function g. Now, $f \in B^1$ means that $f'' \in A^1$, and also $f' \in A^2$ by Lemma 2(b). Therefore we have

$$(\varphi\circ f)'' = (\varphi'\circ f)\cdot f'' + (\varphi''\circ f)\cdot (f')^2 \in A^1$$

for any entire function φ, showing that $\varphi \circ f \in B^1$. □

The following useful criterion, which follows from a change of variables and Lemma 4, also appeared in a recent paper by Walsh [W], and was exploited in [DGV].

Proposition 6. *Let $F : \mathbf{D} \to \Omega$ be a univalent map onto a simply connected domain Ω. Then, for any $1 < p < \infty$, $F \in B^p$ if and only if*

$$\int_\Omega d_\Omega(w)^{p-2}\, dA(w) < \infty\,.$$

For the Bloch space $\mathcal{B}$, the above conditions become the familiar one:

$$\sup_{w\in\Omega} d_\Omega(w) < \infty\,.$$

The next result, which may have some independent interest, will provide a method for constructing unbounded univalent functions in B^p for $1 < p < \infty$ of somewhat different type than the ones constructed in [DGV]. Such functions are precisely the "test functions" which will be needed for our study in certain cases.

Proposition 7. *Let $1 < p < \infty$, let (w_n) be a sequence of complex numbers, and let (r_n) and (h_n) be sequences of positive numbers with the following properties:*

(a) $0 \le \arg w_n \le \pi/4$ *and* $|w_n| \le |w_{n+1}|/2$, $n \in \mathbf{N}$;

(b) $r_n < |w_n|/4$ *and* $|h_n| < \min\{r_n, r_{n+1}\}/3$, $n \in \mathbf{N}$.

Then the domain $\Omega = \cup_{n=1}^\infty [D_n \cup R_n]$ is simply connected, where $D_n = D(w_n, r_n)$ and R_n is the rectangle whose longer symmetry axis is the segment $[w_n, w_{n+1}]$ and whose shorter side has length $2h_n$. Furthermore if F is a Riemann map of $\mathbf{D}$ onto Ω, then $F \in B^p$ if and only if

$$\sum_{n=1}^\infty r_n^p + \sum_{n=1}^\infty |w_{n+1} - w_n|\, h_n^{p-1} < \infty.$$

Proof. It is easy to see that distance to the origin increases as we travel along $[w_n, w_{n+1}]$ from w_n to w_{n+1}, and hence to deduce that $\Omega_N = \bigcup_{n=1}^N [D_n \cup R_n]$ is simply connected for each N. This in turn implies that Ω is simply connected.

Let us define $I(U, V) = \int_U d_V(w)^{p-2}\, dA(w)$, so it follows readily from Proposition 6 that $F \in B^p$ if and only if

$$S \equiv \sum_{n=1}^\infty [I(D_n, \Omega) + I(R_n, \Omega)] < \infty.$$

Notice that there is some duplication of integration here, but by no more than a factor of two. It is also easy to verify that $I(D_n, D_n) \approx r_n^p$ and $I(R_n, R_n) \approx |w_{n+1} - w_n|\, h_n^{p-1}$. Considering separately the cases $p \le 2$ and $p > 2$, we see that $I(D_n, \Omega) \approx I(D_n, D_n)$ (in both cases inequality in one direction is trivial and in the other direction it requires a straightforward estimate). A similar analysis shows the comparability of $I(R_n, \Omega)$ and $I(R_n, R_n)$. Putting together the estimates in this paragraph, the proposition follows. □

Our next result on the action of a superposition operator from a Besov space into the Bloch space is derived from the conformal properties of Riemann maps and will also help us resolve the problem in various related cases. Here and later, a *linear function* means a polynomial f of degree at most 1; we do not insist that $f(0) = 0$.

Theorem 8. *If S_φ acts from B^p into $\mathcal{B}$ for some $1 < p < \infty$, then φ is a linear function.*

Proof. Suppose φ' is non-constant, and let $r_n = 2^{-n}$. By Liouville's theorem, there exists a sequence (w_n) of complex numbers so that $|w_1| > 2$ and

$$\forall\, n \in \mathbf{N}: \qquad |w_{n+1}| \geq 2|w_n|, \quad |\varphi'(w_n)| \geq r_n^{-2}.$$

At least one of the eight octants $\arg^{-1}([(k-1)\pi/4, k\pi/4))$ contains infinitely many of the numbers w_n. By a rotation if necessary, we may therefore assume that $0 \leq \arg w_n < \pi/4$, and so Proposition 7 is applicable. Let $h_n = 2^{-n-2}|w_{n+1} - w_n|^{-1/(p-1)}$, let Ω be the domain in Proposition 7 defined using the data $(w_n, r_n, h_n)_{n=1}^\infty$, and let $F : \mathbf{D} \to \Omega$ be an associated Riemann map. By Proposition 7, we know that $F \in B^p$. Let $F(z_n) = w_n$. It is easily seen that $|z_n| \to 1$ as $n \to \infty$. Applying Lemma 4, we obtain

$$|\varphi'(w_n)|\,|F'(z_n)|\,(1-|z_n|) \approx |\varphi'(w_n)|\, d_\Omega(w_n) \geq C/r_n \to \infty\,,$$

which means that $\varphi \circ F \notin \mathcal{B}$. Contradiction. □

Taking into account the chain of inclusions mentioned earlier, one easily obtains the following result.

Corollary 9. *Let $1 < p < \infty$. Then $S_\varphi(B^p) \subset B^q$ if and only if*

(a) *φ is a linear function, if $p \leq q$.*
(b) *φ is constant, if $p > q$.*

Proof. Obviously, $S_\varphi(B^p) \subset B^q$ implies $S_\varphi(B^p) \subset \mathcal{B}$, whence φ must be linear by Theorem 8. Since the spaces B^p strictly increase with the exponent, the result is now clear. □

Taking into account the inclusions between the Bloch spaces and the four B^∞ type spaces, we obtain another easy consequence of Theorem 8.

Corollary 10. *Let X, Y be spaces of the form B^p, $1 < p \leq \infty$. Then $S_\varphi(X) \subset Y$ if and only if*

(a) *φ is a linear function, whenever $X \subset Y$.*
(b) *φ is constant whenever $X \not\subset Y$.*

3. Spaces D^p of Dirichlet Type

We now turn to a systematic study of the nonlinear superposition operators between the spaces D^p. The results from the previous section immediately take care of some cases. Specifically, the following result follows from Corollary 9 together with the strict inclusion of D^q in D^p when $p < q$.

Corollary 11. *If $p \le 2 \le q < \infty$, $p \ne q$, then $S_\varphi(D^p) \subset D^q$ if and only if φ is constant.*

The following result is similar in spirit to Theorem 1 of [CG] for Bergman spaces. The proof of necessity presented below can be adapted so as to shorten somewhat the Bergman space proof in [CG].

Theorem 12. *For $q \le p < 2$, we have $S_\varphi(D^p) \subset D^q$ if and only if φ is a polynomial of degree at most $\left[\frac{p(2-q)}{q(2-p)}\right]$.*

Proof. For sufficiency, the case $p = q$ is easy so suppose $q < p$. It suffices to verify the statement for $\varphi(z) = z^n$, where $n \in \mathbf{N}$, $n \le s \equiv \frac{p(2-q)}{q(2-p)}$, an inequality which can be rewritten as $\frac{(n-1)pq}{p-q} \le \frac{2p}{2-p}$. This last inequality, together with Hölder's inequality and Lemma 2(a), yields

$$\begin{aligned} \|(f^n)'\|_{A^q} &= n \left(\int_{\mathbf{D}} |f' f^{n-1}|^q \, dA \right)^{1/q} \\ &\le n \|f'\|_{A^p} \left(\int_{\mathbf{D}} |f|^{(n-1)pq/(p-q)} \, dA \right)^{(p-q)/pq} \\ &\lesssim \|f\|_{D^p} \|f\|^{n-1}_{A^{2p/(2-p)}} \lesssim \|f\|^n_{D^p}. \end{aligned} \tag{1}$$

Thus $S_\varphi(D^p) \subset D^q$.

We now turn to necessity. Suppose φ is not a polynomial of degree at most $t \equiv \left[\frac{p(2-q)}{q(2-p)}\right]$, or equivalently that the Taylor expansion of φ about 0 has a non-zero coefficient of order $m > t$. We shall show that $S_\varphi(D^p)$ is not a subset of D^q. Since $(t+1)(2/p-1)+1 > 2/q$, we can choose $\alpha \in (0, 2/p-1)$ such that $m\alpha + 1 > 2/q$. It follows from the Cauchy estimates for φ' that there exist a constant $c > 0$ and an unbounded sequence (w_n) such that $|\varphi'(w_n)| \ge c|w_n|^{m-1}$. Dividing $\mathbf{C} \setminus \{0\}$ into $2N$ closed sectors with pairwise disjoint interiors and of angle π/N each, where $N \ge 2$ is an integer larger than $2/\alpha$, we can always choose one such sector that contains infinitely many points of the form w_n. By an appropriate choice of rotation parameter $\lambda \in \mathbf{T}$, we get a function ψ defined by the equation $\psi(z) = \varphi(\lambda z)$ with the property that for some sequence of points v_n, $|\arg v_n| \le \pi/2N$, we have $|\psi'(v_n)| \ge c|v_n|^{m-1}$. In fact the sequence (v_n) can be chosen to be a subsequence of (w_n/λ). By taking a further subsequence if necessary, we assume that $|v_n| \ge 2^{\alpha n}$.

Let $f(z) = (1-z)^{-\alpha}$. Since $\alpha \in (0, 2/p-1)$, it follows from Lemma 3 that $f \in D^p$. We claim that $\psi \circ f \notin D^q$; note that this claim implies that $S_\varphi(D^p) \not\subset D^q$. To prove the claim, we let $z_n = 1 - v_n^{-1/\alpha}$, so that $\arg(1-z_n) < \pi/4$ and $v_n = f(z_n)$. Now $|1-z_n| = |v_n|^{-1/\alpha} \le 2^{-n}$, so the restriction on $\arg(1-z_n)$ ensures that $1 - |z_n| \ge c_1|1-z_n|$ for some universal $c_1 > 0$. Thus

$$|(\psi \circ f)'(z_n)| = |\psi'(v_n) f'(z_n)| \ge c|v_n|^{m-1}|1-z_n|^{-\alpha-1} \ge c c_1^{m\alpha+1}(1-|z_n|)^{-m\alpha-1}.$$

Since $m\alpha + 1 > 2/q$, it follows from Lemma 1 that $(\psi \circ f)'(z_n)$ grows so fast that $\psi \circ f$ cannot be in A^q, and so $\psi \circ f \notin D^q$.

□

Corollary 13. *If $p < q < 2$, then $S_\varphi(D^p) \subset D^q$ if and only if φ is constant.*

Proof. Since $D^q \subset D^p$, we may apply Theorem 12 to obtain that φ is linear. Since the inclusion $D^q \subset D^p$ is strict, φ must be constant. □

A nonlinear operator is said to be *bounded* if it transforms bounded sets into bounded sets. Continuity is defined in the traditional way.

Corollary 14. *If $q < p < 2$ and $S_\varphi(D^p) \subset D^q$ then the operator $S_\varphi : D^p \to D^q$ is bounded and continuous.*

Proof. We know from Theorem 12 that the mere action of S_φ obliges φ to be a polynomial: $\varphi(z) = \sum_{j=0}^n a_j z^j$, where $n \le \left[\frac{p(2-q)}{q(2-p)}\right]$. By the triangle inequality, we have

$$\|S_\varphi(f) - S_\varphi(f_k)\|_{D^q} \le \sum_{j=1}^n |a_j| \|f^j - f_k^j\|_{D^q}$$

so it will suffice to consider only the case $\varphi(z) = z^n$ and show that

$$\|f^n - f_k^n\|_{D^q} \le M\|f - f_k\|_{D^p}^n .$$

But this is easy: from inequality (1) of Theorem 12 we readily deduce that

$$|f^n(0)| + n^{-1}\|(f^n)'\|_{A^q} \lesssim (|f(0)| + \|f'\|_{A^p})^n .$$

This can be rewritten as $\|f^n\|_{D^q} \le M\|f\|_{D^p}^n$, and both boundedness and continuity follow immediately. □

In particular, we have the following result.

Corollary 15. *Let $q < p < 2$, $n \in \mathbf{N}$, and $n \le \left[\frac{p(2-q)}{q(2-p)}\right]$. If $f_k \to f$ in D^p, then $f_k^n \to f^n$ in D^q (as $k \to \infty$).*

The next case shows a difference in comparison with the Bergman space case.

Proposition 16. *If $2 < p < \infty$ and $q \le p < \infty$, then for any entire function φ we have $S_\varphi(D^p) \subset D^q$.*

Proof. Let $f \in D^p$. By integrating the pointwise bound in Lemma 1(a), we see that f is bounded, and so $\varphi' \circ f$ is bounded on $\mathbf{D}$ for any entire function φ. Thus

$$\left(\int_{\mathbf{D}} |(\varphi' \circ f) \cdot f'|^q \, dA\right)^{1/q} \le C \left(\int_{\mathbf{D}} |f'|^p \, dA\right)^{1/p}, \qquad q \le p. \qquad \square$$

Theorem 17. *If $2 < p < q < \infty$, then $S_\varphi(D^p) \subset D^q$ if and only if φ is constant.*

Proof. Suppose φ is not constant. Fixing $0 < c < 1$ such that $K \equiv |\varphi'(c)| > 0$ and $\beta = 1 - 2/q$, we let $f(z) = c + (1-z)^{1-2/q}$. Then $f \in D^p \setminus D^q$ by Lemma 3 and, since $0 < 1 - 2/q < 1$, f is in the disk algebra. In particular $f(1) = c$ and $|\varphi'(f(r))| \to K$ as $r \to 1^-$. Consequently,

$$\frac{|(\varphi \circ f)'(r)|}{K(1-2/q)(1-r)^{2/q}} \to 1 \quad \text{as } r \to 1^-,$$

and so $|(\varphi \circ f)'(r)|$ grows at least as fast as $(1-r)^{2/q}$ as $r \to 1^-$. By Lemma 1, we deduce that $(\varphi \circ f)'$ cannot be in A^q. Thus $S_\varphi(f) \notin D^q$ and we are done. □

4. Action on the Dirichlet Space

The only remaining case for superposition operators between unweighted Dirichlet-type spaces, namely $S_\varphi : \mathcal{D} \to D^q$ for $q < 2$, requires a more profound study than the other cases. This is essentially due to the fact that $p = 2$ is the critical Sobolev exponent for the plane.

First, we need some definitions. We define a *Young function* to be a convex homeomorphism $\psi : [0, \infty) \to [0, \infty)$. If μ is a positive measure on a set X, and ψ is a Young function, then the associated *Orlicz norm* is given by

$$\|f\|_{\psi(L)(X,\mu)} = \inf \left\{ t > 0 \mid \int_X \psi(|f(x)|/t)\, d\mu(x) \leq 1 \right\}.$$

It is also convenient to define certain classes of entire functions. Specifically, for each $t \geq 1$, $E(t)$ is the class of entire functions f such that

$$\exists\, C, C' > 0 \quad \forall\, z \in \mathbf{C} : \qquad f(z) \leq C \exp(C'|z|^t).$$

Equivalently $f \in E(t)$ if f has order less than t, or order t and finite type.

The integrability of $\exp(c|f|^2)$ has been studied in very general contexts (see for example [Tr], [Mo], [CM], [M], and [Ch]). A whole collection of results with this flavor is usually termed the *Trudinger* or *Trudinger-Moser inequality*. In the context of analytic functions, this inequality essentially goes back to Beurling's doctoral thesis in the 1930's, where he proves [Be, p. 34] the distributional inequality

$$f \in \mathcal{D},\ \|f\|_{\mathcal{D}} \leq 1,\ f(0) = 0 \quad \Longrightarrow \quad |\{\theta \in [0, 2\pi] : |f(e^{i\theta})| > \lambda\}| \leq e^{-\lambda^2 + 1},$$

and also shows that this is sharp in a certain sense. This readily implies the Orlicz estimate

$$\|f\|_{\psi(L)(\mathbf{T},m)} \leq C\|f\|_{\mathcal{D}}, \qquad f \in \mathcal{D},$$

where ψ is the Young function given by $\psi(t) = \exp(t^2) - 1$, and m is Lebesgue measure. Since the space of polynomials is dense[1] in $\mathcal{D}$, and since $\int_{\mathbf{T}} \psi(\alpha P(e^{i\theta}))\, d\theta < \infty$ for every polynomial P, it is not hard to deduce the first statement of the following theorem (the second statement, however, requires much more analysis!).

Theorem 18 [CM], [M]. *For all $f \in \mathcal{D}$ and $\alpha > 0$, we have $\int_0^{2\pi} \exp(\alpha|f(e^{i\theta})|^2)\, d\theta < \infty$. Moreover, $\int_0^{2\pi} \exp(\alpha|f(e^{i\theta})|^2)\, d\theta$ is uniformly bounded for all f in the unit ball of $\mathcal{D}$ if and only if $\alpha \leq 1$.*

We are now ready to prove the main result of this section.

[1]This follows, for instance, from the formula $\|\sum_{j=1}^\infty a_j z^j\|_{\mathcal{D}}^2 = \sum_{j=1}^\infty j|a_j|^2$

Theorem 19. *If $q < 2$ and φ is entire, then $S_\varphi(\mathcal{D}) \subset D^q$ if and only if $\varphi \in E(2)$.*

Proof. We first prove sufficiency. Let $\varphi \in E(2)$. By [Ti, §8.51, p. 265], we also have $\varphi' \in E(2)$, and so there exists $\beta > 0$ such that $|\varphi'(w)| \le e^{\beta\,|w|^2}$ for all sufficiently large $|w|$. Applying first Hölder's inequality and then Theorem 18, we obtain

$$\begin{aligned}\int_{\mathbf{D}} |f'|^q |\varphi' \circ f|^q \, dA &\le \left(\int_{\mathbf{D}} |f'|^2 \, dA\right)^{q/2} \left(\int_{\mathbf{D}} |\varphi' \circ f|^{2q/(2-q)} \, dA\right)^{(2-q)/2} \\ &\le \|f\|_{\mathcal{D}}^q \left(K + \int_{\mathbf{D}} \exp[2q\beta |f|^2/(2-q)] \, dA\right)^{(2-q)/2} < \infty \,,\end{aligned}$$

for all $f \in \mathcal{D}$. Note that we used the following fact:

$$\int_{\mathbf{D}} e^{\beta |f|^2} \, dA = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \int_{\mathbf{D}} |f|^{2n} \, dA \le \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \int_{\mathbf{T}} |f|^{2n} \, d\theta = \int_{\mathbf{T}} e^{\beta |f|^2} \, d\theta \,.$$

To prove necessity, let us take an entire function $\varphi \notin E(2)$. If φ' were in $E(2)$, simple integration would give us $\varphi \in E(2)$. Thus $\varphi' \notin E(2)$ and there exists a sequence $(w_n)_{n=1}^{\infty}$ of complex numbers outside the unit ball such that

$$|\varphi'(w_n)| \ge n \exp(n^2\,|w_n|^2).$$

Clearly, $|w_n| \to \infty$ as $n \to \infty$, so by passing to a subsequence and rotating if necessary, we may assume that $2|w_n| < |w_{n+1}|$ and $0 \le \arg w_n < \pi/4$ for all $n \in \mathbf{N}$. Define the domain Ω as in Proposition 7 so that $0 \in \Omega$ (this is easily achieved by adding a constant to the function φ, which does not change anything), with the auxiliary data

$$r_n = \frac{1}{\sqrt{n}\, \log^2(n+1)}, \quad h_n = \frac{1}{3\,(n+1)\, \log^2(n+2)\, |w_{n+1} - w_n|},$$

and let F be a Riemann map from $\mathbf{D}$ onto Ω with the property $F(0) = 0$. By the case $p = 2$ of Proposition 7, we see that $F \in \mathcal{D} = B^2$. Next, let $z_n = F^{-1}(w_n) \in \mathbf{D}$, and choose a sequence of disks $D_n' = D(z_n, a(1 - |z_n|))$, where $0 < a < 1$ is fixed, in such a way that $F(D_n') \subset D_n = D(w_n, r_n/2)$; this is possible because of Lemma 4.

Suppose N is a large integer. By the geometric growth rate of $|w_n|$, we see that $\sum_{n=1}^{N} |w_n - w_{n-1}|^2$ is uniformly comparable with $|w_N|^2$. By joining the w_N with $w_0 = 0$ via the piecewise linear "join the dots" path through the points, we obtain the following inequalities on the Poincaré metric:

$$\begin{aligned}\lambda_\Omega(0, w_N) &\lesssim \sum_{n=1}^{N} (1 + h_n^{-1} |w_{n+1} - w_n|) \\ &\lesssim N + \left(\sum_{n=1}^{N} |w_{n+1} - w_n|^2\, N\, \log^2 N\right) \\ &\lesssim N(1 + \log^2 N)|w_N|^2 \lesssim N^{3/2} |w_N|^2.\end{aligned}$$

Thus

$$\begin{aligned}
\int_{\mathbf{D}} |\varphi' \circ F|^q \, |F'|^q \, dA &\geq \sum_{n=1}^{N} \int_{D'_n} |\varphi' \circ F|^q \, |F'|^q \, dA \\
&\approx \sum_{n=1}^{N} \left(\int_{D'_n} |\varphi' \circ F|^q |F'|^2 \, dA \right) \left(\frac{r_n}{1-|z_n|} \right)^{q-2} \\
&\approx \sum_{n=1}^{N} \left(\int_{D_n} |\varphi'|^q \, dA \right) r_n^{q-2} \exp\left(-C(2-q)\lambda_\Omega(0, w_n)\right) \\
&\gtrsim \sum_{n=1}^{N} r_n^2 |\varphi'(w_n)|^q r_n^{q-2} \exp\left(-C(2-q)\lambda_\Omega(0, w_n)\right) \\
&\geq \sum_{n=1}^{N} \exp\left((qn^2 - C' n^{3/2}) |w_n|^2 \right) = \infty.
\end{aligned}$$

Above the first estimate is trivial, the second follows from Lemma 4, the third involves a change of variables and the Poincaré metric estimate from the end of Section 1, the fourth uses the area version of the mean value inequality, and the last uses the estimates on $|\varphi'(w_n)|$ and the Poincaré metric from earlier in this lemma. □

Note that it follows from the above proof of sufficiency, together with the second statement of Theorem 18, that S_φ maps the unit ball of $\mathcal{D}$ into a ball of fixed radius in D^q whenever φ is entire of order < 2, or of order 2 and type at most $(2-q)/2q$.

Having now completed the classification of all entire functions φ for which $S_\varphi(D^p) \subset D^q$, for all possible choices of indices $p, q \in (0, \infty)$, let us pause to summarize our results. If $p < q$, the situation is very simple: φ must be constant. The more complicated case $q \leq p$ is summarized by the following table, where E denotes the space of all entire functions, $P(r)$ is the space of all polynomials of degree at most $[r]$, and $E(2)$ is as before.

TABLE 1. The space of symbols $\{\varphi \mid S_\varphi : D^p \mapsto D^q\}$ when $q \leq p$.

	$p < 2$	$p = 2$	$p > 2$
$q < 2$	$P\left(\dfrac{p(2-q)}{q(2-p)}\right)$	$E(2)$	E
$q = 2$	—	$P(1)$	E
$q > 2$	—	—	E

5. Some new inequalities of Trudinger-Moser type

Using techniques similar to the ones employed in [BO], we now derive some inequalities of Trudinger-Moser type for analytic functions. Unlike the results in [BO], however, our inequalities will not always hold in the more general setting of Sobolev functions, a point to which we shall return towards the end of this section. Recall that $Q(z,r)$ denotes the open square of center z and side-length $2r$.

Lemma 20. *There exists a universal constant C such that*

$$\|f-f(z)\|_{L^\infty(Q(z,r))} \le Cr\left(⨍_{Q(z,2r)} |f'|^p \, dA \right)^{1/p},$$

whenever f is holomorphic in $Q(z,2r)$, $z \in \mathbf{C}$, and $p \ge 1$.

Proof. Suppose $w \in Q(z,r)$. By the Sub-mean Value Property, we have

$$\begin{aligned} |f(w) - f_{Q(z,3r/2)}| &\le ⨍_{D(w,r/2)} |f(u) - f_{Q(z,3r/2)}| \, dA(u) \\ &\le \frac{36}{\pi} ⨍_{Q(z,3r/2)} |f(u) - f_{Q(z,3r/2)}| \, dA(u). \end{aligned}$$

In a similar fashion,

$$\begin{aligned} |f_{Q(z,3r/2)} - f(z)| &\le ⨍_{D(z,3r/2)} |f_{Q(z,3r/2)} - f(u)| \, dA(u) \\ &\le \frac{4}{\pi} ⨍_{Q(z,3r/2)} |f(u) - f_{Q(z,3r/2)}| \, dA(u). \end{aligned}$$

It follows that

$$L \equiv \|f-f(z)\|_{L^\infty(Q(z,r))} \le \frac{40}{\pi} ⨍_{Q(z,3r/2)} |f(u) - f_{Q(z,3r/2)}| \, dA(u).$$

But by Cauchy-Schwarz and a classical Poincaré inequality [GT, p. 164], there is a universal constant C_1 such that

$$⨍_{Q(z,3r/2)} |f - f_{Q(z,3r/2)}| \, dA \le C_1 r \left(⨍_{Q(z,3r/2)} |f'|^2 \, dA \right)^{1/2} \le C_1 r \sup_{Q(z,3r/2)} |f'|.$$

Since f' is also analytic, we see as before that

$$\sup_{Q(z,3r/2)} |f'| \le \frac{64}{\pi} ⨍_{Q(z,2r)} |f'| \, dA \le \frac{64}{\pi} \left(⨍_{Q(z,2r)} |f'|^p \, dA \right)^{1/p},$$

and so we are done. □

Our next lemma is a version of the Whitney decomposition, as given in [Sa].

Lemma 21. *Given a proper subdomain Ω of $\mathbf{R}^n$, and a number $A \geq 1$, there exists a constant C dependent only on A and n, and a countable family of pairwise disjoint open cubes $\{Q^k \mid k \in I\}$ such that:*

(i) $\Omega = \bigcup_{k\in I} \overline{Q^k}$;
(ii) $5A \leq \operatorname{dist}(Q^k, \partial\Omega)/\operatorname{diam} Q^k \leq 15A, \quad k \in I$;
(iii) $\sum_{k\in I} \chi_{AQ^k} \leq C\chi_\Omega$.

We use this lemma only in the case $\Omega = \mathbf{D}$, $A = 20$, in which case we fix one such collection $\mathcal{W}$ of Whitney squares. It is also convenient to write $r(Q)$ for the sidelength of a square Q, and $z(Q)$ for the center of a Q. As the reader may readily verify, $r(Q)/r(Q') \in [1/4, 4]$ whenever $Q, Q' \in \mathcal{W}$ are adjacent; it follows that the nine-fold dilate of any square in $\mathcal{W}$ contains all adjacent Whitney squares.

It is convenient to define the radially weighted measures

$$d\mu_\alpha(z) = (1-|z|^2)^\alpha \, dA(z), \qquad \alpha > -1,\ z \in \mathbf{D}.$$

Clearly $r(Q) \approx 1 - |w|^2$ for every $w \in Q \in \mathcal{W}$, and so $\mu_\alpha(Q)/r(Q)^{\alpha+2} \in [1/C, C]$ for some C dependent only on α.

We now state and prove an easy lemma which is well-known to experts.

Lemma 22. *Given $z \in \mathbf{D}$, let $\{Q_i\}_{i=1}^n \subset \mathcal{W}$ be the Whitney squares intersecting the line segment $[0, z]$. Then $n/(1 + \lambda(0, z)) \in [1/C, C]$ for some universal constant C.*

Proof. Without loss of generality we assume that $z \neq 0$, and write $\hat{z} = z/|z|$. Since the Poincaré lengths of the segments $I_j = [(1 - 2^{-j+1})\hat{z}, (1 - 2^{-j})\hat{z}]$ are bounded above and below, it suffices to show that the number of Whitney squares whose closures intersect any one segment I_j is bounded above, and that there is at least one Whitney square that intersects I_k only for $k = j$.

The upper bound follows readily from the fact that if $Q \in \mathcal{W}$ intersects I_j, then the Euclidean length of the intersection of $2Q$ and I_j must be comparable to $|I_j|$; we also need the bounded overlap of the squares $\overline{2Q}$, as assured by Lemma 21(iii). The lower bound follows from the fact that we have picked rather small Whitney squares: since $A = 20$ is “fairly large”, any square whose closure includes the midpoint of I_j cannot intersect I_k for any $k \neq j$. □

We are now ready to state and prove an imbedding theorem which is the main result in this section, and involves the Young function

$$\psi_p(s) = \exp(s^{p/(p-1)}) - 1, \qquad p > 1,$$

The imbedding is rather sharp, a point we shall discuss further at the end of the final section.

Theorem 23. *Given $p \in (1,\infty)$ and $\alpha \in (-1,\infty)$, there exists a constant C such that*

$$\|f\|_{\psi_p(L)(\mathbf{D},\mu_\alpha)} \le C\|f\|_{B^p}, \qquad f \in B^p.$$

Proof. The case of a constant function is trivial so we assume that f is non-constant. We normalize f so that $f(0)=0$ and $\|f\|_{B^p}=1$. We also define j_0 to be the least integer larger than $p-1$.

First, note that by the sublinearity of ψ_p, we have

$$\int_{\mathbf{D}} \psi_p(|f|/C)\,d\mu_\alpha \le C' \Rightarrow \|f\|_{\psi_p(L)(\mathbf{D},\mu_\alpha)} \le CC'.$$

Combining this fact with Hölder's inequality, we see that it is sufficient to prove that for some bounded constant C,

$$\int_{\mathbf{D}} \psi_p^0(|f|/C)\,d\mu_\alpha(x) \equiv \sum_{Q\in\mathcal{W}} \int_Q \psi_p^0(|f|)\,d\mu_\alpha \lesssim 1,$$

where $\psi_p^0(s) = \sum_{j\ge j_0} s^{jp/(p-1)}/j!$.

Poincaré-type inequalities on general John domains were discovered by Bojarski [Bo] and Martio [M]. Although, we are only working with the unit disk, we nevertheless use a common trick for such analysis: we separately control "local" and "global" terms, the former being the above integrals over Q, but with $|f|$ replaced by $|f - f(z(Q))|$, and the latter being the constants $|f(z(Q))|$ which we control by a chaining argument. First we control the local terms; these are not very delicate, so we can take $C=1$.

$$\begin{aligned}
\mathcal{L} &\equiv \sum_{Q\in\mathcal{W}} \int_Q \psi_p^0(|f - f(z(Q))|)\,d\mu_\alpha \\
&= \sum_{j=j_0}^{\infty} \frac{1}{j!} \sum_{Q\in\mathcal{W}} \int_Q |f - f(z(Q))|^{jp/(p-1)}\,d\mu_\alpha \\
&\le \sum_{j=j_0}^{\infty} \frac{1}{j!} \sum_{Q\in\mathcal{W}} \mu_\alpha(Q)\,\|f - f(z(Q))\|_{L^\infty(Q)}^{jp/(p-1)} \\
&\le \sum_{j=j_0}^{\infty} \frac{C_1^{jp/(p-1)}}{j!} \sum_{Q\in\mathcal{W}} \mu_\alpha(Q)\,r(Q)^{jp/(p-1)} \left(⨍_{2Q} |f'|^p \right)^{j/(p-1)} \\
&\le \sum_{j=j_0}^{\infty} \frac{C_2^{jp/(p-1)}}{j!} \sum_{Q\in\mathcal{W}} \mu_\alpha(Q) \left(\int_{2Q} |f'|^p\,d\mu_{p-2} \right)^{j/(p-1)},
\end{aligned}$$

where the inequalities follow from Hölder's inequality, Lemma 20, and the fact that $\mu_{p-2}(Q) \approx r(Q)^p$, and the constants C_1, C_2 depend only on p.

Now $\mu_\alpha(Q)$ is bounded, so it can be ignored. Since $j > p-1$, and the squares $2Q$, $Q \in \mathcal{W}$, have bounded overlap, we have

$$\begin{aligned}\sum_{Q\in\mathcal{W}} \left(\int_{2Q} |f'|^p \, d\mu_{p-2}\right)^{j/(p-1)} &\le \left(\sum_{Q\in\mathcal{W}} \int_{2Q} |f'|^p \, d\mu_{p-2}\right)^{j/(p-1)} \\ &\le \left(C_3 \int_{\mathbf{D}} |f'|^p \, d\mu_{p-2}\right)^{j/(p-1)} = C_3^{j/(p-1)},\end{aligned}$$

for some universal constant C_3. It follows that $\mathcal{L} \le \psi_p(C_2 C_3^{1/p}) \lesssim 1$, as required.

It remains to control the global terms

$$\mathcal{G} \equiv \sum_{j=j_0}^{\infty} \frac{1}{j!} \sum_{Q\in\mathcal{W}} \mu_\alpha(Q) \left(\frac{|f(z(Q))|}{C}\right)^{jp/(p-1)}.$$

Let us fix $Q \in \mathcal{W}$ for the moment, and let $\{Q_i\}_{i=1}^n$ be the squares in $\mathcal{W}$ that intersect $[0, z(Q)]$, ordered in their natural order so that $0 \in \overline{Q_1}$, $Q_n = Q$, and squares with adjacent indices are adjacent. We define an addition square Q_0 to have center 0 and sidelength the same as Q_1. It follows that $Q_{i-1} \subset 9Q_i$ for all $1 \le i \le n$—in fact, we have already noted this property for adjacent Whitney squares, and it is obvious in the remaining case $i = 1$. Let us also write $z_i = z(Q_i)$.

Using Lemma 20 as before, and then Hölder's inequality for sums, we deduce that

$$\begin{aligned}|f(z(Q))| = |f(z_n) - f(z_0)| &\le \sum_{i=1}^n |f(z_i) - f(z_{i-1})| \\ &\le \sum_{i=1}^n \left(\int_{18Q_i} |f'|^p \, d\mu_{p-2}\right)^{1/p} \\ &\le \left(\sum_{i=1}^n \int_{18Q_i} |f'|^p \, d\mu_{p-2}\right)^{1/p} \left(\sum_{i=1}^n 1\right)^{1-1/p}.\end{aligned}$$

Using the bounded overlap of $18Q_i$, we see that the first factor in the last line is bounded. By Lemma 22 and the fact that $\lambda(0,z) \approx \log(100/(1-|z|))$ for all $|z| \ge 1/2$, we see that $|f(z(Q))| \le C_4 \log^{1-1/p}(1/r(Q))$ for some universal constant C_4.

It follows that if we fix $\varepsilon \in (0, \alpha+1)$, and let $C = C_4/\varepsilon^{1-1/p}$, then

$$\begin{aligned}\mathcal{G} &\le \sum_{Q\in\mathcal{W}} \mu_\alpha(Q) \sum_{j=j_0}^{\infty} \frac{(C_4/C)^{jp/(p-1)} \log^j(1/r(Q))}{j!} \\ &\le \sum_{Q\in\mathcal{W}} \mu_\alpha(Q) r(Q)^{-\varepsilon} \approx \sum_{Q\in\mathcal{W}} \mu_{\alpha-\varepsilon}(Q) \approx 1. \qquad \square\end{aligned}$$

Finally in this section, let us mention related results and generalizations of Theorem 23. First, related results that hold for arbitrary Sobolev functions, and more general weights and Euclidean domains, are investigated in [BO]. In particular Theorem 1.2 in that paper implies our Theorem 23 in the case $p \geq 2$. There are, however, easy counterexamples among Sobolev functions when $p < 2$. Nevertheless, one could use the methods of [BO] to get a generalization of Theorem 23 (for analytic functions) that handles a much larger class of weights and domains.

Secondly, one could relax the assumption that the functions are analytic, replacing it by an assumption that they are harmonic (in any of a large class of Euclidean domains), or that they are solutions of some (rather general) elliptic equation. Basically, these two claims follow from the weak manner in which analyticity was used: it was needed only to get the local estimate in Lemma 20, and similar estimates can be derived for solutions to elliptic equations (although the proof is then quite different!). This idea is pursued further in the case of Poincaré inequalities in [BK].

6. From Besov spaces to weighted Dirichlet spaces

In Section 4, we characterized $E(2)$, the entire functions of order less than 2, or of order two and finite type, as the functions which take the Dirichlet space into the intersection of all larger D^q spaces. We now consider the second question mentioned in the introduction: characterize the entire functions of other possible orders in a similar way. Theorem 3 of [CG] characterizes the entire functions of order at most p as symbols of the superposition operators acting from $\bigcup_{p<q} A^q$ to the Bergman-Nevanlinna (area) class BN of functions f in the disk for which $\log^+ |f|$ is area-integrable. However, this answer is not completely satisfactory, as this union of Bergman spaces does not have a satisfactory natural norm topology. Theorem 4 of the same paper says membership in $E(p)$ is a sufficient condition for being a superposition operator from A^p into BN. By considering a different scale of spaces, we get a "necessary and sufficient result" of this type between Banach function spaces. More precisely, we shall see that each $E(t)$ is the space of symbols of superposition operators from B^p into D^q_β, for suitable indices p, q, and β.

The space of polynomials is dense in B^p for every $1 < p < \infty$; this follows, for instance, from the representation of Besov space as a mixed-norm space given by the special case $q = p$, $t = 1 - 1/p$ of Lemma 1.2 in [Bu]. As in Section 4 for $\mathcal{D}$, we deduce that for all $p > 1$, $\alpha > -1$, $C > 0$, and $f \in B^p$, we have

$$\forall\, p > 1, \alpha > -1, C > 0, f \in B^p: \qquad \int_{\mathbf{D}} \exp\left(c|f|^{p/(p-1)}\right)\, d\mu_\alpha < \infty\,. \tag{2}$$

Note that here the integration is performed with respect to a weighted area measure, unlike in the classical Trudinger-Moser inequality. This by no means affects the similarity of the argument, since in Theorem 19 we had to convert the integral over the circle into the integral over the disk by embedding the Hardy space into the Bergman space.

We now prove the following generalization of Theorem 19, which shows that we can distinguish between the classes $E(t)$ for each $t > 1$ via their action as superposition operators.

Theorem 24. *Let* $1 < p < \infty$, $0 < q \le p$, *and* $\beta > q\,(p-1)/p - 1$. *Then* $S_\varphi(B^p) \subset D^q_\beta$ *if and only if* $\varphi \in E(p/(p-1))$.

Proof. To prove sufficiency, we assume that $\varphi \in E(p/(p-1))$. As before, there exists some η such that for all sufficiently large $|w|$, we have $|\varphi'(w)| \le \exp(\eta\,|w|^{p/(p-1)})$. Considering first the case $q < p$, we apply Hölder's inequality and (2) to deduce that

$$\begin{aligned}\int_{\mathbf{D}} |\varphi' \circ f|^q |f'|^q (1-|z|)^\beta \, dA &= \int_{\mathbf{D}} |\varphi' \circ f|^q (1-|z|)^{\frac{\beta - q(p-2)}{p}} |f'|^q (1-|z|)^{\frac{q(p-2)}{p}} \, dA \\ &\le \left(\int_{\mathbf{D}} |f'|^p (1-|z|)^{p-2} \, dA \right)^{q/p} \times \\ &\times \left(\int_{\mathbf{D}} |\varphi' \circ f|^{\frac{pq}{p-q}} (1-|z|)^{(\beta - \frac{q(p-2)}{p})\cdot \frac{p}{p-q}} \, dA \right)^{(p-q)/p} \\ &\le \|f\|^q_{B^p} \int_{\mathbf{D}} \exp\left(\eta p q |f|^{p/(p-1)}/(p-q) \right) (1-|z|)^\alpha \, dA \\ &< \infty\,,\end{aligned}$$

where $\alpha = (\beta p - pq + 2q)/(p-q) > -1$ because of our assumptions on β and q.

The remaining case is $q = p$, $\beta > p-2$. Defining $\gamma(s) = [(2+\beta)s/p] - 2$, we have $D^s_{\gamma(s)} \subset D^p_\beta$ for all $s < p$, a fact which follows from the corresponding result for weighted Bergman spaces as given in [BKV, Theorem 1.3]. By the case $q < p$, we have $S_\varphi(B^p) \subset D^s_{\gamma(s)}$, whenever $p/(3+\beta-p) < s < p$. Since $p/(3+\beta-p) < p$, we may choose such a number s, and deduce that $S_\varphi(B^p) \subset D^s_{\gamma(s)} \subset D^p_\beta$.

The proof of necessity requires only minor changes to the proof of necessity in Theorem 19. First, we choose points w_n so that $\varphi'(w_n) > n \exp\left((n\,|w_n|)^{p/(p-1)}\right)$. Then we choose

$$r_n = \frac{1}{n^{1/p} \log^{2/(p-1)}(n+1)}, \qquad h_n = \frac{r_{n+1}}{3\left(n^{1/p}|w_{n+1} - w_n|\right)^{1/(p-1)}},$$

and get the estimate $\lambda_\Omega(0, w_n) \le C_{p,t} n^t |w_n|^{p/(p-1)}$, for every $t > 1/(p-1)$. We omit the rest of the details, which are practically identical. □

The bounds on q and β in this theorem are sharp. First let us show that the inequality for β in the above theorem cannot be replaced by an equality. For $q < p$, we see this by taking $\varphi \equiv 1$ and using the fact that $B^p \subset D^q_\beta$ only when $\beta > q(p-1)/p - 1$; this last fact in turn follows from a non-containment result for weighted Bergman spaces (for which see the comments on sharpness that precede Theorem 1.3 in [BKV]). For $p = q$, we instead take $\varphi(z) = z^2$ and use the fact that $B^p \not\subset D^{2p}_{p-2}$; this fact again follows from the same comments on sharpness in [BKV].

As for the sharpness of q, we take $\varphi \equiv 1$ and suppose $q > p$. The same comments in [BKV] also imply that $B^p \subset D^q_\beta$ only if $\beta \ge q-2$. Since $q - 2 > q(p-1)/p - 1$, we have a counterexample if we take β close to the critical index $q(p-1)/p - 1$.

The following corollary for operators to unweighted Dirichlet spaces follows immediately from Theorem 24.

Corollary 25. *Let* $1 < p < \infty$, $0 < q < \min\{p, p/(p-1)\}$. *Then* $S_\varphi(B^p) \subset D^q$ *if and only if* $\varphi \in E(p/(p-1))$.

Theorem 24 tells us that if $t \in (1,\infty)$, $0 < q \le t/(t-1)$, and $\beta > q/t-1$, then $E(t)$ is the class of superposition operators from $B^{t/(t-1)}$ to D^q_β. Since $E'(t) \equiv \bigcap_{s>t} E(s)$ is the space of all functions of order at most t, we deduce the following corollary.

Corollary 26. *Let* $1 < t < \infty$, $0 < q \le t/(t-1)$, *and* $\beta \ge q/t - 1$. *Then* $S_\varphi\left(\bigcup_{p<t/(t-1)} B^p\right) \subset D^q_\beta$ *if and only if* $\varphi \in E'(t)$. *In particular, this applies to the case* $\beta = 0$, $q = \min\{t, t/(t-1)\}$.

Note that, whereas $E(t)$ is the space of functions φ satisfying a bound of the form $\varphi(w) \le C\exp(C'|w|^t)$, a definition of $E'(t)$ requires a sequence of such bounds. Thus it seems unreasonable to hope that $E'(t)$ is the class of all superposition operators from one nice function space to another, and the above corollary is probably as good as one can expect to get. In a related vein, Cámera and Giménez [CG, Theorems 3 and 7] prove that $E(t)$ is the space of symbols of superposition operators from both $\bigcup_{p>t} A^p$ and $\bigcup_{p>t/2} H^p$ into BN, the Nevanlinna area class of analytic functions f in the disk for which $\log^+|f|$ is integrable with respect to dA.

Finally, we note that the proof of Theorem 24 reduces the task of showing that the Young function in Theorem 23 is sharp to the construction of entire functions with certain rates of growth. Let us just address the sharpness of the parameter p. To prove this, it suffices to know that there are entire functions φ_r of each possible order $t > 0$. Indeed for each $t > 0$, the function

$$\varphi_t(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{1/t}}$$

has order t; see [Ti, §8.4, p. 255]. Since $\varphi_r \in E(t) \setminus E(s)$, $0 < s < r < t$, it follows that $E(s) \subsetneq E(t)$ whenever $0 < s < t$. As a consequence, Theorem 23 becomes false if we change ψ_p to ψ_s for any $1 < s < p$, since otherwise one could modify the proof of Theorem 24 to deduce that $S_\varphi(B^p) \subset D^q_\beta$ for all $\varphi \in E(s/(s-1)) \supsetneq E(p/(p-1))$, thus contradicting the statement of Theorem 24.

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