WEAK SLICE CONDITIONS, PRODUCT DOMAINS, AND QUASICONFORMAL MAPPINGS

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Abstract. We investigate geometric conditions related to Hölder imbeddings, and show, among other things, that the only bounded Euclidean domains of the form $U \times V$ that are quasiconformally equivalent to inner uniform domains are inner uniform domains.

0. Introduction

Two Euclidean domains are *K-quasiconformally equivalent* if there is a K-quasiconformal mapping from one onto the other. Determining what domains are quasiconformally equivalent to a ball or other nice Euclidean domain is an important and open problem when $n \geq 3$. Some partial results are known, notably those of Gehring and Väisälä [GV], [V4]; see also [R]. In [V4], Väisalä classifies cylinders in $\mathbb{R}^3$ that are quasiconformally equivalent to a ball.

Inner uniform domains, as defined by Väisälä [V5], satisfy a uniformity condition with respect to the inner Euclidean metric. These domains form a class intermediate between uniform and John domains and, in particular, they include all Lipschitz domains; see Section 1 for definitions. We prove the following theorem which indicates that this class is well suited to the study of quasiconformal equivalence.

Theorem 0.1. *Suppose that $\Omega = U \times V \subset \mathbb{R}^n \times \mathbb{R}^m$ is a bounded domain, $n, m \in \mathbb{N}$. The following are equivalent:*

(i) *Ω is quasiconformally equivalent to an inner uniform domain;*
(ii) *Ω is an inner uniform domain;*
(iii) *Both U and V are inner uniform domains.*

In particular, since balls are inner uniform, a bounded product domain $\Omega = U \times V$ must be inner uniform if it is quasiconformally equivalent to a ball (this criterion alone, however, is not sufficient as we explain in Remark 4.11).

The following two theorems show that among product domains, inner uniformity is closely connected with the concept of *broadness*, as introduced by Väisälä [V4]; the inner 0-wSlice$^+$ condition, defined in Section 2, is a technical assumption satisfied in particular by inner uniform domains and their quasiconformal images.

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Theorem 0.2. *If $\Omega = U \times V \subset \mathbb{R}^n \times \mathbb{R}^m$, $n, m \in \mathbb{N}$, is a bounded inner 0-wSlice$^+$ domain, then Ω is broad if and only if it is inner uniform.*

Theorem 0.3. *Suppose that $\Omega = U \times V \subset \mathbb{R}^n \times \mathbb{R}^m$ is bounded, and quasiconformally equivalent to a broad inner 0-wSlice$^+$ domain G. Then Ω is inner uniform.*

Obviously, one can remove every instance of the word "inner" from the above theorems if Ω is assumed to be quasiconvex (i.e., the Euclidean and inner Euclidean metrics are comparable). However it is easy to construct non-quasiconvex counterexamples to the non-inner versions of these theorems. In the case of Theorem 0.2, though, the counterexamples are for one implication only since inner uniform domains are always broad [BHK, Example 6.5(b)].

The rest of the paper is organized as follows. After some preliminaries, we introduce the slice conditions in Section 2. In Section 3, we show that a large class of domains satisfy the various weak slice conditions. In Section 4, we classify bounded product domains satisfying weak slice conditions and prove the above theorems. We examine some further results in Section 5 and, finally, we discuss some open problems in Section 6.

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1. Preliminaries

1.1. Notation.

We adopt two common conventions. First, we drop parameters if we do not wish to specify their values; for instance, we define C-uniform domains, but often talk about uniform domains. Second, we write $C = C(x, y, \dots)$ to mean that a constant C depends only on the parameters $x, y, \dots$.

If $S \subset \mathbb{R}^n$ is measurable, then $|S|$ is the Lebesgue measure of S, and u_S is the average value of a function u on S. We write $A \lesssim B$ if $A \le CB$ for some constant C dependent only on allowed parameters; we write $A \approx B$ if $A \lesssim B \lesssim A$. We write $A \wedge B$ and $A \vee B$ for the minimum and maximum, respectively, of the quantities A and B. Unless otherwise stated, Ω and G are proper subdomains of $\mathbb{R}^n$.

Let $x, y \in U \subsetneq \mathbb{R}^n$. We denote by $\delta_U(x)$ the distance from x to ∂U, and by $\Gamma_U(x, y)$ the class of rectifiable paths $\lambda : [0, t] \to U$ for which $\lambda(0) = x$, $\lambda(t) = y$. If $\alpha \in \mathbb{R}$, γ is a rectifiable path in U, and ds is arclength measure, we define

$$\operatorname{len}_{\alpha,U}(\gamma) = \int_\gamma \delta_U^{\alpha-1}(z)\, ds(z),$$
$$d_{\alpha,U}(x, y) = \inf_{\gamma \in \Gamma_U(x,y)} \operatorname{len}_{\alpha,U}(\gamma),$$

Of course, $d_{\alpha,U}(x, y) = \infty$ if x, y lie in different path components of U. We are mainly interested in $d_{\alpha,U}$ when $\alpha \in [0, 1]$ and U is a domain; $d_{\alpha,U}$ is then a metric. Note that $d_{\alpha,U}$-geodesics may fail to exist if $\alpha > 0$ [BS, Proposition 1.2], but they do exist when U is a domain and $\alpha = 0$ [GO].

We write len in place of $\operatorname{len}_{1,U}$, the Euclidean length of a path. Note that $\operatorname{len}_{0,U}$ and $d_{0,U}$ are the well-known *quasihyperbolic length and distance*, and $d_{1,U}$ is the *inner Euclidean metric*.

For brevity, we abuse notation by writing, for instance, $\mathrm{len}_{\alpha,U}(\gamma \cap S)$ for the $d_{\alpha,U}$-length of those parts of a path γ lying in a subset S of U. We write $[x,y]$ for the line segment joining a pair of points in $\mathbb{R}^n$, and $[x \to y]$ for the path parametrized by arclength that goes from x to y along $[x,y]$.

Given $x \in U$, $E, F \subset U$, and a metric ρ on U, we write $d_\rho(E,F)$ for the ρ-distance between E and F, $\mathrm{dia}_\rho(E)$ for the ρ-diameter of E, and $B_\rho(x,r) = \{y \in U : d_\rho(x,y) < r\}$. If $\rho = d_{1,U}$, we instead write $d_U(E,F)$, $\mathrm{dia}_U(E)$, and $B_U(x,r)$ for these concepts, while if ρ is the Euclidean metric (and so $U = \mathbb{R}^n$), we write $d(E,F)$, $\mathrm{dia}(E)$, and $B(x,r)$. We write $d_U = d_{1,U}$; in particular, $d_{\mathbb{R}^n}$ is the Euclidean metric. Note that distance to the boundary of U is the same with respect to $d_{\mathbb{R}^n}$ and d_U, and that $B_U(x,r) = B(x,r)$ if $r \le \delta_U(x)$. We define the inradius of U, $r(U) = \sup_{x\in U} \delta_U(x)$.

1.2. Uniform domains and mean cigar domains.

Let $C \ge 1$ and let d be the Euclidean metric. A domain G is a *C-uniform domain* if for every $x, y \in G$, there is a *C-uniform path*, i.e., a path $\gamma \in \Gamma_G(x,y)$ of length l and parametrized by arclength for which $l \le Cd(x,y)$, and $t \wedge (l-t) \le C\delta_G(\gamma(t))$. An *inner C-uniform domain* is defined similarly except that $d = d_G$. Uniform domains include all bounded Lipschitz domains, as well as some domains with fractal boundary, such as the interior of a von Koch snowflake. All uniform domains are inner uniform, and a slit disk is a standard example of an inner uniform domain that is not uniform. For more on inner uniform domains, see [V5].

Suppose that $0 \le \alpha \le 1 \le C$ and let $d : G \times G \to [0,\infty)$. We say that G is an *$(\alpha, C; d)$-mCigar domain* if for every pair $x, y \in G$, there is a *$(\alpha, C; d)$-mCigar path*, i.e., a path $\gamma \in \Gamma_G(x,y)$ such that

$$\begin{aligned} &\mathrm{len}_{\alpha,G}(\gamma) \le Cd(x,y)^\alpha \qquad 0 < \alpha \le 1, \\ &\mathrm{len}_{0,G}(\gamma) \le C \log[1 + d(x,y)/(\delta_G(x) \wedge \delta_G(y))], \qquad \alpha = 0. \end{aligned}$$

In particular, if d is the Euclidean metric, we simply say that G is an *(α, C)-mCigar domain*, while if $d = d_G$, we say that G is an *inner (α, C)-mCigar domain*. α-mCigar conditions for $0 < \alpha < 1$ imply the existence of a path λ that satisfies a type of cigar condition on average; see [BK2, Lemma 2.2] and Lemma 4.6 below. In practice we shall not use this terminology for $\alpha = 1$: we prefer to use the more common term *C-quasiconvex domain* rather than $(1, C)$-mCigar domain.

Uniform domains are α-mCigar domains for all α. Gehring and Osgood [GO] showed that the classes of 0-mCigar domains and uniform domains coincide, and Väisälä [V4; 2.33] showed that the classes of inner 0-mCigar and inner uniform domains coincide. The class of (inner) α'-mCigar domains includes the class of (inner) α-mCigar domains if and only if $\alpha \le \alpha'$. The Euclidean version is dealt with in [L] and [BK2]; inclusion follows similarly in the inner case and the counterexamples in [L] also handle the inner version. Thus mCigar domains include domains with rough (even fractal) boundary. Note that the class of inner uniform and inner mCigar domains contain their Euclidean analogues (strictly, since a planar slit disk is in all of the inner classes but none of the Euclidean classes).

We refer the reader to [BK2], [GM], and [L] for more information about α-mCigar domains; these domains are called "weak cigar domains" in [BK2] and "Lip_α extension domains" in [GM] and [L] when $\alpha > 0$. The last name derives from the fact that for $\alpha > 0$, G is α-mCigar if and only if all functions defined on G which are locally Lipschitz of order α are globally Lipschitz of order α; see [GM].

2. Slice domains

The conditions defined in Section 1 rather strongly restrict the geometry. For instance, among planar domains, inner uniform domains cannot have external cusps, while uniform and mCigar domains can have neither internal nor external cusps. By contrast, the slice conditions that we define in this section are all quite weak, at least in two dimensions: they are satisfied by any domain quasiconformally equivalent to a uniform domain and hence by all simply-connected planar domains.

We first discuss weak slice conditions, as first defined in [BS]. The adjective "weak" refers to the fact that for all α, an α-wSlice condition is implied by the analogous "strong" slice condition which we define later; see [BS, Lemma 2.8].

Suppose $0 \le \alpha < 1 \le C$ and let d be a metric on G satisfying $d_{\mathbb{R}^n} \le d \le d_G$. Then G is an $(\alpha, C; d)$-*wSlice domain* if every pair $x, y \in G$ satisfies the following $(\alpha, C; d)$-*wSlice condition*: there exist a path $\gamma \in \Gamma_G(x, y)$, pairwise disjoint open subsets $\{S_i\}_{i=1}^m$ of G, $m \ge 0$, and numbers $d_i \in [\operatorname{dia}_d(S_i), \infty)$ such that for all $1 \le i \le m$:

$$\operatorname{len}(\lambda \cap S_i) \ge d_i/C, \qquad \text{for all } \lambda \in \Gamma_G(x, y); \tag{WS-1}$$

$$\operatorname{len}_{\alpha,G}(\gamma) \le C(\delta_G^\alpha(x) + \delta_G^\alpha(y) + \sum_{i=1}^m d_i^\alpha); \tag{WS-2}$$

$$(B(x, \delta_G(x)/C) \cup B(y, \delta_G(y)/C)) \cap S_i = \emptyset. \tag{WS-3}$$

If d is the Euclidean metric, we say that G is an (α, C)-*wSlice domain*, while if $d = d_G$, we say that G is an *inner* (α, C)-*wSlice domain*; these are the two metrics that mainly interest us. Note that the metric d enters the definition only in limiting the size of the numbers $\{d_i\}$, and that for $\alpha = 0$, (WS-2) simply says that $\operatorname{len}_{0,G}(\gamma) \le C(2 + m)$.

Roughly speaking, a wSlice condition for a pair of points x, y limits the amount of floating boundary and slab-shaped regions in the domain that lie "between" x and y; by a "slab-shaped" region, we mean a piece of the domain which is much larger in two coordinate directions than a third such as $(0, 1) \times (0, 1) \times (0, \epsilon)$ for some small $\epsilon > 0$. The "tolerance level" of an α-wSlice domain for floating boundary components and slab-shaped regions is lower for smaller α. In particular it follows from Theorem 4.1 that the product of an externally cusped domain and an interval is never an α-wSlice domain for any $\alpha \in [0, 1)$. The reader should feel more comfortable with the geometry of this condition after working through the examples in Section 6, and reading the statements of results in Section 4.

As discussed in [BS, 2.1], we can essentially take $d_i = \operatorname{dia}_d(S_i)$ in the definition, but allowing inequality is sometimes convenient. A significant difference between the $\alpha = 0$ and $\alpha > 0$ cases is that, whereas (WS-3) is an essential part of the definition for $\alpha = 0$ (lest every domain be a $(0; d)$-wSlice domain), it can be dropped when $\alpha > 0$ (as shown in Theorem 5.1). Modulo a change in the value of C by a factor at most 4, it is shown in [BS, 2.1] that we may add the following condition to the definition of an $(\alpha, C; d)$-wSlice condition for x, y (and all $1 \le i \le m$):

$$\operatorname{len}_{\alpha,G}(\gamma \cap S_i) \le C d_i^\alpha. \tag{WS-4}$$

Given a path λ intersecting a slice S_i, let λ^i denote the component of $\lambda \cap S_i$ with maximal d-diameter. We define an $(\alpha, C; d)$-wSlice$^+$ domain to be an $(\alpha, C; d)$-wSlice domain in which the slice data satisfy the following extra pair of conditions for all $1 \le i \le m$:

$$\mathrm{dia}_d(\lambda^i) \ge d_i/C, \qquad \text{for all } \lambda \in \Gamma_G(x, y); \tag{WS-1$^+$}$$

$$\exists\, z_i \in S_i : \quad B_i \equiv B(z_i, d_i/C) \subset S_i. \tag{WS-5}$$

(α, C)-*wSlice*$^+$ and *inner* (α, C)-*wSlice*$^+$ *domains* are defined in the obvious way. (For (WS-5), d_i comes from the inner metric but the ball is Euclidean.) We need the extra conditions (WS-1$^+$) and (WS-5) for some of our proofs but intuitively one should think of wSlice$^+$ domains as being very similar to wSlice domains. In fact, we believe it likely that the classes of $(\alpha; d)$-wSlice and $(\alpha; d)$-wSlice$^+$ domains coincide; see the discussion before Open Problem A in Section 6.

We recall Lemma 2.3 of [BS].

Lemma 2.1. *If the data γ, $\{S_i, d_i\}_{i=1}^m$ satisfy (WS-1) and (WS-3) for the pair $x, y \in G$, and $d_i > 0$, then* $\mathrm{dia}(S_i) \ge 2\delta_G(z)/(C+1)$*, for all $z \in S_i$ and $1 \le i \le m$. Furthermore, if $d_i \ge \mathrm{dia}(S_i)$ and $|x - y| \ge (\delta_G(x) + \delta_G(y))/2$, then there exists a constant $C' = C'(C, \alpha)$ such that*

$$\delta_G^\alpha(x) + \delta_G^\alpha(y) + \sum_{k=1}^m d_i^\alpha \le C' \,\mathrm{len}_{\alpha,G}(\lambda), \qquad \lambda \in \Gamma_G(x, y). \tag{2.2}$$

We next define "strong" slice conditions. Suppose $C \ge 1$ and let d be a metric on G satisfying $d_{\mathbb{R}^n} \le d \le d_G$. Then G is a $(C; d)$-*Slice domain* if every pair $x, y \in G$ satisfies the following $(C; d)$-*Slice condition*: there exist a path $\gamma \in \Gamma_G(x, y)$ and pairwise disjoint open subsets $\{S_i\}_{i=0}^j$ of G, with $d_i \equiv \mathrm{dia}_d(S_i) < \infty$, such that:

(i) $x \in S_0$, $y \in S_j$, and x and y are in different components of $G \setminus \overline{S_i}$, for all $0 < i < j$.
(ii) $\mathrm{len}(\lambda \cap S_i) \ge d_i/C$, for all $0 < i < j$ and $\lambda \in \Gamma_G(x, y)$.
(iii) For all $t \in [0, 1]$, we have $B\left(\gamma(t), C^{-1}\delta_G(\gamma(t))\right) \subset \bigcup_{i=0}^j \overline{S_i}$. Also, for all $0 \le i \le j$, there exists $x_i \in \gamma_i$, such that $x_0 = x$, $x_j = y$, and $B\left(x_i, C^{-1}\delta_G(x_i)\right) \subset S_i$.
(iv) For all $0 \le i \le j$ and $z \in \gamma_i \equiv \gamma([0, 1]) \cap S_i$, we have $d_i \le C\delta_G(z)$.

If d is the Euclidean metric, we say that G is a C-*Slice domain*, while if $d = d_G$, we say that G is an *inner* C-*Slice domain*. The (Euclidean) Slice condition was defined in [BK2, Definition 3.1] (for a fixed y but uniformly in x).

The d-Slice condition for a pair of points implies an $(\alpha; d)$-wSlice condition[1] for the same pair of points, quantitatively; see [BS, Lemma 2.8]. However, if $\alpha > 0$, then there are α-wSlice domains which are not Slice domains; see [BS, Proposition 4.5]. The d-Slice condition is quite similar to the $(0, d)$-wSlice but even less tolerant of "slab-shaped" regions, as discussed after Open Problem C in Section 6. Although a 0-wSlice *condition* does not necessarily quantitatively imply an Slice condition, we suspect that the classes of Slice and 0-wSlice *domains* coincide.

[1] We suspect but cannot prove that a d-Slice condition implies an (α, d)-wSlice$^+$ condition; it certainly implies (WS-5) because of (iii) and the fact that the slices are left unchanged in the proof that a Slice condition implies an α-wSlice condition.

3. Inner uniform and inner slice domains

In this section, we show that inner uniform domains and their quasiconformal images satisfy certain inner slice conditions.

Theorem 3.1. *Let $\alpha \in [0,1)$ and let f be a K-quasiconformal mapping from an inner C-uniform domain $G \subset \mathbb{R}^n$ onto Ω. Then Ω is an inner C'-Slice domain and an inner (α, C')-wSlice$^+$ domain for some $C' = C'(C, n, K, \alpha)$.*

Suppose that E, F are disjoint subsets of a domain $G \subset \mathbb{R}^n$. The *conformal modulus*, $\operatorname{mod}(E, F; G)$, of the pair E, F relative to G is defined to be the infimum of $\int_G \rho^n$, as $\rho : G \to [0, \infty]$ ranges over the class of Borel functions for which every line integral over a path $\gamma : [0,1] \to G$ joining E and F is at least 1. We refer the reader to [V2] for the fundamentals of conformal modulus and quasiconformal mappings.

We say that a domain $G \subset \mathbb{R}^n$ is *ϕ-broad*[2] if

$$\phi(t) \equiv \inf\{\operatorname{mod}(E, F; G) \,:\, \Delta_G(E, F) \le t\} > 0, \qquad t > 0,$$

where E, F designate non-degenerate disjoint continua in G and

$$\Delta_G(E, F) \equiv \frac{d_G(E, F)}{\operatorname{dia}_G(E) \wedge \operatorname{dia}_G(F)}$$

denotes the *relative inner distance* between E and F.

Before proving Theorem 3.1, we need some lemmas. The first is a special case of results of Bonk, Heinonen and Koskela (see Example 6.5(b) in [BHK]); in the terminology of that paper, G is broad if it is Loewner with respect to d_G.

Lemma 3.2. *An inner C-uniform domain $G \subset \mathbb{R}^n$ is ϕ-broad, with ϕ dependent only on C and n.*

Lemma 3.3. *Suppose $G \subset \mathbb{R}^n$ is a domain and $E, F \subset G$ are disjoint compact subsets in G and $\Delta_G(E, F) \ge 2$. Then there exists a constant $C = C(n)$ such that*

$$\operatorname{mod}(E, F; G) \le C(\log \Delta_G(E, F))^{-n+1}.$$

Proof. Without loss of generality, we assume that $\operatorname{dia}_G(E) \le \operatorname{dia}_G(F)$. Let us fix a point $x \in E$ and write $r = \operatorname{dia}_G(E)$, $R = d_G(E, F)$, so that $\Delta_G(E, F) = R/r \ge 2$. Let $N = \lfloor \log_2 R/r \rfloor$, let $A_i = B_G(x, 2^i r) \setminus B_G(x, 2^{i-1} r)$ for each $1 \le i \le n$ and define the function $\rho : G \to [0, \infty)$ by the equation

$$\rho(x) = \begin{cases} 1/(2^{i-1} N r), & x \in A_i, 1 \le i \le N \\ 0, & \text{otherwise.} \end{cases}$$

Clearly ρ is an allowable modulus test function and, since $|A_i|$ is dominated by the measure of a Euclidean ball of radius $2^i r$, it follows that $\operatorname{mod}(E, F; G) \lesssim N^{-n+1}$. The lemma now follows readily. □

Our next lemma implies that an inner α-mCigar domain is an inner α-wSlice$^+$ domain and, if $\alpha = 0$, it is also an inner Slice domain.

[2]This concept was introduced by Väisälä [V4]. Our definition looks a little different but is equivalent to Väisälä's in the Euclidean setting according to [HK, Theorem 3.6].

Lemma 3.4. *Suppose that $0 \le \alpha < 1$ and that $G \subsetneq \mathbb{R}^n$. If there is an inner (α, C_1)-mCigar path for the points $x, y \in G$, then the pair x, y satisfies an inner (α, C_2)-wSlice$^+$ condition for some $C_2 = C_2(C_1, \alpha, n)$. If $\alpha = 0$, x, y also satisfies an inner C_3-Slice condition for some $C_3 = C_3(C_1, n)$.*

Proof. Without loss of generality, $\delta_G(y) \le \delta_G(x)$. We write $B_w = B(w, \delta_G(w)/2)$ for $w \in G$. Suppose that $z \in B_x$. If $\alpha > 0$, then

$$d_{\alpha,G}(x,z) \le \mathrm{len}_{\alpha,G}([x \to z]) < \int_{\delta_G(x)/2}^{\delta_G(x)} t^{\alpha-1}\,dt = \left(\frac{\delta_G(x)}{2}\right)^{\alpha}\left(\frac{2^\alpha - 1}{\alpha}\right) < \delta_G^\alpha(x).$$

By a separate calculation, we see that $d_{\alpha,G}(x,z) < \delta_G^\alpha(x)$ even for $\alpha = 0$. Thus if B_x and B_y overlap, then $d_{\alpha,G}(x,y) < \delta_G^\alpha(x) + \delta_G^\alpha(y)$, so x, y satisfy an inner $(\alpha, 1)$-wSlice$^+$ condition with zero slices. We may therefore suppose that B_x and B_y are disjoint and so $d_G(x,y) \ge (\delta_G(x) + \delta_G(y))/2$.

Define annuli $S_i \equiv B_G(y, 2^{i-2}\delta_G(y)) \setminus \overline{B_G(y, 2^{i-3}\delta_G(y))}$ for every $i \in \mathbb{N}$. Let $m \ge 2$ be the smallest integer for which S_{i+1} intersects $B(x, \delta_G(x)/2)$, and let $d_i = 2^{i-1}\delta_G(y)$. Consider the slice data γ, $\{S_i, d_i\}_{i=1}^m$, where γ is any inner (α, C_1)-mCigar path for x, y. First (WS-3) is automatically true, and (WS-1$^+$) is true because the d_G-diameter of each annulus is comparable to its thickness.

Suppose $\alpha > 0$. Since γ is an inner (α, C_1)-mCigar path, we have

$$\mathrm{len}_{\alpha,G}(\gamma) \le C_1 d_G(x,y)^\alpha < C_1(d_m + \delta_G(x))^\alpha,$$

which implies (WS-2). If instead $\alpha = 0$, note that $m \ge \log_2(8 d_G(x,y)/\delta_G(y))$. Since $d_G(x,y) > \delta_G(y)/2$, the inner 0-mCigar property of γ then implies (WS-2).

We have now proved that all conditions other than (WS-5) hold with some preliminary constant value $C = C_4$. To prove (WS-5) we shall discard some of the slices, leaving enough of them that (WS-2) remains true with $C = 2C_4$. For $1 \le i \le m$, let $f_i : G \to \mathbb{R}$ be defined by $f_i(z) = d_G(y,z)/2^{i-3}\delta_G(y)$. Thus $S_i = f_i^{-1}((1,2))$, and we also define the thinner annuli $S_i' = f_i^{-1}([4/3, 5/3]) \subset S_i$ and their "inner and outer boundaries", $I_i = f_i^{-1}(4/3)$, $O_i = f_i^{-1}(5/3)$. Since I_i and O_i are separated by an inner Euclidean distance $d_i/12$, and γ must pass from one to the other on its way through S_i', we see that

$$\mathrm{len}_{\alpha,G}(\gamma \cap S_i') \ge M_i^{\alpha-1} d_i/12, \tag{3.5}$$

where M_i is the maximum value of δ_G on S_i'. Let $z = z_i \in S_i'$ be any point for which $\delta_G(z) = M_i$; this will be the point z_i in (WS-5) for appropriate i.

We partition the set of integers $i \in [1, m]$ into two sets: the set of *good indices* $\mathcal{G}$ for which $d_i/M_i \le K$, and the set of *bad indices* $\mathcal{B}$ for which $d_i/M_i > K$, where the cut-off value K equals $(2^\alpha \cdot 24 C_1)^{1/(1-\alpha)}$. Since (WS-5) readily follows for any value of i for which $d_i/M_i \lesssim 1$, it suffices to find a value K, dependent on allowable parameters, such that (WS-2) remains true with $C = 2C_4$ if we sum up only over good indices on the right-hand side.

Consider first the case $\alpha > 0$. We may as well assume that $\delta_G^\alpha(x) \le \sum_{i=1}^m d_i^\alpha$ since otherwise x, y satisfy an inner $(\alpha, 2C_4)$-wSlice$^+$ condition (with $m = 0$). By simple geometry, we see that $d_G(x,y) \le d_m + \delta_G(x)/2 \le 2d_m$, and so

$$\begin{aligned} 2^\alpha \cdot 24C_1 \sum_{i\in\mathcal{B}} d_i^\alpha = K^{1-\alpha} \sum_{i\in\mathcal{B}} d_i^\alpha &< \sum_{i\in\mathcal{B}} M_i^{\alpha-1} d_i \\ &\le 12 \operatorname{len}_{\alpha,G}(\gamma) \\ &\le 12C_1 d_G(x,y)^\alpha \\ &\le 12C_1 \left(2\sum_{i=1}^m d_i\right)^\alpha \\ &\le 2^\alpha \cdot 12C_1 \sum_{i=1}^m d_i^\alpha, \end{aligned}$$

where the second inequality follows from (3.5), and the third from the α-mCigar condition. It follows that $\sum_{i\in\mathcal{B}} d_i^\alpha \le \sum_{i\in\mathcal{G}} d_i^\alpha$, and so (WS-2) holds $C = 2C_4$ for the set of good indices $\mathcal{G}$ alone.

As for the case $\alpha = 0$, we have $d_G(x,y) \le 2d_m \le 2^m \delta_G(y)$, and so since $m \ge 2$,

$$(12C_1)^{-1} \sum_{i=1}^m M_i^{-1} d_i \le C_1^{-1} \operatorname{len}_{0,G}(\gamma) \le \log\left(1 + \frac{d_G(x,y)}{\delta_G(y)}\right) \le \log(1+2^m) \le m.$$

It follows that (WS-2) holds with $C = 2C_4$ for the set of good indices alone.

We omit the proof of the last statement of the lemma, as it merely involves making straightforward adjustments to the proof for the Euclidean case, which is Lemma 3.3(a) of [BK2]. □

Theorem 3.6. *Suppose f is a K-quasiconformal mapping from a ϕ-broad inner $(0,C)$-wSlice$^+$ (or inner C-Slice) domain $G \subset \mathbb{R}^n$ onto Ω. Then Ω is an inner $(0,C')$-wSlice$^+$ domain (or inner C'-Slice, respectively) domain for some $C' = C'(C, \phi, n, K)$.*

If f, G, and Ω are as in Theorem 3.1, then Lemma 3.2 tells us that G is broad and the $\alpha = 0$ case of Lemma 3.4 tells us that Ω is an inner 0-wSlice$^+$ domain. Thus Theorem 3.1 follows from Theorem 3.6, at least when $\alpha = 0$. The $\alpha > 0$ case requires little extra effort. First, according to [BS, Lemma 2.8], an inner Slice domain is an inner α-wSlice domain, quantitatively, for all $\alpha \in [0,1)$, so the Slice part already implies most of the α-wSlice$^+$ part of Theorem 3.1. It remains to verify (WS-5) and (WS-1$^+$). The former immediately follows from the Slice condition, while the latter is implicit in the proof of the Slice part of Theorem 3.6.

Recall that 0-wSlice$^+$ domains are 0-wSlice domains that satisfy two extra conditions, (WS-1$^+$) and (WS-5). (WS-1$^+$) will play an important role in the proof of Theorem 3.6 but, by contrast, the proof would work as well if (WS-5) were not part of the definition of an inner 0-wSlice$^+$ domain; it will, however, play an important role in Section 4 when proving the theorems stated in the introduction.

In proving Theorem 3.6, we will make use of a few basic properties of quasiconformal mappings which we describe here. Suppose that f is a K-quasiconformal mapping from G onto Ω, where G,

Ω are domains in $\mathbb{R}^n$. Then f^{-1} is K'-quasiconformal, where $K' = K'(K,n)$. If $B = B(x,r) \subset G$ with $r = C\delta(B, \partial G)$ for some $1 > C > 0$, then for any $y \in fB$, we have $c'\delta_\Omega(y) \le \operatorname{dia}_\Omega fB \le C'\delta_\Omega(y)$ and $B(f(x), c'\delta_\Omega(f(x))) \subset fB$, where c' and C' depend only on C, K, n; furthermore we can choose c', C' tending to 0 as $C \to 0$. Briefly, quasiconformal mappings send Whitney balls to Whitney type objects. K-quasiconformal mappings quasipreserve conformal modulus (i.e., they preserve it up to a multiplicative constant dependent on K and n) and they also quasipreserve large quasihyperbolic distance, in the sense that $1 + d_{0,G}(x,y)$ and $1 + d_{0,\Omega}(f(x), f(y))$ are comparable. For details of these and other properties of quasiconformal mappings, we refer the reader to Theorem 18.1 and other parts of [V2], [V3, 2.4], and [GO, Theorem 3].

Proof of Theorem 3.6. Given $x', y' \in \Omega$, let γ, $\{S_i, d_i\}_{i=1}^m$ be $(0,C)$-wSlice$^+$ data for the pair x, y, where $x \equiv f^{-1}(x')$ and $y \equiv f^{-1}(x')$. Since we are working with an α-wSlice$^+$ condition with $\alpha = 0$, we may *a fortiori* take $d_i = \operatorname{dia}_G(S_i)$. Here and throughout the proof, our notation for objects associated with G and corresponding objects associated with Ω differs only by the use of superscript primes in the latter case.

Multiplying the size of C by 4 if necessary, we may also assume that (WS-4) holds. If $m = 0$, then x', y' satisfy a 0-wSlice$^+$ condition with $m' = 0$ (since f quasipreserves large quasihyperbolic distance). We may therefore assume that $m > 0$. Let $\gamma_i = \gamma([0,1]) \cap S_i$, let γ^i be a component of γ_i of inner diameter at least d_i/C, as guaranteed by (WS-1$^+$), and let $\delta_i = \delta_G(z^i)$ for some fixed but arbitrary point $z^i \in \gamma^i$. By elementary estimation, we see that the quasihyperbolic length of any component K of γ_i must be at least $\log(\delta_G(z')/\delta_G(z''))$ for any pair of points $z', z'' \in K$. Thus (WS-4) implies that $\delta_G(z) \approx \delta_i$, $z \in \gamma^i$. By (WS-1$^+$) and (WS-4), it follows that $d_i/\delta_i \lesssim \operatorname{len}_{0,G}(\gamma^i) \le C$, while the first statement in Lemma 2.1 says that $d_i/\delta_i \gtrsim 1$. Consequently, $\delta_G(z) \approx d_i$, $z \in \gamma^i$.

Fix $x_i \in \gamma^i$ and let $x_i' = f(x_i)$, for each $1 \le i \le m$. For a constant $C_0' > 3$ to be chosen later, let $B_i' = B_\Omega(x_i', C_0'\delta_\Omega(x_i'))$ and $S_i' = f(S_i) \cap B_i'$. Writing $m' = m$, $d_i' = \operatorname{dia}_\Omega(S_i')$, and choosing γ' to be a quasihyperbolic geodesic in Ω, we claim that γ', $\{S_i', d_i'\}_{i=1}^{m'}$ are $(0,C')$-wSlice$^+$ data for x', y', as long as $C' > C_0'$ are both suitably large.

Since f maps Whitney balls to Whitney type objects, the slice data for x', y' inherit the conditions (WS-3) and (WS-5) from G (in general not with the same constant, of course). Since f quasipreserves large quasihyperbolic distance, the slice data for x', y' inherit condition (WS-2) from G. It remains to prove (WS-1$^+$).

We claim that x' and y' lie in separate components of $\Omega \setminus S_i'$, provided that C_0' is large enough. Suppose that they lie in the same component, and so there exists a path $\lambda' \in \Gamma_\Omega(x', y')$ which does not intersect S_i'. Let $\lambda = f^{-1} \circ \lambda'$, let λ^i be as in (WS-1$^+$), and define $F \equiv \overline{\lambda^i}$, $F' = fF$, $E = \overline{B(x_i, c\delta_G(x_i))}$, and $E' = fE$, where $c = c(K,n)$ is the largest constant in $(0, 1/2]$ for which $E' \subset \overline{B(x_i', \delta_\Omega(x_i')/2)}$. Then $\operatorname{dia}_G(F) \approx d_i$, $d_G(E,F) \lesssim d_i$, and by the quasiconformality of f, $\operatorname{dia}_G(E) \approx d_i$. Thus $\Delta_G(E,F) \lesssim 1$, and so $\operatorname{mod}(E,F;G) \ge \epsilon = \epsilon(\phi, C, n, K) > 0$.

Now $d_\Omega(E', F') \ge (C_0' - 1/2)\delta_\Omega(x_i')$ and $\operatorname{dia}_\Omega(E') \le \delta_\Omega(x_i')$, and so $\Delta_\Omega(E', F') \ge C_0' - 1/2$. Thus by Lemma 3.3 and the quasiconformality of f,

$$\operatorname{mod}(E,F;G) \approx \operatorname{mod}(E', F'; \Omega) \lesssim [\log(C_0' - 1/2)]^{-n+1}.$$

Since $\operatorname{mod}(E,F;G) \ge \epsilon$, we get an upper bound for C_0' in terms of ϕ, C, n, and K; we may assume that this upper bound is at least 3. For any C_0' larger than this bound, the claim follows.

We fix C_0' to be a little more than twice as large as this bound, so that $f(S_i) \cap (1/2)\overline{B_i'}$ separates x' from y'. Let $\lambda' \in \Gamma_\Omega(x', y')$, $\lambda \equiv f^{-1} \circ \lambda'$, let λ^i be as in (WS-1$^+$), and define $F = \overline{\lambda^i}$, $F' = fF$. We wish to show that $\operatorname{dia}_\Omega(F') \gtrsim d_i'$. We may assume that $F' \subset S_i'$ since otherwise F' contains points in both $\Omega \setminus B_i'$ and $(1/2)B_i'$, and so $\operatorname{dia}_\Omega(F') \geq C_0' \delta_\Omega(x_i')/2 \geq d_i'/4$.

Now for each $z \in \gamma^i$, we have $d_i \approx \delta_G(z) \lesssim \operatorname{dia}_G(\gamma^i)$, so we can choose a connected compact subset E_0 of γ^i for which $\delta_G(z) \lesssim \operatorname{dia}_G(E_0) \leq \delta_G(z)/2$ for all $z \in E_0$. Letting $E_0' = fE_0$, it follows that $d_i' \approx \delta_\Omega(z') \approx \operatorname{dia}_\Omega(E_0')$ for each $z' \in E_0'$. We choose continua $E_1', E_2' \subset E_0'$ such that $\operatorname{dia}_\Omega(E_1'), \operatorname{dia}_\Omega(E_2') \geq \operatorname{dia}_\Omega(E_0')/4$ and $d_\Omega(E_1', E_2') \geq \operatorname{dia}_\Omega(E_0')/4$. If $d_\Omega(F', E_j') \leq \operatorname{dia}_\Omega(E_0')/10$ for $j = 1, 2$, then $\operatorname{dia}_\Omega(F') \gtrsim \operatorname{dia}_\Omega(E_0') \approx d_i'$ as required. Suppose therefore that $d_\Omega(F', E_j') > \operatorname{dia}(E_0')/10$ for some $j \in \{1, 2\}$. We write $E' = E_j'$, $E = f^{-1}E'$. Note that $\operatorname{dia}_G(F) \approx d_i$, $d_G(E, F) \lesssim d_i$, and by quasiconformality of f, $\operatorname{dia}_G(E) \approx d_i$. Thus by Lemma 3.2 we obtain

$$\operatorname{mod}(E', F'; \Omega) \approx \operatorname{mod}(E, F; G) \gtrsim 1.$$

But $\operatorname{dia}_\Omega(E') \approx d_i'$, $d_\Omega(E', F') \gtrsim d_i'$, and so by Lemma 3.3, $\operatorname{dia}_\Omega(F') \gtrsim d_i$.

The proof for the Slice version is similar, so we omit it. □

4. Product domains

One of the main lessons of this section is that (inner) slice conditions are rather restrictive when imposed upon product domains. This stands in contrast to Section 3, where we saw that the various slice conditions are very weak, at least in the plane. We note that simply-connected planar counterexamples are easily constructed to each of the product domain results in this section if we remove the product domain hypothesis.

Our main theorem in this section is as follows.

Theorem 4.1. *Suppose that $0 \leq \alpha < 1$ and that $\Omega = U \times V \subset \mathbb{R}^n \times \mathbb{R}^N$ is a bounded domain, $n, N \in \mathbb{N}$. The following are equivalent:*

(i) *Ω is an inner (α, C_1)-wSlice$^+$ domain;*
(ii) *Both U and V are inner (α, C_2)-mCigar domains;*
(iii) *Ω is an inner (α, C_3)-mCigar domain.*

The constants C_i depend only on each other and on α, $\operatorname{dia}_\Omega(\Omega)/r(\Omega)$, n, and N.

A result of Lappalainen [L, 6.7] says that, for every $0 < \alpha < \beta < 1$, there exists a planar domain D^* which is an (inner) β-mCigar domain but not an (inner) α-mCigar domain; D^* happens to be bounded, quasiconvex, and simply-connected. Lappalainen's result extends to the case $0 = \alpha < \beta < 1$ since a 0-mCigar domain is a uniform domain and so any β-mCigar domain which is not a $(\beta/2)$-mCigar domain is certain not a 0-mCigar domain. Taking $U = D^*$ and letting V be the unit ball in $\mathbb{R}^{n-2}$, we thus get the following corollary of Theorem 4.1.

Corollary 4.2. *For any $0 \leq \alpha < \beta < 1$ and $3 \leq n \in \mathbb{N}$, there exists an (inner) β-wSlice$^+$ domain $\Omega \subsetneq \mathbb{R}^n$ which is not an (inner) α-wSlice$^+$ domain but is homeomorphic to a ball.*

Note that α-wSlice domains may be inner unbounded even if they are bounded (e.g., many simply-connected planar domains with a spiralling cusp). If however Ω is assumed to be inner

bounded in Theorem 4.1, then the reader can verify from the proof that the inner α-wSlice$^+$ condition in this theorem can be weakened to an inner α-wSlice condition. Lappalainen's examples are certainly inner bounded, so the same examples show that for any $0 \leq \alpha < \beta < 1$ and $3 \leq n \in \mathbb{N}$, there exists an (inner) β-wSlice domain $\Omega \subset \mathbb{R}^n$ which is not an (inner) α-wSlice domain but is homeomorphic to a ball.

To prove Theorem 4.1, we shall need some lemmas.

Lemma 4.3. *Let Ω be an inner (α, C)-mCigar domain, $0 \leq \alpha < 1$. For every $x, y \in \Omega$, there exists an inner (α, C)-mCigar path γ such that all initial and final segments of γ are inner $(\alpha, 2C)$-mCigar paths (for the segment endpoints).*

Proof. Fixing $x, y \in \Omega$, we may assume that $|x - y| \geq \delta_\Omega(x) \vee \delta_\Omega(y)$, since otherwise $[x \to y]$ has minimal $d_{\alpha,\Omega}$-length among all paths connecting x and y, and so all segments of this line segment are (α, C)-mCigar paths. Let $B_x \equiv B(x, \delta_\Omega(x)/2)$ and $B_y \equiv B(y, \delta_\Omega(y)/2)$.

By symmetry, it suffices to prove the result only for initial segments. Consider first the case $\alpha > 0$. Let $\epsilon = \epsilon_x \wedge \epsilon_y$, where $\epsilon_z = ((3/2)^\alpha - 1)\delta_\Omega^\alpha(z)/2\alpha$ for $z \in \{x, y\}$. The desired path γ will be an inner (α, C)-mCigar path for x, y with some extra properties. First, we assume that $\mathrm{len}_{\alpha,\Omega}(\gamma) < d_{\alpha,\Omega}(x, y) + \epsilon$. Since the $d_{\alpha,\Omega}$-mimimal length paths from x to any $x_1 \in \partial B_x$, and from y to any $y_1 \in \partial B_y$, are line segments, we may also assume that the only subarc of γ lying in either B_x or B_y is a single line segment. Finally by reparametrization, we may assume that $\gamma|_{[0,1/4]}$ and $\gamma|_{[3/4,1]}$ are the line segments in question, from x to $x' \in \partial B_x$ and from $y' \in \partial B_y$ to y, respectively, and that both of these line segments are traversed by γ at a constant Euclidean speed.

By direct calculation, it is easy to check that $\gamma|_{[0,t]}$ is an inner $(\alpha, f_\alpha(t))$-mCigar path for $t \leq 1/4$, with $f_\alpha(t) = [1 - (1 - 2t)^\alpha]/\alpha(2t)^\alpha$; this largest constant is attained by picking x' so that $\delta_\Omega(x') = \delta_\Omega(x)/2$. Since f_α is increasing on $[0, 1/4]$, we have $f_\alpha(t) \leq f_\alpha(1/4) = (2^\alpha - 1)/\alpha$, $t \in [0, 1/4]$. By calculus, we see that $f_\alpha(1/4) < f_1(1/4) = 1$, $\alpha \in (0, 1)$. Thus these initial segments are (inner) $(\alpha, 1)$-mCigar paths.

To go from x to $\gamma(t)$, $t > 1/4$, one must first exit B_x, and so $d_{\alpha,\Omega}(x, \gamma(t)) \geq \min_{u \in \partial B_x} d_{\alpha,\Omega}(x, u) \geq 2\epsilon_x$. Suppose for the purposes of contradiction that $\gamma|_{[0,t]}$ is not an inner $(\alpha, 2C)$-mCigar path for the pair $x, \gamma(t)$. The $d_{\alpha,\Omega}$-length of an inner (α, C)-mCigar path for $x, \gamma(t)$ is less than half that of $\gamma|_{[0,t]}$, and so shorter by an amount in excess of ϵ_x. Thus splicing the (reparametrized) shorter path into γ in place of $\gamma|_{[0,t]}$, we get a new path, contradicting the near-minimal $d_{\alpha,\Omega}$-length of γ.

Taking $\epsilon = \log\sqrt{3/2}$, the proof when $\alpha = 0$ is similar, so we omit it. Alternatively, it follows from the fact that quasihyperbolic geodesics in an inner $(0, C_1)$-mCigar domain are inner C_2-uniform paths for some $C_2 = C_2(C_1)$; see [V4, 2.29]. $\square$

Lemma 4.4. *If $0 \leq \alpha < 1$ and $\Omega = U \times V \subset \mathbb{R}^n \times \mathbb{R}^N$ is a bounded inner (α, C)-wSlice$^+$ domain, then Ω is also inner bounded, and $\mathrm{dia}_\Omega(\Omega) \leq C' \,\mathrm{dia}(\Omega)$, where $C' = C'(\alpha, C, \mathrm{dia}(\Omega)/r(\Omega), n+N)$.*

Proof. Without loss of generality, we may assume that $C \geq 4$ and that $\mathrm{dia}(\Omega) = 1$ (the latter because of the scale invariance of the hypotheses and conclusion). By symmetry it suffices to prove that $\mathrm{dia}_U(U) \lesssim 1$. We choose $v_0 \in V$ such that $\delta_0 \equiv \delta_V(v_0) = r(V)$. Note that $r(\Omega) \leq \delta_0 \leq 1/2$ and that $d_U(u_1, u_2) = d_\Omega((u_1, v_0), (u_2, v_0))$.

Suppose that there exist points $u, w \in U$ such that $d_U(u,v) > 1$. Writing $x = (u, v_0)$, $y = (w, v_0)$, we assume that inner (α, C)-wSlice$^+$ data for x, y are γ, $\{S_i, d_i\}_{i=1}^m$, with the indexing chosen so that $\{d_i\}_{i=1}^m$ is non-decreasing. It is also convenient to define $d_0 = 0$ and $d_{m+1} = \infty$. Let $m_0 \in [0, m]$ be the unique integer for which $d_{m_0} < \delta_0/2 \le d_{m_0+1}$.

Using only (WS-1), we claim that $d_1 \ge 2(\delta_\Omega(x) \wedge \delta_\Omega(y))/C$, and that there exist constants $C_1, t > 0$, dependent only on C, such that $d_i \ge C_1 2^{(i-j)t} d_j$ whenever $j < i \le m_0$. We first construct two paths λ^+ and λ^- from x to y, each consisting of three segments. The first segment of λ^+ is $[x \to x']$, where $x' = (u, v_0 + v') \in \partial B(x, \delta_0/2)$. The second segment, from x' to $y' = (w, v_0 + v')$, has constant V-component, and the final segment is (a reparametrization of) $[y' \to y]$. The path λ^- is defined in a similar fashion except that we replace v' by $-v'$ throughout.

Let $i \le m_0$. By (WS-1), both λ^+ and λ^- intersect S_i on a set of length at least d_i/C; we denote the sets of intersection by S_i^+ and S_i^-, and write $S_i' = S_i^+ \cup S_i^-$. Since $d_i < \delta_0/2 < 1/4$, it follows that S_i' (in fact, all of S_i) is contained in either $B(x, \delta_0/2)$ or $B(y, \delta_0/2)$. The argument is the same in both cases, so we assume that $S_i' \subset B(x, \delta_0/2)$. Since S_i^+ and S_i^- lie outside $B(x, \delta_\Omega(x)/C)$, and on opposite sides of (u, v_0), we have $d_i \ge 2\delta_\Omega(x)/C$, giving the first half of our claim. For the same reason, we actually have $S_i' \subset B(x, d_i)$. In particular, if $d_i \in (a/2, a]$ for some positive number $a \le \delta_0/2$, then S_i intersects both λ^+ and λ^- on sets of length at least $a/2C$ lying in $B(x,a) \cup B(y,a)$. Slices are disjoint, and the total intersection of either λ^+ or λ^- with $B(x,a) \cup B(y,a)$ has length $2a$, so there can be at most $4C$ such slices S_i. The second half of our claim now follows.

Suppose that $\alpha > 0$. To prove inner boundedness of U, we find a bound for $\operatorname{len}(\gamma)$. Since $\operatorname{dia}(\Omega) = 1$, we have $\operatorname{len}(\gamma) \le \operatorname{len}_{\alpha,\Omega}(\gamma)$. Thus it suffices to bound $\sum_{i=1}^m d_i^\alpha$. The geometric growth of $\{d_i\}_{i=1}^{m_0}$ ensures that

$$\sum_{i=1}^{m_0} d_i^\alpha \lesssim d_{m_0}^\alpha \le (\delta_0/2)^\alpha.$$

By (WS-5), we have $(d_i/C)^{n+N} \le |S_i| \le |\Omega| \le 1$, and so $d_i \le C$ for all i. If $i \ge m_0$ then $d_i \ge \delta_0/2$ and so $|S_i| \ge (\delta_0/2)^{n+N}$. Since the slices are disjoint, we deduce that $m - m_0 \le (2/\delta_0)^{n+N}$. Thus

$$\sum_{i=m_0+1}^{m} d_i^\alpha \le (m - m_0) d_m^\alpha \le 2^{n+N} \delta_0^{-n-N} C^\alpha.$$

It follows that $\sum_{i=1}^m d_i^\alpha \lesssim 1$, as desired.

Suppose instead that $\alpha = 0$. It is not hard to show that

$$d_{0,\Omega}(x,y) \gtrsim \log\left(1 + \frac{d_\Omega(x,y)}{\delta_\Omega(x) \wedge \delta_\Omega(y)}\right) \tag{4.5}$$

For the Euclidean version of this inequality, see [GP, Lemma 2.1], whose proof also handles this inner version; see also [V4, 2.5]. As in the case $\alpha > 0$, we have $m - m_0 \le 2^{n+N}/\delta_0^{n+N}$. The size and growth properties of $\{d_i\}_{i=1}^{m_0}$ obtained above imply that $m_0 \lesssim 1 + \log[1/(\delta_\Omega(x) \wedge \delta_\Omega(y))]$. Thus

$$d_{0,\Omega}(x,y) \lesssim 2 + m \lesssim 1 + \log\left(\frac{1}{\delta_\Omega(x) \wedge \delta_\Omega(y)}\right) \lesssim \log\left(1 + \frac{1}{\delta_\Omega(x) \wedge \delta_\Omega(y)}\right).$$

Comparing this last inequality with (4.5), we deduce that $d_\Omega(x,y) \lesssim 1$. $\square$

The Euclidean version of the next lemma is part of the $\beta = \alpha$ case of [BK2, Lemma 2.2][3]. We omit a proof, as it is entirely analogous to the Euclidean case.

Lemma 4.6. *Let $0 < \alpha < 1$ and let $\gamma : [0,l] \to \Omega$ be an inner (α, C)-mCigar path, parametrized by arclength, for the points x, y in a domain $\Omega \subsetneq \mathbb{R}^n$. Let us denote by $r : [0,l] \to (0,\infty)$ the non-decreasing rearrangement of $t \mapsto \delta_\Omega(\gamma(t))$. Then there exists exists a constant $C_0 = C_0(C,\alpha)$ such that $\operatorname{len}(\gamma) \le C_0 d_\Omega(x,y)$ and $r(t) \ge C_0^{-1}(t d_\Omega(x,y)^{-\alpha})^{1/(1-\alpha)}$. In particular, $r(cl) \ge C_0^{-1} c^{1/(1-\alpha)} d_\Omega(x,y)$ for all $c > 0$.*

Proof of Theorem 4.1.
$(i) \Rightarrow (ii)$: Assuming that Ω is an inner (α, C_1)-wSlice$^+$ domain, it suffices by symmetry to prove that U is an inner α-mCigar domain. Fix a point $v_0 \in V$ such that $\delta_0 \equiv \delta_V(v_0) = r(V)$. Let γ, $\{S_i, d_i\}_{i=1}^m$ be inner (α, C_1)-wSlice$^+$ data for a pair of points $x = (u, v_0), y = (w, v_0) \in \Omega$. We may assume that $d_\Omega(x,y) > (\delta_\Omega(x) \vee \delta_\Omega(y))/2$, since otherwise a line segment satisfies an α-mCigar condition. We index the slices so that $\{d_i\}_{i=1}^m$ is non-decreasing; it is also convenient to define $d_0 = 0$ and $d_{m+1} = \infty$. Let $m_0 \in [0,m]$ be the unique integer for which $d_{m_0} < s \le d_{m_0+1}$, where $s = (\delta_0 \wedge d_\Omega(x,y))/2$.

As in Lemma 4.4, we see that $d_1 \ge 2(\delta_\Omega(x) \wedge \delta_\Omega(y))/C_1$ and that there exist constants $C', t > 0$, dependent only on C, such that $d_i \ge C' 2^{(i-j)t} d_j$ whenever $j < i \le m_0$. Fixing a path $\lambda \in \Gamma_\Omega(x,y)$ such that $\operatorname{len}(\lambda) \le 2 d_\Omega(x,y)$, (WS-1) implies that

$$\sum_{i=1}^{m} d_i \le 2C d_\Omega(x,y). \tag{4.7}$$

Thus $m - m_0 \le 2C d_\Omega(x,y)/s \le 2C \operatorname{dia}_U(U)/\delta_0$. Thus by Lemma 4.4, $m - m_0 \lesssim 1$.

Consider the case $\alpha = 0$. By the size and growth properties of $\{d_i\}_{i=1}^{m_0}$, we see that

$$m_0 \lesssim 1 + \log\left(\frac{d_\Omega(x,y)}{\delta_\Omega(y) \wedge \delta_\Omega(x)}\right) \lesssim \log\left(1 + \frac{d_\Omega(x,y)}{\delta_\Omega(y) \wedge \delta_\Omega(x)}\right)$$

Since also $m - m_0 \lesssim 1$, (WS-2) now implies an inner 0-mCigar condition for x, y. When we project from Ω to U, Euclidean length cannot increase, and distance to the boundary cannot decrease. Therefore we deduce an inner $(0, C_2)$-mCigar condition for u, w, with $C_2 = C_2(T)$, where T denotes the data $(\alpha, C_1, \operatorname{dia}_\Omega(\Omega)/r(\Omega), n+N)$.

Consider next the case $\alpha > 0$. Suppose, for the purposes of contradiction that U is not an inner α-mCigar domain. For each $k \in \mathbb{N}$, there exist points u_k and w_k for which $d_{\alpha,U}(u_k, w_k) \ge k d_U(u_k, w_k)^\alpha$; also let $x_k = (u_k, v_0)$ and $y = (w_k, v_0)$. Regardless of the values of k, α, we must have $2d_\Omega(x_k, y_k) \ge \delta_\Omega(x_k) \vee \delta_\Omega(y_k)$ since otherwise by consideration of the segment $[x_k \to y_k]$, the points u_k and w_k would violate the previous inequality. But

$$d_{\alpha,\Omega}(x_k, y_k) \ge d_{\alpha,U}(u_k, w_k) \ge k d_U(u_k, w_k)^\alpha = k d_\Omega(x_k, y_k)^\alpha, \tag{4.8}$$

[3]The version there is stated for α-mCigar domains Ω but the proof merely uses the fact that that there exists an α-mCigar path for a particular pair of points. Also Ω is assumed to be bounded, but this is not used.

and so $d_{\alpha,\Omega}(x_k, y_k) \geq k2^{-\alpha}(\delta_\Omega(x_k) \vee \delta_\Omega(y_k))$. Let γ, $\{S_i, d_i\}_{i=1}^m$, be inner (α, C_1)-wSlice$^+$ data for the pair x_k, y_k, with $\{d_i\}_{i=1}^m$ non-decreasing; for ease of notation, the dependence on k is implicit. Taking $k > 3 \cdot 2^\alpha C_1$, it follows from (WS-2) that $\mathrm{len}_{\alpha,\Omega}(\gamma) \leq 3C_1 \sum_{i=1}^m d_i^\alpha$. Combining this inequality with (4.7) and (4.8), we get

$$\frac{\sum_{i=1}^m d_i^\alpha}{(\sum_{i=1}^m d_i)^\alpha} \geq \frac{\mathrm{len}_{\alpha,\Omega}(\gamma)}{3C_1 (\sum_{i=1}^m d_i)^\alpha} \geq \frac{\mathrm{len}_{\alpha,\Omega}(\gamma)}{3C_1^{1+\alpha} \cdot 2^\alpha \cdot d_\Omega(x_k, y_k)^\alpha} \geq \frac{k}{3C_1^{1+\alpha} \cdot 2^\alpha}. \tag{4.9}$$

But the growth rate of the $\{d_i\}_{i=1}^{m_0}$ and the bound on $m - m_0$ imply that both $\sum_{i=1}^{m_0} d_i^\alpha$ and $\sum_{i=m_0+1}^m d_i^\alpha$ are no more than a constant multiple of $(\sum_{i=1}^m d_i)^\alpha$. Taking k to be larger than some constant $C_2 = C_2(T)$, we get the desired contradiction to (4.9).

(ii) ⇒ (iii): Assume first that $\alpha > 0$. By the triangle inequality, it suffices to verify the inner α-mCigar condition for pairs of points $x, y \in \Omega$ with one common coordinate; by symmetry, we may assume that $x = (u, v)$, $y = (w, v)$. Let us fix a point $v_0 \in V$ such that $\delta_0 \equiv \delta_V(v_0) = r(V)$. Let $\mu : [0, 1] \to V$ and $\gamma : [0, 1] \to U$ be inner (α, C_2)-mCigar paths from v to v_0 and from u to w respectively, where μ has the additional properties guaranteed by Lemma 4.3 (applied to V). Letting $L = \mathrm{len}(\gamma)$, $\gamma_1 = \gamma|_{[0,1/2]}$, $\gamma_2 = \gamma|_{[1/2,1]}$, $z = \gamma(1/2)$, we may assume that γ is parametrized so that $\mathrm{len}(\gamma_1) = \mathrm{len}(\gamma_2) = L/2$.

Suppose also that $L \leq 2\,\mathrm{len}(\mu)$. We wish to define an inner α-mCigar path $\Lambda \in \Gamma_\Omega(x, y)$. We choose $\Lambda(t) = (\gamma(t), \lambda(t))$, where λ is a path in V which starts and finishes at v but, in between times, moves along μ and back. More precisely, for $0 \leq t \leq 1/2$, λ coincides with a reparametrized initial segment of μ, with the parametrization chosen so that $\mathrm{len}(\gamma|_{[0,t]}) = \mathrm{len}(\lambda|_{[0,t]})$. For $1/2 \leq t \leq 1$, λ traces its way back along the curve of μ in such a way that $\mathrm{len}(\gamma|_{[t,1]}) = \mathrm{len}(\lambda|_{[t,1]})$.

Since $\delta_\Omega((a, b)) = \delta_U(a) \wedge \delta_V(b)$, $a \in U, b \in V$, we obtain

$$\begin{aligned}\mathrm{len}_{\alpha,\Omega}(\Lambda) &< \int_\Lambda \delta_U(\gamma(t))^{\alpha-1}\, ds(t) + \int_\Lambda \delta_V(\lambda(t))^{\alpha-1}\, ds(t) \\ &= \sqrt{2}(\mathrm{len}_{\alpha,U}(\gamma) + \mathrm{len}_{\alpha,V}(\lambda)).\end{aligned}$$

Now $\mathrm{len}_{\alpha,U}(\gamma) \leq C_2 d_U(u, w)^\alpha = C_2 d_\Omega(x, y)^\alpha$. By Lemma 4.6 we may assume that $L \lesssim d_U(u, w) = d_\Omega(x, y)$, and so by Lemma 4.3 applied to the segments γ_1 and γ_2,

$$\mathrm{len}_{\alpha,V}(\lambda)/4C_2 \leq d_V(v, \lambda(1/2))^\alpha \leq (L/2)^\alpha \lesssim d_\Omega(x, y)^\alpha.$$

The inner α-mCigar condition for Λ now follows.

The construction for $L > 2\,\mathrm{len}(\mu)$ is similar: $\lambda(t)$ moves along $\mu([0, 1])$ from v at the same speed as before, except now it reaches v_0 at some $t = t_0 < 1/2$. Similarly, there is some number $t_1 > 1/2$ such that $\mathrm{len}([t_1, 1]) = \mathrm{len}(\mu)$. The path λ is now continued so that $\lambda(t) = v_0$ for that $t_0 \leq t \leq t_1$, and finally for $t_1 \leq t \leq 1$, $\lambda(t)$ moves back along μ to v at the same speed as before. The estimates are the same as before except for

$$\int_{\Lambda|_{[t_0,t_1]}} \delta_V(\lambda(t))^{\alpha-1}\, ds(t) \leq L\delta_V(v_0)^{\alpha-1} \lesssim L^\alpha.$$

We must still consider the $\alpha = 0$ case. Since the 0-mCigar condition is quantitatively equivalent to uniformity [V4, 2.33], it suffices to verify that if U and V are uniform, then Ω is uniform. Let $v_0 \in V$ be as in the $\alpha > 0$ case, but now we seek to find a uniform path between a pair of points (u_1, v_1) and (u_2, v_2). Let $\gamma : [0, l_1] \to U$ and $\mu : [0, l_2] \to V$ be uniform paths parametrized by arclength for the pairs of points u_1, u_2 and v_1, v_2 in their respective domains; without loss of generality $l_1 \geq l_2$. Let $\nu : [0, l_3] \to V$ be a uniform path in V, parametrized by arclength, for the pair $\mu(l_2/2), v_0$. We now define a new path $\lambda : [0, l_1] \to V$ linking v_1 and v_2. If $l_1 \leq l_2 + 2l_3$, then

$$\lambda(t) = \begin{cases} \mu(t), & 0 \leq t \leq l_2/2, \\ \nu(t - l_2/2), & l_2/2 \leq t \leq l_1/2, \\ \nu(l_1 - l_2/2 - t), & l_1/2 \leq t \leq l_1 - l_2/2, \\ \mu(t - l_1 + l_2), & l_1 - l_2/2 \leq t \leq l_1. \end{cases}$$

while if $l_1 \geq l_2 + 2l_3$, then the definition is similar except that λ "rests" at v_0 for an interval of length $l_1 - l_2 - 2l_3$ before turning back. We leave it to the reader to verify that the path $\Lambda = (\gamma, \lambda)$ is a uniform path in Ω for the pair (u_1, v_1) and (u_2, v_2), with quantitative dependence only on allowed parameters, namely $\mathrm{dia}_\Omega(\Omega)/r(\Omega)$, n, N, and the uniformity constants for U, V.

$(iii) \Rightarrow (i)$: This follows from Lemma 3.4. $\square$

Proof of Theorems 0.1, 0.2, and 0.3. We first prove Theorem 0.1. Trivially (ii) implies (i). Since an inner 0-mCigar domain is just an inner uniform domain, the equivalence of (ii) and (iii) follows from Theorem 4.1. If Ω is K-quasiconformally equivalent to an inner C-uniform domain then Theorem 3.1 ensures that it is an inner $(0, C_1)$-wSlice$^+$ domain, with $C_1 = C_1(C, n + N, K)$, and so Theorem 4.1 tells us that (i) implies (ii).

Theorem 0.3 follows similarly by combining Theorem 3.6 and Theorem 4.1. As for Theorem 0.2, one direction is given by Lemma 3.2, while the other follows from Theorem 0.3 with $G = \Omega$. $\square$

Remark 4.10 The implication (i) $\Rightarrow$ (ii) of Theorem 0.1 also follows from recent work of Bonk, Heinonen, and Koskela; see [BHK, Remark 7.34]. Their methods (based around Gromov hyperbolicity) are however quite different and do not apply to the $\alpha > 0$ case of Theorem 4.1.

Remark 4.11 Theorem 0.1 does not tell us what product domains are quasiconformally equivalent to a ball. In fact, Väisälä [V4] showed that if G is a simply-connected proper subdomain of the plane, then $G \times \mathbb{R}$ is quasiconformally equivalent to a ball if and only if there is a BLD (bounded length distortion) mapping from G to a disk or a half-plane. It is not hard to modify his proof to show that for a bounded domain G, $G \times (0, 1)$ is quasiconformally equivalent to a ball if and only if there is a BLD (bounded length distortion) mapping from G to a disk. It follows that there are inner uniform domains of product type that are not quasiconformally equivalent to a ball. For instance, the planar domain U bounded by a von Koch snowflake is a uniform domain but, because its boundary is not locally rectifiable, no such BLD mapping can exist and consequently $\Omega = U \times \mathbb{R}$ is uniform but not quasiconformally equivalent to a ball.

Remark 4.12 The 0-wSlice$^+$ hypothesis cannot be removed from Theorems 0.2 and 0.3. For example, let B^k denote the unit ball in $\mathbb{R}^k$, let $n > 1$, and consider the product domain $\Omega = B \setminus N$, where $B = B^n \times B^m$ and $N = A \times B^m$, $A = \bigcup_{j=1}^{\infty} A_j$, and A_j consists of $(j!)^{n-1}$ points on the

sphere $S_j = \{|z| = 1 - 2^{-j}\} \subset \mathbb{R}^n$, spaced so that the distance from any $x \in S_j$ to A_j is at most $C/j!$, for some $C = C(n)$. Clearly B is broad, and we claim that Ω is also broad. To see this, let E, F be non-degenerate disjoint continua in Ω. Since $B \setminus \Omega$ has Hausdorff dimension at most $m < n + m - 1$, it is an *null set for extremal distance* [V1] and so $\operatorname{mod}(E, F; B) = \operatorname{mod}(E, F, \Omega)$. The restriction of d_B to $\Omega \times \Omega$ coincides with d_Ω, and so $\Delta_B(E, F) = \Delta_\Omega(E, F)$. The claim now follows readily. However $B^n \setminus A$, and hence Ω, is not inner uniform since a path from the origin to a point x of norm nearly 1 must pass through very narrow bottlenecks as it approaches x.

5. Further results

We first use some of the ideas developed in the last section to prove, as promised in Section 2, that (WS-3) can be removed from the definition of (inner) α-wSlice conditions when $\alpha > 0$ without changing the class of domains; we shall need this result in the final section.

Theorem 5.1. *Suppose that $0 < \alpha < 1$, $x, y \in G \subset \mathbb{R}^n$, and that $\gamma \in \Gamma_G(x, y)$, $\{S_i, d_i\}_{i=1}^m$ satisfy (WS-1) and (WS-2), with $d_i \geq \operatorname{dia}_d(S_i)$ for some metric satisfying $d_{\mathbb{R}^n} \leq d \leq d_G$. Then x, y satisfy an $(\alpha, C'; d)$-wSlice condition for some $C' = C'(C, \alpha)$, with slice data γ', $\{T_i, e_i\}_{i=1}^M$ satisfying*

$$C' \left(\delta_G^\alpha(x) + \delta_G^\alpha(y) + \sum_{i=1}^{M} e_i^\alpha \right) \geq C \left(\delta_G^\alpha(x) + \delta_G^\alpha(y) + \sum_{i=1}^{m} d_i^\alpha \right)$$

Proof. Without loss of generality $\delta_G(x) \geq \delta_G(y)$. We may assume that $|x - y| > \delta_G(x)$, since otherwise the conclusion is true with $M = 0$ and $\gamma' = [x \to y]$. Writing $B_z = B(z, \delta_G(z)/16C)$ and $\widetilde{B}_z = B(z, \delta_G(z)/2)$, for $z \in \{x, y\}$, we note that $\widetilde{B}_x$ and $\widetilde{B}_y$ are disjoint. The first step is to define new slices $S_i' = S_i \setminus \overline{B_x \cup B_y}$, and leave the numbers d_i unchanged. Certainly, these new slices satisfy (WS-3), but (WS-1) may now fail. We discard any slice S_i' for which (WS-1) still fails even after we replace C by $2C$. Renumbering the remaining pairs (S_i', d_i), we get new slice data $\gamma' \equiv \gamma$, $\{T_i, e_i\}_{i=1}^M$.

By construction, the new data satisfy (WS-3), and (WS-1) with constant $2C$. It remains to prove (WS-2) (with C replaced by some C'). If S_i' is a discarded slice then there must exist some path $\lambda \in \Gamma_G(x, y)$ whose intersection with S_i' has length less than $d_i/2C$. Now (WS-1) for S_i tells us that $\operatorname{len}(\lambda \cap S_i \cap \overline{B_x \cup B_y}) > d_i/2C$. If we alter λ so that for $z = x, y$ the only segment of γ lying in $\overline{B_z}$ is a single line segment (of length $\delta_G(z)/16C$), but otherwise leave λ unchanged, this inequality must remain true. Thus $(\delta_G(x) + \delta_G(y))/16C \geq d_i/2C$, and so $d_i \leq \delta_G(x)/4$. If $S_i \cap \overline{B_x}$ is non-empty, then S_i must lie fully in $\widetilde{B}_x$. On the other hand, if $S_i \cap \overline{B_x}$ is empty, then $S_i \cap \overline{B_y}$ is non-empty and $d_i \leq \delta_G(y)/4$. In either case, we deduce that S_i lies fully in either $\widetilde{B}_x$ or $\widetilde{B}_y$.

Let us enumerate the discarded slices and the corresponding numbers as $\{S_{j_i}', d_i'\}_{i=1}^k$, with $d_i' \equiv d_{j_i}$. We choose the enumeration so that $(d_i')_{i=1}^k$ is non-decreasing. As in the proof of Lemma 4.4, we obtain the growth estimate

$$d_i' \geq C_1 2^{(i-j)t} d_j', \qquad \text{for } 1 \leq j < i \leq k.$$

In fact to get this estimate, the two paths used should be as follows. The first one, λ^+, starts off as any line segment of length $\delta_\Omega(x)/2$ emanating from x, and ends as any line segment of length $\delta_\Omega(y)/2$ ending at y, the middle part of the path being any path joining the outer endpoints of these two segments in Ω which stays outside $\widetilde{B}_x \cup \widetilde{B}_y$. The second path λ^- has the same construction except that the initial and final line segments are in directions opposite to those of the λ^+.

The growth estimate and (WS-1) now give

$$\sum_{i=1}^{k}(d_i')^\alpha \lesssim (d_k')^\alpha \sum_{j=1}^{\infty} 2^{-tj} \lesssim (d_k')^\alpha \lesssim \delta_\Omega^\alpha(x) + \delta_\Omega^\alpha(y).$$

Thus

$$\delta_G^\alpha(x) + \delta_G^\alpha(y) + \sum_{i=1}^{M} e_i^\alpha \gtrsim \delta_G^\alpha(x) + \delta_G^\alpha(y) + \sum_{i=1}^{m} d_i^\alpha$$

and so if we replace C by an appropriate C', then the remaining slices satisfy all three conditions (WS-1), (WS-2), and (WS-3). $\square$

Next we wish to state a John-Separation version of Theorem 4.1, but let us begin with two definitions that we need.

Let us fix a constant $C \geq 1$ and a point x_0 in the domain G.. A *C-John path for x (with respect to x_0)* is a path $\gamma \in \Gamma_G(x, x_0)$, $\gamma : [0, l] \to G$, which is parametrized by arclength such that $\delta(\gamma(t)) \geq t/C$ for all $t \in [0, l]$. We say that G is a *C-John domain (with respect to x_0)* if there exists a C-John path (with respect to x_0) for all $x \in G$.

Let C, x_0 be as above and let $B_z = B(z, C\delta_G(z))$, $z \in G$. As defined in [BK1], a *C-Separation path for x (with respect to x_0)* is a path $\gamma : [0, 1] \to G$, $\gamma \in \Gamma_G(x, x_0)$, such that for each $t \in [0, 1]$, any path from a point in $\gamma([0, t]) \setminus B_{\gamma(t)}$ to x_0 must intersect $\partial B_{\gamma(t)}$. We say that G is a *C-Separation domain (with respect to x_0)* if there exists a C-Separation path (with respect to x_0) for all $x \in G$. A C-John domain is a C-Separation domain (since $\gamma([0, t]) \setminus B_{\gamma(t)}$ is empty) but there are many more Separation domains, including all quasiconformal images of uniform domains [BK1].

Theorem 5.2. *Suppose that $\Omega = U \times V \subset \mathbb{R}^n \times \mathbb{R}^N$ is a bounded domain, $x_0 = (u_0, v_0) \in \Omega$, and $n, N \in \mathbb{N}$. The following are equivalent:*

(i) *Ω is a C_1-Separation domain with respect to x_0;*
(ii) *Both U and V are C_2-John domains with respect to u_0 and v_0 respectively;*
(iii) *Ω is a C_3-John domain with respect to x_0.*

The constants C_i depend only on each other and on n, N, and $\operatorname{dia}_\Omega(\Omega)/d_\Omega(x_0)$.

Proof. We omit the easy verifications of the implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Supposing that Ω satisfies (i), we shall prove (ii). We may assume that $C_1 > 2$ and, by symmetry, it suffices to show that U is a John domain with respect to u_0. We claim that the first coordinate projection γ_1 of any C_1-separation path γ for the point $x = (u, v_0)$ must be a C_2-John path for u, with $C_2 = C_2(C_1, \operatorname{dia}(\Omega)/\delta_\Omega(x_0))$. To see this, we write $r(t) = C_1\delta_\Omega(\gamma(t))$, $B_t = B(\gamma(t), r(t))$. If

$\gamma([0,t]) \subset \overline{B_t}$, then γ_1 satisfies the $2C_1$-John condition for x at $\gamma_1(t)$, so we shall assume that $\gamma([0,t]) \not\subset \overline{B_t}$. We may also assume that $r(t) < \delta_V(v_0)/6$ since otherwise the claim follows with $C_2 = 6C_1 \operatorname{dia}(V)/\delta_V(x_0) \le 6C_1 \operatorname{dia}(\Omega)/\delta_\Omega(x_0)$.

If $|\gamma(t)-x_0| \le \delta_\Omega(x_0)/2$, then $r(t) \ge C_1\delta_\Omega(x_0)/2$. On the other hand, if $|\gamma(t)-x_0| > \delta_\Omega(x_0)/2$ and $x_0 \in \overline{B_t}$, then $r(t) \ge \delta_\Omega(x_0)/2$. Both of these contradict the bound on $r(t)$, so we conclude that $x_0 \notin \overline{B_t}$.

Suppose that $x \notin \overline{B_t}$. We construct two paths $\lambda^+, \lambda^- \in \Gamma_\Omega(x, x_0)$ by first moving in a straight line from (u, v_0) to points $x^+ = (u, v_0+w)$, $x^- = (u, v_0-w)$, respectively, where $w \in \mathbb{R}^N$ is chosen so that $2r(t) < |w| < \delta_V(v_0)$. The second segment of each paths has constant second coordinate and rectifiable first coordinate finishing at a point with first coordinate u_0, and the last segment of each is a straight line segment back to (u_0, v_0).

Now ∂B_t must intersect both λ^+ and λ^-. But ∂B_t cannot intersect the middle segment of either path since its distance from the other path exceeds $2r(t)$. Neither can it intersect the first segments, or the last segments, of both paths, since it would then follow that either $x \in B_t$ or $x_0 \in B_t$. Finally suppose that ∂B_t intersects the first segment of one path at (u, v_1), say, and the last segment of the other at (u_0, v_2), say. Thus

$$2r(t) > |v_1 - v_2| > 2(|v_1 - v_0| \wedge |v_2 - v_0|),$$

and so $2B_t$ contains either x or x_0. The claim follows as before.

We are left to consider the case where $x \in \overline{B_t}$. By assumption, there is a point $\hat{x} \in \gamma([0,t]) \setminus \overline{B_t}$, which by continuity we may assume to lie in the annulus $2B_t \setminus \overline{B_t}$. We now define a pair of paths $\lambda^+, \lambda^- \in \Gamma_\Omega(\hat{x}, x_0)$ by first moving in a straight line from $\hat{x}$ to points $x^+ = \hat{x} + (0, w)$, $x^- = \hat{x} + (0, -w)$, respectively, where $w \in \mathbb{R}^N$ is chosen so that $4r(t) < |w| < \delta_V(v_0) - 2r(t)$. As before, the second segment of each path has constant second coordinate and rectifiable first coordinate finishing at a point with first coordinate u_0, and the last segment of both paths is a straight line segment back to x_0. This claim now follows as in the previous case. □

We now discuss the case of unbounded domains $\Omega \subsetneq \mathbb{R}^n$. As we shall see below, most of the implications in Theorem 4.1 fail if we simply drop the boundedness assumption, but we do have the following theorem.

Theorem 5.3. *Suppose that $0 \le \alpha < 1$ and that $\Omega = U \times V$ where $U \subsetneq \mathbb{R}^n$, $V \subsetneq \mathbb{R}^N$, $r(U) = r(V) = \infty$, and $n, N \in \mathbb{N}$. The following are equivalent:*

(i) *Ω is is an inner (α, C_1)-wSlice$^+$ domain;*
(ii) *Both U and V are inner (α, C_2)-mCigar domains;*
(iii) *Ω is an inner (α, C_3)-mCigar domain.*

The constants C_i depend only on each other and on α, n, and N.

Sketch of proof. Let Ω satisfy the hypotheses and that it is an inner (α, C)-wSlice$^+$ domain. By symmetry, it suffices to prove an inner α-mCigar condition for U. We fix points $u, w \in U$, and choose a path $\gamma \in \Gamma_U(u, w)$ such that $L_\alpha \equiv \operatorname{len}_{\alpha,U}(\gamma) \le 2d_{\alpha,U}(u, w)$. Let $L_1 = \operatorname{len}(\gamma)$ and let M denote the largest value of $\delta_U(z)$ on the image of γ. Now choose v_0 so that $\delta_0 \equiv \delta_V(v_0) > M \vee 2CL_1$. Let $x = (u, v_0)$, $y = (w, v_0)$, and define the path Λ by $\Lambda(t) = (\gamma(t), v_0)$. It follows

that $\mathrm{len}_{\alpha,\Omega}(\Lambda) = L_\alpha$ and that $d_{\alpha,\Omega}(x,y) = d_{\alpha,U}(u,w)$. We deduce that the pair x, y possesses $(\alpha, 2C)$-wSlice$^+$ data of the form Λ, $\{S_i, d_i\}_{i=1}^m$, with the indexing chosen so that $\{d_i\}_{i=1}^m$ is non-decreasing. By (WS-1), we see that $d_m \leq \delta_0/2$. Arguing as in Lemma 4.4, it then follows that the numbers d_i satisfy a geometric growth condition and the inner α-mCigar condition for U now follows as before.

As for the implication (ii) $\Rightarrow$ (iii), assume u, v, w are as in the corresponding part of the proof of Theorem 4.1, and let γ be an (α, C_2)-mCigar path from u to w of length L, say. Choosing v_0 so that $|v - v_0|$ exceeds $L/2$, the proof then follows as before. The implication (iii) $\Rightarrow$ (i) follows from Lemma 3.4. $\square$

The assumption $r(U) = r(V) = \infty$ can be weakened in the above theorem, although it cannot be dropped since we shall give counterexamples in the case where only one domain is unbounded (it might suffice for both domains to be unbounded but we cannot prove this). The assumption $r(U) = r(V) = \infty$ can be dropped altogether from the implication (iii) $\Rightarrow$ (i), and for (ii) $\Rightarrow$ (iii) above, it suffices that U and V are both unbounded (but this is hardly more general, since it is easy to see that an inner α-mCigar domain must have infinite inradius if it has infinite inner diameter). Finally for (i) $\Rightarrow$ (ii), the following substitute assumption suffices (we leave to the reader the straightforward task of adapting the proof).

The following condition is satisfied by both $W = U$ and $W = V$ for some constant $c \in (0,1)$: for every $A > 0$, there exists a point $w_0 \in W$ and paths λ^+, λ^- parametrized by arclength and of total length A, such that $\lambda^+(0) = \lambda^-(0) = w_0$, and for every $t \in (0, A]$, the distances from $\lambda^+(t)$ to the image of λ^-, and from $\lambda^-(t)$ to the image of λ^+ are both at least ct.

Of course any domain W satisfying such a condition but having finite inradius is certainly not an inner α-mCigar domain. For a typical example of such a domain, we first let μ^+, μ^- be the Archimidean spirals given in polar coordinates by $\mu^+(\theta) = (\theta, \theta)$, $\mu^-(\theta) = (\theta, \theta + \pi)$, both for all $t \geq 0$, and let W be the planar domain consisting of all points in the unit disk together with all points within a distance $1/10$ of the union of the images of μ^+ and μ^-. Then for each $A > 0$, we can take $w_0 = 0$ and λ^+, λ^- to be suitably reparametrized initial segments of μ^+, μ^-.

For an arbitrary pair of domains $U \subsetneq \mathbb{R}^n$, $V \subsetneq \mathbb{R}^N$, the implications (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) hold (the former because of Lemma 3.4, while the latter is easy), but we now give three counterexamples which show that the other four possible implications fail. In all examples, $\Omega \equiv U \times V$, and U is the open interval $(0,1)$, which is of course an inner α-mCigar domain for every $\alpha \in [0,1)$.

First, we see that $V = (0, \infty)$ is uniform, and so an inner α-mCigar domain for every $\alpha \in [0,1)$. Moreover Ω is simply connected, and so an inner α-wSlice$^+$ domain by Theorem 3.1 and the Riemann mapping theorem. However Ω is not an inner α-mCigar domain for any such α. This neither (i) nor (ii) imply (iii).

Next taking $V = (0,1) \times (0, \infty)$, we see that V is not an inner α-mCigar domain. However Ω is an inner α-wSlice$^+$ domain for every α. In fact if $x, y \in \Omega$ and $|x - y| < 6$, then zero slices suffice. Suppose instead that $|x - y| \geq 6$, with $x_3 < y_3$, where x_3, y_3 are the third coordinates of x, y respectively. Then $y_3 \geq x_3 + 4$ and we take as slices all cylinders $(0,1) \times (0,1) \times (i, i+1)$, $i \in \mathbb{N}$, for which $x_3 + 1 \leq i \leq y_3 - 2$; we leave the verifications to the reader. Thus (i) does not imply (ii).

Finally, $V = (0,\infty)\times(0,\infty)$ is uniform and so an inner α-mCigar domain for every α. However Ω is not an inner α-wSlice domain. In fact for any constant C, the points (u, v_1) and (u, v_2) fail to satisfy an (α, C)-wSlice condition if $u = 1/2$, $v_1 = (t, t)$, $v_2 = (t, 2t)$, and $t \geq t_0$ for some sufficiently large number $t_0 = t_0(C)$. We leave the verification of this to the reader, with the hint that the techniques of Lemma 4.4 can again be adapted to this purpose. Thus (ii) does not imply (i).

6. Open problems

In this final section, we discuss the basic relationships between the various slice[4] conditions. We use the term *zero-point implications* for implications between slice conditions for a fixed pair of points. Note that all slice conditions hold for a fixed pair of points if we choose a sufficiently large constant, so zero-point implications are only of interest if we insist that the implied slice constant depends quantitatively only on the assumed slice constant and other reasonable parameters such as the dimension. We also discuss *one-point implications* involving *one-point slice conditions*, where the slice condition is assumed to be true uniformly for one fixed point $x = x_0$ and all y in the domain; we call the classes of domains satisfying such conditions *one-sided slice domains*. Finally, we discuss *two-point implications* involving two-point slice conditions, where the slice condition is assumed to be true uniformly for all pairs x, y in the domain; as in previous sections, we use the term *slice domains* to refer to the associated domains.

We shall first note some quantitative zero-point implications; these immediately imply the corresponding one- and two-point implications. Most other quantitative zero-point implications will be seen to be false and the corresponding one-point implications are also false. Actually, these facts are essentially equivalent since a counterexample to a one-point implication immediately gives a counterexample to a quantitative zero-point implication, while the opposite direction involves the usual trick of gluing successively worse appendages either to each other or to a central subdomain. By contrast, we have few answers as to whether or not the corresponding two-point implications are true.

As a convenient reference, we include the following diagram of some of the basic quantitative zero-point implications among the various slice conditions that have been used in this paper.

$$\begin{array}{ccccc} \text{Inner Slice} & \Longrightarrow & \text{Inner } \alpha\text{-wSlice} & \Longleftarrow & \text{Inner } \alpha\text{-wSlice}^+ \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{Slice} & \Longrightarrow & \alpha\text{-wSlice} & \Longleftarrow & \alpha\text{-wSlice}^+ \end{array}$$

As mentioned at the end of Section 2, the two left-to-right implications were established in [BS]. The remaining implications are immediate consequences of the definitions. The authors conjecture that the second and third columns of this diagram coincide; see Open Problem A below and the accompanying discussion. Eliminating the third column, the counterexamples in this section together with those in [BS, Section 5] show that the four remaining one-point implications cannot be reversed (and so the zero-point implications cannot be reversed with quantitative dependence).

[4]Below, the term *slice* is used to refer generically to Slice, α-wSlice, α-wSlice$^+$, and all other slice conditions.

In fact, we even have a one-sided Slice domain which is not a one-sided inner α-wSlice (after Open Problem D) and a one-sided inner α-wSlice domain that is not a one-sided Slice domain ([BS] for $\alpha > 0$; the example after Open Problem B for $\alpha = 0$). We also know that the one-sided α-wSlice(+) conditions are incomparable for different values of the slice parameter α (see the example after Open Problem B for one direction and Corollary 4.2 for the other).

At this point, the authors know much less in terms of being able to reverse the two-point versions of the four implications discussed above. We do know that the two left to right implications cannot be reversed when $\alpha > 0$ (a counterexample appears in [BS]) but we conjecture that these arrows can be reversed in case $\alpha = 0$. The same examples in [BS] prove the diagonal non-implications

$$\begin{aligned} \alpha\text{-wSlice} &\not\Longrightarrow \text{Inner Slice}, \qquad \alpha > 0; \\ \text{Inner } \alpha\text{-wSlice} &\not\Longrightarrow \text{Slice}, \qquad \alpha > 0. \end{aligned}$$

But when $\alpha = 0$, we again conjecture that these implications are valid. Below we give some more details on these open questions and related examples.

For any constants C, C', it is not hard to concoct a set of slices for a pair of points x, y that satisfies the (α, C)-wSlicecondition, but not the (α, C')-wSlice$^+$condition. For instance, let us begin with annular slices $\{S_i\}_{i=1}^m$, as given by Lemma 3.4, for a pair x, y in a ball. Cut each annulus S_i into $2N$ equally thin subannuli for some $N \in \mathbb{N}$, and redistribute each of these subannuli in alternating order into two new slices S_i' and S_i''. The set of new slices $\{S_i', S_i''\}_{i=1}^m$ still satisfy the α-wSlice condition (although we must double the size of C to ensure (WS-1)) but no longer satisfy the extra wSlice$^+$ conditions with any given constant if N is very large. However in this and all other examples we have constructed, there always exists a "better" set of slices which demonstates that the pair x, y satisfies an α-wSlice$^+$condition. We suspect that in fact that the logically weaker α-wSlice condition implies the α-wSlice$^+$ condition quantitatively, but this seems hard to prove.

Open Problem A. If the pair $x, y \in G \subset \mathbb{R}^n$ satisfies an (α, C)-wSlice condition, show that it also satisfies an (α, C')-wSlice$^+$ condition for some $C' = C'(C, \alpha, n)$.

Let $0 \leq \alpha, \beta < 1$. For the class of α-mCigar domains to contain the class of β-mCigar domains it is necessary and sufficient that $\alpha \geq \beta$; see [L] and [BK2]. One might suspect that an analogous result might be true for wSlice$^+$ (or wSlice) domains. Indeed, Corollary 4.2 gives us necessity. We suspect that sufficiency is also valid, but proving this appears to be difficult.

Open Problem B. Suppose $0 \leq \beta < \alpha < 1$. Show that a (β, C)-wSlice$^+$ domain is an (α, C')-wSlice$^+$ domain for some $C' = C'(C, \alpha, \beta, n)$.

The analogous result for mCigar domains is rather easy. In fact, an β-mCigar condition for a fixed pair $u, v \in G$ implies an (α, C')-mCigar condition for u, v, with $C' = C'(C, \alpha, \beta, n)$, as can be seen from the proof of [BK2, Proposition 2.4]. By contrast, the wSlice$^+$ variant for a fixed pair of points cannot be true. Indeed, given $0 < \beta < \alpha < 1$, we now describe a bounded domain $G \subset \mathbb{R}^3$ which is a one-sided β-wSlice$^+$ domain (with respect to $x_0 \in G$), but it is not a one-sided α-wSlice$^+$ domain (with respect to x_0). It is not hard to modify this example to handle also the case $\beta = 0$.

Our counterexample $G \subset \mathbb{R}^3$ is got by gluing together a sequence of open rectangular boxes F_n, L_n ($n \geq 0$) of dimensions $R_n \times R_n \times r_n$ and $S_n \times s_n \times s_n$, respectively, where $R_n = 2^{-n}$, $r_n = 2^{-n(1-\alpha)^{-1}}$, $S_n = 2^{-2n}$, $s_n = 2^{-n((1-\alpha)^{-1}+(1-\beta)^{-1})}$; note that for large n, r_n is much smaller than R_n and s_n is much smaller than S_n so that F_n is a flat box and L_n is a long box. For each n, we choose a line segment of length R_n (and S_n) linking the centers of opposite faces of F_n (and L_n, respectively) and call this the main axis of this box. G is then defined by gluing these boxes together according to the order $F_0, L_0, F_1, L_1, \ldots, F_n, L_n, \ldots$, so that all the main axes line up to form a single main axis (of symmetry) for G. Let $f_{k,t}$ and $l_{k,t}$ denote the $d_{t,G}$-length of the main axis of F_k and L_k respectively, for $k \geq 0$.

We claim that $l_{k,\beta} \approx f_{k,\beta} \approx 2^{k(\alpha-\beta)/(1-\alpha)}$, whereas $f_{k,\alpha} \approx 1$ and $l_{k,\alpha} \approx 2^{-k(\alpha-\beta)/(1-\beta)}$. Let us first consider $l_{n,\beta}$, for large n. It is easy to see that if define a truncated box by chopping off a cube (of sidelength s_n) from both ends of L_n, then the $d_{\beta,G}$-length of the part of the main axis lying in the truncated box is comparable to $s_n^{\beta-1} S_n \approx l_{n,\beta}$. The length of the parts of the axis that were chopped off is at most comparable to

$$\int_0^{r_n} t^{\beta-1}\, dt \approx r_n^\beta,$$

which is much smaller. The estimate for $l_{n,\alpha}$ is similar. For $f_{n,\alpha}$ and $f_{n,\beta}$, the estimates are derived in a similar fashion once we chop off a box of size $r_n \times R_n \times r_n$ from both ends of F_n in such a way that these little boxes cover the ends of the main axis of F_n. This establishes our claim.

The choice of x_0 is not important; we may as well take it to be the center of F_0. There is no difficulty in choosing slices for x_0, x when $x \in F_n \cup L_n$ for some small n, since $d_{t,G}(x, x_0)$ is bounded in such cases ($t = \alpha$ or $t = \beta$), so zero slices will suffice. Let us look at the case where $x = x_n$ is the center of F_n for large n; it is easy to adjust the arguments to handle other points. Notice that the $d_{\beta,G}$ geodesic γ_n from x_0 to x_n is simply $[x_0 \to x_n]$. For the β-wSlice$^+$ condition, we slice up the boxes L_k, $0 \leq k < n$, perpendicular to their main axes into cubes of sidelength s_n, discarding any remnant at one end of L_i which is too small to make another cube. Gathering together all these slices, it is easy to see that (WS-1$^+$), (WS-3), and (WS-5) hold. Almost all the $d_{\beta,G}$-length of $\gamma_n \cap L_k$, $0 \leq k < n$, lies in some slice. Since also $l_{k,\beta} \approx f_{k,\beta}$, (WS-2) follows easily.

Suppose for the purposes of contradiction that an α-wSlice$^+$ condition also holds for the pair x_0, x_n, uniformly in n. We show that this is untenable for large n. This is rather tricky but the idea is simple: flat boxes, unlike long boxes, cannot be "nicely sliced", which causes a problem since most of the $d_{\alpha,G}$-distance between x_0 and x_n consists of flat boxes.

We denote by F_k' and L_k' the parts of a box F_k or L_k, respectively, that lie within a distance $s_k/2$ of a face of that box that is glued to another box, and by T_k^- and T_k^+ the transitional part of $\overline{L_k \cup F_k} \cap G$ or $\overline{L_k \cup F_{k+1}} \cap G$ that lies within a distance s_k of a glued face of one of its component boxes. We first modify the slices so that there only two types of slices: *nice slices* which are contained in a single F_k' or L_k', and *transitional slices* that are contained in either T_k^- or T_k^+ for some k. This can be done (with a controlled change in the slice constant) by replacing each original slice S with $S \cap F_k$, $S \cap L_k$, $S \cap T_k^-$, or $S \cap T_k^+$, for some k; we leave the details to the reader.

Let us fix a box B from among the boxes intersecting $[x_0, x_n]$. Take x to be the point in the box to the immediate left of B which lies on $[x_0, x_n]$ and whose Euclidean distance from B equals $r(B)$ (note that $r(B)$ is r_k or s_k for some $0 \leq k \leq n$, depending on whether B is a flat or a long box), and take y to be the corresponding point in the box to the immediate right of B. There are two endpoint cases where these definitions do not make sense: if $B = F_0$, instead let $x = x_0$ and if $B = F_n$, instead let $y = x_n$. The $d_{\alpha,G}$-length of the line segment joining x and y is easily seen to be comparable to the $d_{\alpha,G}$-length of the main axis of B, which we call $l_\alpha(B)$ for short. By construction, the nice slices in B also satisfy (WS-1) and (WS-3) for the pair x, y, so Lemma 2.1 implies that the contribution to the sum in (WS-2) of the numbers d_i that correspond to these slices is at most some constant multiple of $l_\alpha(B)$. Similarly, for large n, the transitional slices between two adjacent boxes B_1, B_2 cannot contribute more than a small multiple of $l_\alpha(B_1) + l_\alpha(B + 2)$.

Since $f_{k,\alpha} \approx 1$ is much larger than $l_{k,\alpha}$ for large k, the last estimates imply that the contributions of the nice slices contained in F_k must be bounded below, at least for some fixed fraction of the numbers $0 \leq k \leq n$. However (WS-1) implies that nice slices in F_k must have diameter comparable with R_k. It follows that their number is bounded and that their total contribution can be at most comparable with R_k^α. Since R_k^α is much smaller than 1, we get a contradiction.

Note that above we have only used the wSlice conditions, not (WS-1$^+$) or (WS-5), so as to emphasise that the peculiarity of this example is not because of the latter extra conditions. The proof that an α-wSlice$^+$ condition does not uniformly hold for pairs x_0, x_n is a little easier if we use (WS-5). Also note that G is not a β-wSlice$^+$ domain, as can be shown by considering the β-wSlice$^+$ condition for points near either end of F_n for large n.

Open Problem C. Show that a $(0, C)$-wSlice domain is a C'-Slice domain for some $C' = C'(C, n)$.

According to Corollary 4.2, the classes of α-wSlice$^+$ domains are distinct for all $\alpha > 0$, and according to [BS, Proposition 4.5] there are domains that are α-wSlice$^+$ domains for all $\alpha > 0$, but not Slice domains. However, even if the first two open problems can be made into theorems, Open Problem C remains unresolved. Furthermore, taking $0 = \beta < \alpha = 1/2$, the counterexample G to the one-point variant of Open Problem B is also a counterexample to the one-point variant of this problem since, as mentioned in Section 2, any Slice condition implies an α-wSlice condition quantitatively.

Open Problem D. Suppose $0 \leq \alpha < 1$. Show that an (α, C)-wSlice domain (or C-Slice domain) is an inner (α, C')-wSlice domain (or inner C'-Slice domain, respectively) for some $C' = C'(C, n)$.

Note that if this can be shown then the class of $(\alpha; d)$-wSlice domains is the same for every metric d lying between the Euclidean and inner Euclidean metrics.

Yet again, there are counterexamples for the one-point variant of this problem. Consider for example the planar domain $G = (0, 1)^2 \cup (\bigcup_{k=1}^{\infty} R_k)$, where

$$R_k = \left((2^{-k} - 2^{-sk}, 2^{-k} + 2^{-sk}) \times [1, 1 + 2^{-k})\right) \setminus \left(\{2^{-k}\} \times [1, 1 + 2^{-k-1}]\right)$$

for some $s > 2$; note that G consists of the unit square with disjoint narrow slitted rectangles attached. Taking $u_0 = (1/2, 1/2)$ and v to be arbitrary, we claim that the pair u_0, v satisfies any

of the Euclidean slice conditions with a constant independent of $v = (v_1, v_2)$, but that it does not uniformly satisfy any inner Slice condition, nor any inner α-wSlice condition if $s > 1/(1-\alpha)$.

In the positive direction, we sketch only the α-wSlice$^+$ condition for $\alpha > 0$; the case $\alpha = 0$ and the Slice condition are left as exercises. The cases where $v \in (0,1)^2$, or $v \in R_k$ with $v_2 - 1 \lesssim 2^{-sk}$, are easily handled since $d_{\alpha,G}(u_0, v)$ is then bounded so u_0, v satisfy an α-wSlice$^+$ condition with zero slices. Suppose instead that $v \in R_k$ and $v_2 > 1 + 4 \cdot 2^{-sk}$. For each $i \in \mathbb{N}$ define

$$S_i \equiv R_k \cap (\mathbb{R} \times (1 + 2^{-sk+1}(i-1), 1 + 2^{-sk+1} i)), \qquad i \in \mathbb{N}.$$

Letting $\gamma \in \Gamma_G(u_0, v)$ be such that $\mathrm{len}_{\alpha,G}(\gamma) < 2 d_{\alpha,G}(u_0, v)$, and letting m be the integer such that $v \in S_{m+2}$, it is straightforward to verify a uniform α-wSlice$^+$ condition for u_0, v with slice data γ, $\{S_i, \mathrm{dia}(S_i)\}_{i=1}^m$.

For the negative results, it suffices to show that for every $\alpha \in [0,1)$, $s > 1/(1-\alpha)$, and $C > 1$, there always exists $v \in G$ such that the pair $u_0 = (1/2, 1/2), v$ fails to satisfy the inner (α, C)-wSlice condition. We consider only the case $\alpha > 0$; the case $\alpha = 0$ is left as an exercise. We write $v_k = (2^{-k} + 2^{-sk-1}, 1 + 2^{-k-1})$, $k \in \mathbb{N}$. We claim that if the data γ, $\{S_i, d_i\}_{i=1}^m$ satisfies (WS-1) and (WS-3) for the pair $u_0, v_k \in G$, and $d_i \geq d_G(S_i)$, then $\Sigma \equiv \sum_{i=1}^m d_i^\alpha \lesssim 1$. Since $d_{\alpha,G}(u_0, v_k) \approx 2^{k(s(1-\alpha)-1)}$ grows arbitrarily large as $k \to \infty$, it follows from this claim that pairs u_0, v_k cannot uniformly satisfy any inner α-wSlice condition.

We may as well assume that the slices S_i are contained in $(0,1)^2 \cup R_k$, since if we remove those parts of S_i lying in R_j, $j \neq k$, it follows that (WS-1) must still be true with the same constant C. Let Σ_1 be the subsum of Σ corresponding to those slices contained entirely in

$$A_k \equiv G \cap [\mathbb{R} \times (0, 1 + 2^{-sk+1})].$$

The subset of slices contained in A_k, together with the corresponding numbers d_i, forms a set of data satisfying (WS-1) and (WS-3) for the pair of points u_0, v_k', with $v_k' = (2^{-k} + 2^{-sk-1}, 1 + 2^{-sk+2})$. Since $d_{\alpha,G}(u_0, v_k') \lesssim 1$, it follows from Lemma 2.1 that $\Sigma_1 \lesssim 1$.

Next let Σ_2 be the subsum corresponding to those slices contained entirely in R_k. Since we can move from u_0 to v_k by going up either side of the slit in R_k, an argument similar to that used in the proof of Theorem 5.1 shows that the numbers d_i satisfy a geometric growth condition. It readily follows that $\Sigma_2 \lesssim 1$.

Finally, we consider slices S_i that intersect both $(0,1)^2$ and $R_k \setminus A_k$. If by replacing S_i by $S_i \cap A_k$ (but leaving d_i unchanged) we get a would-be slice that satisfies (WS-1) with C replaced by $2C$, then we can include the term d_i^α in Σ_1. Assume instead that $S_i \cap A_k$ fails to satisfy (WS-1) even with C replaced by $2C$. Let $w_0 = (x_0, 1)$ be the point of first entry into R_k of a path λ_0 for which this version of (WS-1) fails. Since $\mathrm{len}(\lambda_0 \cap S_i) \geq d_i/C$ and $\mathrm{len}(\lambda_0 \cap S_i \cap A_k) < d_i/2C$, it follows that

$$\mathrm{len}(\lambda_0 \cap S_i \cap (0,1)^2) < \frac{d_i}{2C} - 2^{-sk+1}.$$

Suppose that there exists $\lambda_1 \in \Gamma_G(u_0, v_k)$ such that $\mathrm{len}(\lambda_1 \cap S_i \cap R_k) < d_i/2C$. We define a path $\lambda \in \Gamma_G(u_0, v_k)$ as follows: λ coincides with λ_0 as far as the point w_0, then it traverses a path in $G \cap [\,(0,1) \times (0,1]\,]$ of length at most 2^{-sk+1} from w_0 to the last point of entry of λ into R_k, and finally it traverses the final segment of λ_1. Such a path λ would satisfy $\mathrm{len}(\lambda \cap S_i) < d_i/C$ in contradiction to (WS-1). Thus if we replace S_i by $S_i \cap R_k$, and C by $2C$, then (WS-1) is still satisfied, and so we may include the term d_i^α in Σ_2. There are no remaining terms, so our claim is proved and we are done.

References

[BHK] M. Bonk, J. Heinonen, and P. Koskela, *Uniformizing Gromov hyperbolic spaces*, preprint.

[BK1] S.M. Buckley and P. Koskela, *Sobolev-Poincaré implies John*, Math. Research Letters **2** (1995), 577–593.

[BK2] S.M. Buckley and P. Koskela, *Criteria for Imbeddings of Sobolev-Poincaré type*, Internat. Math. Res. Notices (1996), 881–901.

[BS] S.M. Buckley and A. Stanoyevitch, *Weak slice conditions and Hölder imbeddings*, preprint.

[GM] F.W. Gehring and O. Martio, *Lipschitz classes and quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 203–219.

[GO] F.W. Gehring and B. Osgood, *Uniform domains and the quasihyperbolic metric*, J. Analyse Math. **36** (1979), 50–74.

[GP] F.W. Gehring and B. Palka, *Quasiconformally homogeneous domains*, J. Analyse Math. **30** (1976), 172–199.

[GV] F.W. Gehring and J. Väisälä, *The coefficients of quasiconformality of domains in space*, Acta Math. **114** (1965), 1–70.

[HK] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), 1–61.

[L] V. Lappalainen, *Lip_h-extension domains*, Ann. Acad. Sci. Fenn. Ser. A I Math. Diss. **56** (1985), 1–52.

[R] S. Rickman, *Quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **13** (1988), 371-385.

[V1] J. Väisälä, *On the null-sets for extremal distances*, Ann. Acad. Sci. Fenn. Ser. A I **322** (1962), 12pp.

[V2] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics 229, Springer-Verlag, Berlin, 1970.

[V3] J. Väisälä, *Quasimöbius maps*, J. Analyse Math. **44** (1984/85), 218–234.

[V4] J. Väisälä, *Quasiconformal mappings of cylindrical domains*, Acta Math. **162** (1989), 201–225.

[V5] J. Väisälä, *Relatively and inner uniform domains*, Conf. Geom. Dyn. **2** (1998), 56–88.

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