Manifolds Carrying Large Scalar Curvature

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Abstract

Let $W = S \otimes E$ be a complex spinor bundle with vanishing first Chern class over a simply connected spin manifold M of dimension ≥ 5. Up to connected sums we prove that W admits a twisted Dirac operator with positive order-0-term in the Weitzenböck decomposition if and only if the characteristic numbers $\hat{A}(TM)[M]$ and $\mathrm{ch}\,(E)\hat{A}(TM)[M]$ vanish. This is achieved by generalizing [2] to twisted Dirac operators.

1 Introduction

A key point in the Lichnerowicz argument, showing that the $\hat{A}$ genus is an obstruction to the existence of a metric with positive scalar curvature, is the fact that the scalar curvature appears as the order-0-term in the Weitzenböck decomposition of the ordinary Dirac Laplacian $\mathcal{D}^2$. It was shown in [2], [8] that positive scalar curvature can be preserved under surgeries in codimension ≥ 3. Within the class of simply connected spin manifolds of dimension ≥ 5 the cobordism relation is generated by surgeries of this type. Therefore all such manifolds admitting a metric of positive scalar curvature could be determined by computations in the spin cobordism ring (see [2], [8], [6], [7]).

Here we extend this to general Dirac operators (see [1], [4]). The role of scalar curvature is taken by the order-0-term in the Weitzenböck decomposition of a twisted Dirac operator. This term is positive if the scalar curvature is larger than a certain norm of the curvature endomorphism of the coefficient bundle. First we prove a surgery theorem for the order-0-term in the the Weitzenböck decomposition of twisted Dirac Laplacians $\mathcal{D}^2_\nabla$ (Theorem 1). Next we consider complex spinor bundles with trivial first Chern class over simply connected spin manifolds of dimension ≥ 5. Up to connected sums, we determine all spinor bundles within this class, which admit a Dirac operator with positive order-0-term in its Weitzenböck decomposition (Theorem 2). This is done by a computation in the cobordism ring $\sum_{n,k} \Omega_n^{spin}(BSU(k)) \otimes \mathbb{Q}$.

2 Statement of Results

Let W be a complex spinor bundle over a spin manifold M. Then W is a twisted spinor bundle $W = S \otimes E$, where S is the spinor bundle associated to the irreducible representation of the Clifford algebra and E is a complex vector bundle, see [1], [4]. To a Riemannian metric g on M and a Hermitian connection ∇ on E there is naturally associated the twisted Dirac operator $\mathcal{D}_\nabla$ acting on sections of W. The Weitzenböck decomposition of its Dirac laplacian $\mathcal{D}_\nabla^2$ reads ([1], [4])

$$\mathcal{D}_\nabla^2 = D^* D + \frac{1}{4} s + \sum_{i,j} e_i e_j \otimes R_{e_i, e_j} \ ,$$

the sum being taken over an orthonormal basis $\{e_i\}$ of the tangent space of M. Here D is the covariant derivative on W induced from the connection ∇ and the Levi-Civita connection on M. By s we denote the scalar curvature of M and by R the curvature tensor of ∇. We also define $\mathcal{E}(\nabla) := 4 \sum_{i,j} e_i e_j \otimes R_{e_i, e_j}$ and $\|\mathcal{E}(\nabla)\|(x)$ to be minus the smallest eigenvalue of the bundle endomorphism $\mathcal{E}(\nabla)$ at the point $x \in M$.

Assume additionally that M is simply connected, $\dim M \geq 5$ and $c_1(E) = 0$. We will show that rationally, i.e. after eventually passing to a suitable connected sum multiple of (M, E), the following are equivalent:

1. *M admits a Riemannian metric g and E a Hermitian connection ∇, such that $s(g) > \|\mathcal{E}(\nabla)\|$ on M. We will then say that (M, E) admits* large scalar curvature*.*
2. *Both S and W admit an invertible Dirac operator.*
3. *The characteristic numbers $\hat{A}(TM)[M]$ and $\mathrm{ch}(E)\hat{A}(TM)[M]$ vanish.*

As $\|\mathcal{E}(\nabla)\|$ is always nonnegative, the implication (1) $\Rightarrow$ (2) is immediate from the Weitzenböck decomposition. By the index theorem we have (2) $\Rightarrow$ (3). So we are left with the implication (3) $\Rightarrow$ (1).

Therefore we will first extend the surgery theorem for scalar curvature (cf. [2], [8]) to show that positivity of $s + \mathcal{E}(\nabla)$ can be preserved under surgeries of codimension at least 3.

Theorem 1 *Let $E \to M$ be a vectorbundle over the smooth manifold M. Assume that there is a Riemannian metric g on M and a unitary connection ∇ on E with $s(g) > \|\mathcal{E}(\nabla)\|$. If the manifold M' is produced from M by surgery in codimension more than 2 and such that the vector bundle E extends over the trace of the surgery giving a vector bundle E' over M' then there are a Riemannian metric g' on M' and a unitary connection ∇' on E' with $s(g') > \|\mathcal{E}(\nabla')\|$.*

Now we look at simply connected spin manifolds M of dimension $\dim M \geq 5$ endowed with a complex vectorbundle E with vanishing first Chern class. Then E — and the spinor bundle $W = S \otimes E$ — are trivial over embedded 2-spheres. As in [2] we obtain that any cobordism can be replaced by a sequence of surgeries of codimension ≥ 3. Hence we can decide from the cobordism class of (M, E) in $\Omega_n^{spin}(BSU(k))$, whether it admits large scalar curvature. We have

Theorem 2 *Let $E \to M$ be a $SU(r)$-vectorbundle over the smooth simply connected spin manifold M of dimension ≥ 5. Then the following are equivalent:*

1. *For some q the q-fold connected sum $(M, E)\# \ldots \#(M, E)$ carries a metric g and a connection ∇ with $s(g) > \|\mathcal{E}(\nabla)\|$.*
2. *$\hat{A}(TM)[M] = 0$ and $\mathrm{ch}\,(E)\hat{A}(TM)[M] = 0$.*

3 Proof of Theorem 1

Consider surgery on an embedded sphere $S^k \cong S \subset M^{k+l}$, $n = k + l$, with trivial normal bundle and such that the restriction to S of the vector bundle E is trivial. M' is then obtained by cutting out a tubular neighbourhood $f : S^k \times D^l \hookrightarrow M$ of S and glueing back $D^{k+1} \times S^{l-1}$ along the boundary $S^k \times S^{l-1}$. In the end M' will be described as a submanifold of $Z := M \times [0, \delta] \cup_f D^{k+1} \times D^l$

Let $S^k \times D^l$ carry the metric and the connection induced via f from M. We can extend these data to all of $D^{k+1} \times D^l$, such that in the vicinity of the boundary $S^k \times D^l$ they are compatible to a product structure of a collar neighbourhood. The metric and connection on Z are then obtained by glueing this handle $D^{k+1} \times D^l$ with the product metric and connection on $M \times [0, \delta]$.

Let $\varrho \leq \mathcal{R}$ be sufficiently small constants (e.g. less that the injectivity radius of Z) and denote by $d(\cdot, S)$ the distance from S. Define $N_r := \{x \in M \mid d(x, S) \leq r\}$ and $Y_\rho = \partial N_\rho$. If $\rho \leq \mathcal{R}$ then the exponential map provides diffeomorphisms $D^{n-k} \times S^k \cong \rho D\nu(S, M) \to N_\rho$ and $S^{n-k-1} \times S^k \cong \rho S\nu(S, M) \to Y_\rho$. Pick a decreasing real function $\phi(\rho)$ defined for $\rho \geq \varrho$, vanishing for $\rho \geq \mathcal{R}$ and such that all derivatives of its inverse function $\chi = \phi^{-1}$ vanish at $\phi(\varrho)$. Let $\delta := \phi(\varrho)$ and $\psi(x) := \phi(d(x, S))$, $x \in M$. The result of the surgery is

$$M' = \{(m, t) \mid \phi(d(m, S)) = t\} \cup_f \{x \in D^{k+1} \times D^{n-k} \mid d(x, S^k \times D^{n-k} = \varrho\} \; .$$

We will show that one can find ϱ and ϕ such that on M' we have $s - \mathcal{E}$ positive.

The calculations in 3.1 are much the same as in [2] and merely included for the reader's convenience.

3.1 Scalar Curvature of M'

M' is glued together from the graph X of ψ on $M \setminus N_\varrho$ and a handle. We express the scalar curvature of $X \subset M \times \mathbb{R}$ at (m,t) in terms of the second fundamental form T of the submanifolds $\psi^{-1}(t) \subset M$ at m. This is a straightforward calculation based on the Gauß equation.

Denote the derivation in direction of the $\mathbb{R}$-factor by ∂_t and the gradient of ψ by $\partial\psi$. Let $r := -\partial\psi/|\partial\psi| = \partial\psi/\phi'$ and $\hat{n} := (-\partial\psi, \partial_t)/\sqrt{1+|\partial\psi|^2} = (-\phi' r, \partial_t)/\sqrt{1+\phi'^2}$ be the normal unit vectors to $\psi^{-1}(t)$ and X respectively. For a vector $v \in T_m M$ define $\overline{v} := (v, v(\psi)\,\partial_t) \in T_{(m,\psi(m))}X$.

At a point $(m,t) \in X$ choose an orthonormal basis $v_1, \ldots, v_{n-1}$ of the orthogonal complement of the gradient $\partial\psi$ in $T_m M$. We work in the orthonormal basis

$$\left(\overline{v}_1, \ldots, \overline{v}_{n-1}, \frac{\overline{\partial\psi}}{|\overline{\partial\psi}|} = -\frac{(r, \phi'\,\partial_t)}{\sqrt{1+\phi'^2}} \right)$$

of $T_{(m,t)}X$.

First we compare the second fundamental form T of the submanifolds $\psi^{-1}(t) \subset M$ at m with the second fundamental form $\overline{T}$ of $X \subset M \times \mathbb{R}$ at (m,t). For v, w perpendicular to $\partial\psi$ we obtain

$$\begin{aligned}
\overline{T}(\overline{v}, \overline{w}) &= \langle \nabla_{\overline{v}} \overline{w} \mid \hat{n} \rangle = \langle \nabla_v w \mid (-\phi' r) \rangle / \sqrt{1+\phi'^2} \\
&= T(v,w) \frac{-\phi'}{\sqrt{1+\phi'^2}} \\
\overline{T}\left(\overline{v}, \overline{\partial\psi}/|\overline{\partial\psi}|\right) &= -\langle \nabla_{\overline{v}} (r, \phi'\,\partial_t) \mid (-\phi' r, \partial_t) \rangle / \left(1+\phi'^2\right) \\
&= \left(\phi' \langle \nabla_v r \mid r \rangle - v(\phi')\right) / \left(1+\phi'^2\right) \\
&= \left(\phi' v(|r|^2)/2 - v(\phi')\right) / \left(1+\phi'^2\right) = 0 \\
\overline{T}\left(\overline{\partial\psi}/|\overline{\partial\psi}|, \overline{\partial\psi}/|\overline{\partial\psi}|\right) &= \left\langle \nabla_{(r,\phi'\,\partial_t)} (r, \phi'\,\partial_t) \mid (-\phi' r, \partial_t) \right\rangle / \left(1+\phi'^2\right)^{3/2} \\
&= \left(\langle \nabla_r r \mid -\phi' r \rangle + r(\phi')\right) / \left(1+\phi'^2\right)^{3/2} \\
&= \frac{\phi''}{\left(1+\phi'^2\right)^{3/2}}
\end{aligned}$$

The Gauss formula then yields for the sectional curvature $\overline{K}$ of the submanifold X:

$$\begin{aligned}
\overline{K}(\overline{v}, \overline{w}) &= K(v,w) + \frac{\phi'^2}{1+\phi'^2} \left(T(v)T(w) - T(v,w)^2\right) \\
\overline{K}\left(\overline{v}, \overline{\partial\psi}/|\overline{\partial\psi}|\right) &= K^{M\times\mathbb{R}}\left(v, \overline{\partial\psi}/|\overline{\partial\psi}|\right) - \frac{\phi'\phi''}{\left(1+\phi'^2\right)^2} T(v)
\end{aligned}$$

$$= \frac{1}{1+\phi'^2} K(v,r) - \frac{\phi'\phi''}{\left(1+\phi'^2\right)^2} T(v)$$

Taking sums over the basis above we end up with

$$\begin{aligned} \overline{s} &= s + \frac{\phi'^2}{1+\phi'^2} \sum_{i,j} \left(T(v_i)T(v_j) - T(v_i,v_j)^2 \right) \\ &\quad - \frac{\phi'^2}{1+\phi'^2}\, 2 \sum_i K(v_i,r) \qquad\qquad (3.1) \\ &\quad - \frac{\phi'\phi''}{\left(1+\phi'^2\right)^2}\, 2 \sum_i T(v_i) \ . \end{aligned}$$

We will need the asymptotic behaviour when approaching S of the functions on $N_{\mathcal{R}} \setminus N_{\varrho}$ defined by the sums in (3.1):

Lemma 3.2 *As $\rho = d(x,S) \to 0$ the asymptotic behaviour of the functions $\mathcal{A} := \sum_{i,j}(T(v_i)T(v_j) - T(v_i,v_j)^2) = (\mathrm{Tr}\, T)^2 - \mathrm{Tr}\, T^2$, $\mathcal{B} := -2\ \sum_i T(v_i) = 2\ \mathrm{Tr}\, T$ and $\mathcal{C} := 2\ \sum_i K(v_i,r) = 2\ \mathrm{Ric}\,(r)$ is*

$$\mathcal{A}(x) = a_2\ \rho^{-2} + a_1(x)\ \rho^{-1} + a_0(x), \quad \mathcal{B}(x) = b_1 \rho^{-1} + b_0(x)\ ,$$

with bounded functions $a_1(x)$, $a_0(x)$ and $b_0(x)$ and positive constants a_2 and b_1. $\mathcal{C}$ also extends to a bounded function on $N_{\mathcal{R}}$.

In fact since the codimension l of the submanifold S is ≥ 3 we have $a_2 = (l-1)(l-2)/2 > 0$ and $b_1 = l - 1 > 0$.

Proof: Consider the diffeomorphism $S \times \mathbb{R}^l = \nu(S,M) \to N_{\mathcal{R}}$ given by the exponential map i.e. mapping $(p,v) \mapsto \exp_p v$. For unit speed curves $p(t)$ in S and A_t in $SO(l)$ define vectorfields $h = \frac{d}{dt} \exp_{p(t)} v$, $u = \frac{d}{dt} \exp_p A_t v$, $\tilde{r} = \frac{d}{dt} \exp_p tv$. Then for small $\rho = |v|$ expand $|u| = \rho + b_u \rho^2$ and $|\tilde{r}| = \rho + b_r \rho^2$ with smooth functions b_u, b_r. We compute

$$T(\frac{u}{|u|}) = \frac{1}{|u|^2 |\tilde{r}|} \left\langle \nabla_u u \mid \tilde{r} \right\rangle = -\frac{1}{2|u|^2|\tilde{r}|} \tilde{r}(|u|^2)$$

because u and $\tilde{r}$ commute and are mutually perpendicular. Since $r = \tilde{r}/\left|\tilde{r}\right| = \frac{\partial}{\partial \rho}$, we infer from the asymptotics of $|u|$, that this is

$$\frac{-1}{2|u|^2} \frac{\partial}{\partial \rho} \left(\rho + b_u \rho^2 \right)^2 = -\frac{1}{\rho} + O(1)\ .$$

A similiar computation shows that $T(h/|h|)$ and $T(u/|u|, h/|h|)$ are bounded. The Lemma then follows from polarisation. •

The scalar curvature of Y_ρ is also obtained from the Gauß formula (substitute $\phi'' = 0$ and $\phi' = \infty$ in (3.1)). Hence for small ρ we get:

$$s^{Y_\rho} = s + \mathcal{A} - \mathcal{C} = a_2\ \rho^{-2} + a_1\ \rho^{-1} + a_0 - \mathcal{C}\ .$$

3.2 The Curvature Endomorphism

The manifold X can also be viewed as obtained from $M \setminus N_\varrho$ by blowing up the metric in direction of r. More precisely X is isometric to $(M \setminus N_\varrho, \overline{g})$ with

$$\overline{g}(v,w) := g(v,w) + g(v,\partial\psi)g(\partial\psi,w) = g(v,w) + |\partial\psi|^2 g(v,r)g(r,w)\ .$$

Especially the length of r becomes $\sqrt{1+\phi'^2}$. The transition matrix between the metrics g and $\overline{g}$ gives an isomorphism between the spinor bundles of (M,g) and of $(M,\overline{g})$. The pull back via this isomorphism of the curvature endomorphism of $(M,\overline{g})$ to the spinor bundle over (M,g) is:

$$\begin{aligned} \overline{\mathcal{E}} &= 4\sum_{i,j} v_i v_j \otimes R_{v_i,v_j} + \frac{4}{\sqrt{1+\phi'^2}}\sum_i r v_i \otimes R_{r,v_i} \\ &= \mathcal{E} - 4\left(1 - \frac{1}{\sqrt{1+\phi'^2}}\right)\sum_i r v_i \otimes R_{r,v_i} \end{aligned} \tag{3.3}$$

and its smallest eigenvalue is estimated by

$$\begin{aligned} \left\|\overline{\mathcal{E}}\right\| &\leq \|\mathcal{E}\| + 4\left(1 - \frac{1}{\sqrt{1+\phi'^2}}\right)\left\|\sum_i r v_i \otimes R_{r,v_i}\right\| \\ &\leq \|\mathcal{E}\| + 4\,\frac{\phi'^2}{1+\phi'^2}\left\|\sum_i r v_i \otimes R_{r,v_i}\right\| \end{aligned} \tag{3.4}$$

Herein $\mathcal{D} := 4\ \|\sum_i r v_i \otimes R_{r,v_i}\|$ extends to a bounded function on $N_\mathcal{R}$.

3.3 Solution of The Differential Inequality

Finally we need to solve the differential estimate $\overline{s} - \left\|\overline{\mathcal{E}}\right\| > 0$. From (3.1), (3.4) and Lemma 3.2 we infer that

$$\begin{aligned} \overline{s} - \left\|\overline{\mathcal{E}}\right\| &\geq s - \|\mathcal{E}\| + \frac{\phi'^2}{1+\phi'^2}\left(\mathcal{A} - \mathcal{D} - \mathcal{C}\right) + \frac{\phi'\phi''}{\left(1+\phi'^2\right)^2}\mathcal{B} \\ &= s - \|\mathcal{E}\| + \frac{\phi'^2}{1+\phi'^2}\left(a_2\rho^{-2} + a_1(x)\rho^{-1} + a_0(x) - \mathcal{D} - \mathcal{C}\right) \\ &\quad + \frac{\phi'\phi''}{\left(1+\phi'^2\right)^2}\left(b_1\rho^{-1} + b_0(x)\right) \end{aligned}$$

So we have solved the problem on X if we can find a decreasing function ϕ on $[\varrho, \mathcal{R}]$ such that this expression is positive. Furthermore we need that ϕ vanishes identically near $\mathcal{R}$ and that all derivatives of its inverse function $\chi = \phi^{-1}$ vanish at $\phi(\varrho)$ so that X will inherit a product metric and connection near its boundary.

Eventually after taking an even smaller value of $\mathcal{R}$, we pick positive constants a, b such that on $N_{\mathcal{R}}$ the estimates $a\rho^{-2} \leq a_2\rho^{-2} + a_1(x)\rho^{-1} + a_0(x) - \mathcal{D} - \mathcal{C}$ and $b \leq b_1\rho^{-1} + b_0(x)$ hold. Furtermore let $\epsilon := \min(s - \|\mathcal{E}\|) > 0$. Then it suffices to solve

$$\epsilon + \phi'^4 a\rho^{-2} + \phi'\phi'' b\rho^{-1} > 0 \tag{3.5}$$

Consider the the differential equation $\phi'^4\rho^{-2}\ a/2 + \phi'\phi''\rho^{-1}\ b = 0$ and its solutions

$$\phi_C(\rho) = \int_\rho^{\mathcal{R}} \frac{1}{\sqrt{\frac{a}{b}\log x + C}}\ dx$$

defined for $\rho \geq \varrho := e^{-Ca/b}$ for some $C \in \mathbb{R}$. For a sufficiently large value of C we can find a decreasing solution of $\epsilon + \phi'\phi'' b\rho^{-1} > 0$ in the intervall $[\mathcal{R}/2, \mathcal{R}]$ which vanishes identically near $\mathcal{R}$ and extends ϕ_C smoothly from $[\varrho, \mathcal{R}/2]$ to $[\varrho, \mathcal{R}]$ to ensure the proper boundary condition at $\rho = \mathcal{R}$. At the other boundary (3.5) for the inverse function χ reads $\epsilon\chi^2\chi'^4 + a - b\chi\chi'' \geq 0$. Let χ_C be the inverse function of ϕ_C on $[0, \phi(\varrho)]$ extended by the constant ϱ to all of $\mathbb{R}^+$. Then we have $a/2 - b\chi_C(y)\chi_C''(y) = 0$ for all $y \neq \phi(\varrho)$. But χ_C can clearly be smoothed keeping $a - b\chi(y)\chi''(y) \geq 0$.

3.4 The result of glueing

In the above we could make ϱ arbitrarily small. By the remark after Lemma 3.2 we thus may assume $s - \|\mathcal{E}\|$ positive on the handle $H_\varrho := \{x \in D^{k+1} \times D^l \mid d(x, D^{k+1} \times \varrho S^{l-1}) = \varrho\}$. Since both X and $D^{k+1} \times D^l$ were produced to carry product metric and connection near their boundary, we can glue $M' = X \cup H_\varrho$ metrically and obtain the desired metric and connection over M'. This proves theorem 1. •

4 Proof of Theorem 2

We will exhibit representatives (M, E) admitting large scalar curvature in every cobordism class in $\Omega_n^{spin}(BSU(r)) \otimes \mathbb{Q}$ with vanishing characteristic numbers $\hat{A}(TM)[M]$ and $\mathrm{ch}\,(E)\hat{A}(TM)[M]$. In the sequel cobordism classes will always be understood rationally, i.e. tensored with $\mathbb{Q}$, but this will be supressed in the notation. We will produce suitable generators of $\Omega_n^{spin}(BSU(r))$ first.

This vectorspace is trivial for n odd. The cobordism classes $X = (M, E) \in \Omega_n^{spin}(BSU(r))$ are detected by the characteristic numbers

$$c_J p_I(X) = c_J(E) p_I(TM)[M]\ ,$$

where $c_J = c_r^{j_r} \cdots c_2^{j_2}$ and $p_I = p_r^{i_s} \cdots p_1^{i_1}$ for $J = (j_r, \ldots, j_2)$, $I = (i_s, \ldots, i_1)$ with $2(rj_r + \cdots + 2j_2) + 4(si_s + \cdots i_1) = n$. First we will define $X^n(J) \in$

$\Omega^{spin}_{2n}(BSU(r))$ such that the matrix $(c_{J'}(X^n(r,J)))_{J',J}$ has full rank. We will construct appropriate bundles over products of the sphere S^2 and the complex projective spaces $\mathbb{C}\mathrm{P}^{2n+1}$:

For $J=(j_r,\dots,j_2)$ with $\sum_{i=2}^r ij_i = n$ we define $(r\times n)$-matrices M^r_J. If $r\geq 4$ let

$$M^r_J := \left(\begin{array}{ccccc} \underbrace{\begin{matrix}1&\cdots&1\\-1&\cdots&-1\end{matrix}}_{j_r} & & & 0 & \\ & \underbrace{\begin{matrix}1&\cdots&1\\-1&\cdots&-1\end{matrix}}_{j_r+j_{r-1}} & & & \\ & & \ddots & & \\ & 0 & & \underbrace{\begin{matrix}1&\cdots&1\\-1&\cdots&-1\end{matrix}}_{j_r+j_{r-1}+\ldots+j_3} & \\ & & & & \underbrace{\begin{matrix}1&\cdots&1\\-1&\cdots&-1\end{matrix}}_{j_r+j_{r-1}+\ldots+j_3+2j_2} \end{array}\right)$$

For $r=3$ and $J=(j_3,j_2)$ with $j_3>1$ let

$$M^3_J = \begin{pmatrix} \underbrace{\begin{matrix}1&\cdots&1\\-1&\cdots&-1\\0&\cdots&0\end{matrix}}_{j_3} & \begin{matrix}1\\0\\-1\end{matrix} & \underbrace{\begin{matrix}0&\cdots&0\\1&\cdots&1\\-1&\cdots&-1\end{matrix}}_{j_3+2j_2-1} \end{pmatrix}$$

Then let

$$X^n(r,J) = (\underbrace{S^2\times\cdots\times S^2}_{n}, E(M^r_J)), \tag{4.1}$$

with

$$E(M^r_J) = \bigoplus_{i=1}^{r} \gamma_1^{\epsilon_{1,i}}\otimes\cdots\otimes\gamma_n^{\epsilon_{n,i}} \tag{4.2}$$

for $M^r_J = (\epsilon_{\mu,\nu})_{\mu=1\ldots n,\nu=1\ldots r}$. Here γ_q is the canonical complex line bundle over the qth factor S^2 in (4.1). Slightly abusing the system of notation above define

$$X^{2n+2}(2,(n+1)) := (\mathbb{C}\mathrm{P}^{2n+1}\times S^2, (\eta\otimes\gamma)\oplus(\eta^{-1}\otimes\gamma^{-1})) \tag{4.3}$$

and

$$X^{2n+1}(3,(a,b,c)) := (\mathbb{C}\mathrm{P}^{2n+1}, \eta^a\oplus\eta^b\oplus\eta^b) \tag{4.4}$$

for $a,b,c\in\mathbb{Z}$, $-n\leq a,b,c\leq n$, $a+b+c=0$, if $n\geq 2$. If $n=1$ we take $a=2$, $b=c=-1$. In (4.3) and (4.4) η, γ denote the canonical bundles over $\mathbb{C}\mathrm{P}^{2n+1}$ and S^2.

Lemma 4.5 *The $X^n(r,J)$ above admit large positive scalar curvature with exception of $X^2(2,(2))$ and $X^3(3,((2,-1,-1)))$*

Proof: In [3] Hitchin has proved that $(\mathbb{C}\mathrm{P}^q, \eta^s)$ admits large scalar curvature if $q \geq 2s$ and that $s - \|\mathcal{E}(\eta^s)\| = 0$ if $q = |s| = 1$. It is immediate from the definition that $\|\mathcal{E}(E \oplus F)\| = \max(\|\mathcal{E}(E)\|, \|\mathcal{E}(F)\|)$ and $\|\mathcal{E}(E \otimes F)\| \leq \|\mathcal{E}(E)\| + \|\mathcal{E}(F)\|$. Thus we can estimate

$$\|\mathcal{E}(E(M_J^r))\| \leq \max_i \left(\sum_{q=1}^{n} \epsilon_{q,i} \|\mathcal{E}(\gamma)\| \right) < n \|\mathcal{E}(\gamma)\|$$

because, with the above exceptions, in every row of the matrices M_J^r at least one entry vanishes. Since the scalar curvature of the round S^2 equals $\|\mathcal{E}(\gamma)\|$, we thus get that the scalar curvature of $S^2 \times \ldots \times S^2$ is larger than $\|\mathcal{E}(E(M_J^r))\|$. The cases involving $\mathbb{C}\mathrm{P}^{2n+1}$ are similiar. •

Lemma 4.6 *The matrix* $\left(c_{J'}\left(X^n(s, J)\right)\right)_{J',(s,J)}$, $s \leq r$, *has full rank.*

Proof: We compute the Chern class of the vectorbundle $E(M_J^r)$: Denoting by $x_q = c_1(\gamma_q)$ the generator of the second cohomology group of the qth factor S^2 in (4.1) we obtain from (4.2) that:

$$c_k(E) = \sum_{\mu_1, \ldots \mu_k, \nu_1, \ldots \nu_k} \epsilon_{\mu_1,\nu_1} \cdots \epsilon_{\mu_k,\nu_k} x_{\nu_1} \cdots x_{\nu_k},$$

where the μ_s respectively ν_s in this sum are pairwise distinct. Order the partitions I, J lexicographically. Observing that $x_s^2 = 0$ we get for $r \geq 4$ that

$$c_I(X^n(r, J)) = \begin{cases} 0 & \text{if } I > J \\ \neq 0 & \text{if } I = J \end{cases} \tag{4.7}$$

Thus this part of the matrix is triangular. If $r = 3$, then a straightforward calculations gives that $c_{j_3,j_2}(X^n(3, (j_3, j_2))) = (-1)^{j_3+j_2-1}(j_3-1)(j_3+j_2)j_3!(2j_3+2j_2-1)!$. If n is even then $j_3 \neq 1$ and (4.7) still holds. For the remainder of the matrix we use the manifolds defined in (4.3) and (4.4). For $r = 2$ we clearly have $c_2^{n+1}((\eta \otimes \gamma) \oplus (\eta^{-1} \otimes \gamma^{-1})) = 2(-1)^{j_2} j_2 \neq 0$. We are left with the case $r = 3$ and n odd. The Chernclasses of $\eta^a \oplus \eta^b \oplus \eta^b$ are given by the elemenary symmetric polynomials σ_3, σ_2 in a, b, c. Assume that the polynomial

$$P(a, b, c) := \sum_{j_3, j_2} \alpha_{j_3,j_2} c_{j_3,j_2}(X^n(3, (a, b, c))) = \sum_{j_3, j_2} \alpha_{j_3,j_2} \sigma_3^{j_3} \sigma_2^{j_2}$$

of degree $2n + 1$ vanishes for all a, b, c as after (4.4). Then the polynomial $P(a, b, -a - b)$ vanishes for all $a, b \in \mathbb{Z}$ with $-n \leq a, b, a + b \leq n$. Since it is homogeneous it must be divisible by all $(na + sb)$ and $(sa + nb)$, $s = 0 \ldots n$ and if $n \geq 2$ it must also contain $(a - b)$ hence have degree at least $2n + 2$. Therefore P vanishes on the entire plane $a + b + c = 0$. Since it does not contain σ_1 and since there are no algebraic relations between the elementary symmetric polynomials, the coefficients α_{j_3,j_2} are all 0.

$\bullet$

Let $\mathcal{K}_{n,r} \subset \Omega_{2n}^{spin}(BSU(r))$ be the kernel of those $c_J p_I$ with nontrivial I. We have shown that the span of the $X^n(r,J)$ as above projects onto $\mathcal{K}_{n,r}$. It is well known that $\bigoplus_n \Omega_n^{spin}$ is polynomially generated by the Kummer surface K and the quaternionic projective spaces $\mathbb{HP}^n, n \geq 2$. In view of the direct sum decomposition

$$\Omega_{2n}^{spin}(BSU(r)) = \bigoplus_{p=0}^{n} \mathcal{K}_{p,r} \times \Omega_{2n-2p}^{spin} \tag{4.8}$$

we infer from Lemma 4.6 that there is a basis of $\Omega_n^{spin}(BSU(r))$ consisting of monomials in K, quaternionic projective spaces and one of the $X^n(r,J)$. Among these only K, $X^2(2,(2))$ and $X^3(3,((2,-1,-1)))$ do not admit large scalar curvature. Therefore the only monomials not admitting large scalar curvature are of the form $K^{d/4-1} \times X^2(2,(2))$ or $K^{d/4}$ if the dimension d is divisible by 4 and $K^{(d-2)/4-1} \times X^3(3,((2,-1,-1)))$ if the dimension is $d = 2 \bmod 4$. These monomials are also detected by the characteristic numbers $\hat{A}$ and $\mathrm{ch}\,\hat{A}$.

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