Detours and Gromov hyperbolicity

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0 Introduction

The notion of Gromov hyperbolicity was introduced by Gromov in the setting of geometric group theory [G1], [G2], but has played an increasing role in analysis on general metric spaces [BHK], [BS], [BBo], [BBu], and extendability of Lipschitz mappings [L].

In this theory, it is often additionally assumed that the hyperbolic metric space is proper and geodesic (meaning that closed balls are compact, and each pair of points can be joined by a path whose length equals the distance between the points). These additional assumptions are useful in proofs, and valid for large classes of examples of hyperbolic spaces, for instance Cayley graphs of (finitely generated) hyperbolic groups, and certain important conformal distortions of locally compact length metrics that push the boundary of the space to infinity; see for instance [BHK, 2.8] for the case of a quasihyperbolic metric. However if the underlying metric is not locally compact, as in examples that arise in a Banach space context, then such hyperbolic conformal distortions typically fail to be proper and geodesic (although they are always length spaces).

Without these added assumptions, a few "standard" results for hyperbolic spaces may fail; for one such example, see [GH, 5.13]. However, Väisälä recently proved [V1] that a large part of the theory of hyperbolic spaces goes through if we merely assume the metric is a length metric and not geodesic or proper; Väisälä then applied this theory in a Banach space context [V2].

Our paper adds to the work of [V1] by extending a characterization by Bonk of hyperbolicity in a geodesic context [B] to a length space context; see Theorem 2.1 below. Note that our version says a little more than Bonk's result even in a geodesic space context.

1 Preliminaries

We make the standing assumption that (X, d) is a metric space; any additional properties of (X, d) will be explicitly listed.

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Our notation and terminology is fairly standard, but we record it here for completeness. $B(x,r)$ is the open ball $\{y \in X \mid d(x,y) < r\}$. Given $x, y \in X$, we write $\gamma : x \curvearrowright y$ whenever γ is a rectifiable path from x to y, and we denote by $l(\gamma)$ the length of γ. We denote by $\Gamma(x,y)$ the collection of all paths $\gamma : x \curvearrowright y$. If x, y are two points on an arc γ, then $\gamma[x,y]$ denotes the subarc of γ from x to y; we do not care about the parametrization or direction. An *arc* is an injective path, and an arc $\gamma : x \curvearrowright y$ is said to be a *h-short arc*, $h \geq 0$, if $l(\gamma) \leq d(x,y) + h$. (X,d) is a *length space* if every pair of points in X can be joined by a h-short arc for every $h > 0$; if we can always take $h = 0$, we call (X,d) a *geodesic space*. A path $\gamma : [0,T] \to X$ is a *(α,h)-quasigeodesic segment*, $\alpha \geq 1$, $h \geq 0$, if

$$l(\gamma|_{[s,t]}) \leq \alpha d(\gamma(s),\gamma(t)) + h, \qquad 0 \leq s \leq t \leq T.$$

We do not distinguish notationally between paths and their images, so $x \in \gamma$ means that x is in the image of γ.

(X,d) is *(Gromov) δ-hyperbolic*, $\delta \geq 0$, if

$$\left\langle x,z \right\rangle_p \geq \left\langle x,y \right\rangle_p \wedge \left\langle y,z \right\rangle_p - \delta, \qquad x,y,z,p \in X, \tag{1.1}$$

where $\left\langle x,y \right\rangle_p$ is the *Gromov product* with basepoint $p \in X$:

$$\left\langle x,y \right\rangle_p = (d(x,p) + d(y,p) - d(x,y))/2, \qquad x,y \in X.$$

For more on the basics of hyperbolicity, we refer the reader to [CDP], [GH], and [V1].

(X,d) is *(α,h,C)-geodesically stable* if all (α,h)-quasigeodesics are within a Hausdorff distance C of each other, and (X,d) is *geodesically stable* if for each $\alpha \geq 1$, $h \geq 0$, there exists $C = C(\alpha,h)$ such that (X,d) is (α,h,C)-geodesically stable. Bonk [B] showed that geodesic stability is equivalent with Gromov hyperbolicity in a geodesic space context, and Väisälä extended this to a length space context (this follows from a combination of 3.7, 3.12, and 2.34 in [V1]). We note that both results are quantitative.

A *h-short triangle* $\Delta \subseteq X$ consists of three h-short arcs, $\gamma_1 : x_2 \curvearrowright x_3$, $\gamma_2 : x_3 \curvearrowright x_1$, and $\gamma_3 : x_1 \curvearrowright x_2$, which we call the *sides* of the triangle, while the three endpoints are called the *vertices*. The orientation of the sides is not important. We say that a h-short triangle $\Delta = \gamma_1 \cup \gamma_2 \cup \gamma_3 \subseteq X$ is *δ-slim*, for some $\delta \geq 0$, if

$$d(z, \gamma_{i+1} \cup \gamma_{i+2}) \leq \delta,$$

for all $z \in \gamma_i$, and all i, indices counted modulo 3. By [V1, 2.34 and 2.35], Gromov hyperbolicity of X is equivalent to the property that all h-short triangles are uniformly slim.

In the same way we can define (α,h)-quasigeodesic triangles as triangles with (α,h)-quasigeodesic sides. It follows from geodesic stability that quasigeodesic triangles in δ-hyperbolic spaces are uniformly slim, with a slimness parameter depending on δ, α, and h.

Detour and Excess functions

We call $\lambda \in \Gamma(x,y)$ a *(t,h)-detour* if there is a h-short arc $\gamma \in \Gamma(x,y)$ and a point $z \in \gamma$ such that

$$B(z,t) \cap \lambda = \emptyset.$$

For each $h \geq 0$, we then define the *detour function*

$$G_X^h(t) := \inf\{l(\lambda) \mid \lambda \text{ is a } (t,h)\text{-detour}\}, \qquad t \geq 0. \tag{1.2}$$

We shall also work with the following related function, which we call the *excess function*:

$$E_X^h(t) := \inf\{e_\lambda := l(\lambda) - d(x,y) \mid \lambda \in \Gamma(x,y) \text{ is a } (t,h)\text{-detour}\}, \qquad t \geq 0. \tag{1.3}$$

Here the infimum is over all (t,h)-detours between all pairs of endpoints x, y. If there are no (t,h)-detours, we set $G_X^h(t) = E_X^h(t) := \infty$.

Clearly $E_X^h(t) < G_X^h(t)$ for all $t > 0$. We have $G_X^{h_1}(t) \geq G_X^{h_2}(t)$ for $h_1 \leq h_2$ and similarly for the excess function. Note also that $G_X^h(t) > 0 \implies t > \frac{h}{2}$. The detour function G_X^0 was first defined by Bonk [B].

Consider the case of Euclidean space $X = \mathbb{R}^n$, $n > 1$. It is straightforward to show that $G_X^0(t) = 4t$, $t \geq 0$; indeed the extreme case is given by an isosceles right-angled triangle whose sidelengths are $2t$, $2t$, and $2^{3/2}t$. On the other hand, $E_X^0(t) \equiv 0$, as can be seen by considering isosceles triangles with a very long base and inradius t.

Bonk [B] showed that hyperbolicity, geodesic stability, and the superlinear growth of G_X^0 are all equivalent if (X, d) is a geodesic space; in fact, hyperbolicity implies an exponential growth rate for G_X^0 [B, Proposition 1.1]. We prove the same sort of equivalence in a length space context; of course, G_X^0 must be replaced by G_X^h, since there might not be any geodesic segments. Additionally, we prove that even E_X^h grows exponentially if the space is hyperbolic.

2 The Main Result and its Proof

Our main result is as follows:

Theorem 2.1. *Let (X, d) be a length space. The following conditions are equivalent:*

(a) X is Gromov hyperbolic;

(b) X is geodesically stable;

(c) $\lim_{t\to\infty} t^{-1} G_X^h(t) = \infty$, for some $h > 0$;

(d) $\lim_{t\to\infty} t^{-1} E_X^h(t) = \infty$, for some $h > 0$;

(e) $\liminf_{t\to\infty} t^{-1} \log(E_X^h(t)) > 0$, for some $h > 0$.

A version of this theorem for geodesic spaces was proved by Bonk [B], where it was shown that in this context, (a), (b), and the $h = 0$ version of (c) are all equivalent. Also the equivalence of (a) and (b) in a length space context was proved by Väisälä; see 3.7, 3.12, and 2.34 of [V1].

Proof of Theorem 2.1. As mentioned above, Väisälä [V1] proved that (a) and (b) are equivalent. Clearly (e) $\implies$ (d) $\implies$ (c). The rest of the proof is based on a series of lemmas and a proposition given below. Specifically, (a) $\implies$ (e) will follow from Lemma 2.2, while (c) $\implies$ (a) will follow from Proposition 2.15. $\square$

Lemma 2.2. *Suppose X is δ-hyperbolic and $\lambda \in \Gamma(x,y)$. If $\gamma \in \Gamma(x,y)$ is a (α,h)-quasigeodesic and $d(z,\lambda) \geq R$ for some $z \in \gamma$, then*

$$e_\lambda := l(\lambda) - d(x,y) \geq C_1 e^{C_2 R} - C_3,$$

for some constants C_1, C_2, C_3 depending only on δ, α, h.

Proof. Let $M \equiv M(\delta,\alpha,h)$ be a constant such that (α,h)-quasigeodesic triangles are M-slim. We will show by induction, that for $\mathbb{N} \ni k \geq 0$, any $x,y \in X$ and $\lambda,\gamma \in \Gamma(x,y)$, where γ is (α,h)-quasigeodesic, we have $e_\lambda < 2^k h \implies d(z,\lambda) \leq (k+1)M$ for all $z \in \gamma$. By geodesic stability, the claim is true for $k=0$. So assume it is true for k and that $e_\lambda < 2^{k+1}h$. Then there is a point $v \in \lambda$ such that the subpath $\lambda_1 = \lambda[x,v]$ has excess $e_{\lambda_1} = \frac{1}{2}e_\lambda < 2^k h$. Let $\lambda_2 = \lambda[v,y]$ be the other subpath. By the triangle inequality we have

$$e_{\lambda_1} + e_{\lambda_2} \leq e_\lambda,$$

hence $e_{\lambda_2} \leq \frac{1}{2}e_\lambda < 2^k h$. Let now $\gamma_1 : x \curvearrowright v, \gamma_2 : v \curvearrowright y$ be h-short arcs, then the triangle $\gamma \cup \gamma_1 \cup \gamma_2$ is M-slim. So for $z \in \gamma$ we have $z' \in \gamma_1 \cup \gamma_2$ with $d(z,z') \leq M$ and by induction hypothesis $d(z',\lambda_i) < (k+1)M$, if $z' \in \gamma_i$, which shows the claim for $k+1$.

Therefore

$$d(z,\lambda) \geq (k+1)M \implies e_\lambda \geq 2^k h, \qquad k \in \mathbb{N}.$$

So if λ is an R-detour of γ, i.e. $d(z,\lambda) \geq R$, we have

$$e_\lambda \geq 2^{(\frac{1}{M}R-2)}h - \frac{1}{2}h$$

$\square$

Let us observe a couple of easy estimates that we will use repeatedly. First, if $\gamma : y \curvearrowright z$ is a h-short arc in X, then there is a sort of reverse triangle inequality for points $w \in \gamma$, namely $d(y,w) + d(w,z) \leq d(y,z) + h$. From this it is not hard to deduce (see [V1, 2.9]) that for all $x,y,z \in X$ and every h-short arc $\gamma : y \curvearrowright z$:

$$\langle y,z \rangle_x \leq d(x,\gamma) + \frac{h}{2}, \tag{2.3}$$

and in particular,

$$2\langle y,z \rangle_x \leq h, \qquad x \in \gamma. \tag{2.4}$$

The following is a variation of a well-known lemma for Gromov hyperbolic spaces, designed to suit our purposes. Other variants include [V1, 2.15] and one half of [B, Lemma 1.3].

Lemma 2.5 (Tripod Lemma). *Let $\Delta = \gamma_1 \cup \gamma_2 \cup \gamma_3 \subseteq X$ be a h-short triangle, which is δ-slim. Let $\gamma_1 : x_1 \curvearrowright x_2$ and $\gamma_2 : x_1 \curvearrowright x_3$ be sides with common vertex x_1, and let $y \in \gamma_1$, $z \in \gamma_2$ be two points. Then*

(a) $d(x_1, y) = d(x_1, z) \leq \langle x_2, x_3 \rangle_{x_1} \implies d(y, z) \leq 4\delta + 6h$;

(b) $l(\gamma_1[x_1, y]) = l(\gamma_2[x_1, z]) \leq \langle x_2, x_3 \rangle_{x_1} \implies d(y, z) \leq 4\delta + 7h$.

Proof. Assuming that $d(x_1, y) = d(x_1, z) \leq \langle x_2, x_3 \rangle_{x_1}$, we shall show that the inequality $d(y, z) > 4\delta + 6h$ leads to a contradiction. Part (a) then follows by a continuity argument, and (a) in turn easily implies (b).

By (2.3),

$$d(z, \gamma_1) = \min\{d(z, \gamma_1[x_1, y]), d(z, \gamma_1[y, x_2])\} \geq \min\{\langle x_1, y \rangle_z, \langle y, x_2 \rangle_z\} - \frac{h}{2}$$

But

$$2\langle x_1, y \rangle_z = d(x_1, z) + d(y, z) - d(x_1, y) = d(y, z)$$

and, by the shortness of γ_1 and (2.4),

$$\begin{aligned} 2\langle y, x_2 \rangle_z &= d(y, z) + d(x_2, z) - d(y, x_2) \\ &\geq d(y, z) + d(x_2, z) - (d(x_1, x_2) - d(x_1, y) + h) \\ &= d(y, z) + d(x_2, z) + d(x_1, z) - d(x_1, x_2) - h \\ &= d(y, z) + 2\langle x_2, x_1 \rangle_z - h \\ &\geq d(y, z) - h \end{aligned}$$

From this we then get $d(z, \gamma_1) \geq \frac{1}{2} d(y, z) - h > 2\delta + 2h$. In particular $d(z, x_1) > 2\delta + 2h$, hence there is a point $w \in \gamma_2[x_1, z]$ with $d(w, z) = \delta + \frac{3}{2}h$. By the triangle inequality we have

$$d(w, \gamma_1) \geq d(z, \gamma_1) - (\delta + \frac{3}{2}h) > \delta + \frac{h}{2}$$

and

$$\begin{aligned} d(w, \gamma_3) &\geq d(x_1, \gamma_3) - d(x_1, w) \geq \langle x_2, x_3 \rangle_{x_1} - \frac{h}{2} - d(x_1, w) \\ &> d(x_1, z) - \frac{h}{2} - d(x_1, w) \geq d(z, w) - h - \frac{h}{2} = \delta. \end{aligned}$$

So $d(w, \gamma_1 \cup \gamma_3) > \delta$, contradicting that the triangle is δ-slim, thus establishing (a) and hence (b). □

The following is an adaptation of [B, Lemma 2.2].

Lemma 2.6. *Let (X, d) be a length space and let $\Delta \subseteq X$ be a h-short triangle. If Δ is δ-slim and $\delta' \geq 0$ is a number satisfying $27\delta + 56h < G_X^h(\delta')$ and $\delta' \leq \delta$, then Δ is also $2\delta'$-slim.*

Proof. Let $\alpha : b \curvearrowright c$, $\beta : a \curvearrowright c$ and $\gamma : a \curvearrowright b$ be the sides of a h-short triangle Δ, which is δ-slim. We assume without loss of generality that $x \in \gamma$ and want to show that $d(x, \alpha \cup \beta) \leq 2\delta'$, when δ' satisfies the conditions above. Define

$$e_\gamma := l(\gamma) - d(a,b) = l(\gamma) - \langle b,c \rangle_a - \langle a,c \rangle_b. \tag{2.7}$$

Since γ is h-short, $0 \leq e_\gamma \leq h$. We also assume that $l(\gamma[a,x]) \leq \langle b,c \rangle_a + e_\gamma$; in view of (2.7), this assumption can be made without loss of generality since, if it it false, we can make it true by swapping a and b.

Define $\rho = 4\delta + 7h$ and $\rho' = 5\delta + 10h$. Now let x_1 be the point in $\gamma[a,x]$ with $l(\gamma[x_1,x]) = \rho'$ if $l(\gamma[a,x]) \geq \rho'$ and let $x_1 = a$ otherwise. Similarly define $x_2 \in \gamma[x,b]$ with $l(\gamma_1[x,x_2]) = \rho'$ if $l(\gamma[x,b]) \geq \rho'$ and put $x_2 = b$ otherwise. As in [B] we consider two cases:

Case 1: Assume $l(\gamma[a,x_2]) < \langle b,c \rangle_a + e_\gamma$ and thus $x_2 \neq b$.

Then $l(\gamma[a,x_1]) < \langle b,c \rangle_a$, and so by the Tripod Lemma for a point $y_1 \in \beta$ with $l(\beta[a,y_1]) = l(\gamma([a,x_1])$, we have $d(x_1,y_1) \leq \rho$. If $l(\gamma[a,x_2]) \leq \langle b,c \rangle_a$, then let $y_2 \in \beta$ be the point with $l(\beta[a,y_2]) = l(\gamma[a,x_2])$ and $d(x_2,y_2) \leq \rho$. Otherwise, for $l(\gamma[a,x_2]) \in (\langle b,c \rangle_a, \langle b,c \rangle_a + e_\gamma)$, let $y_2 \in \beta$ be the point with $l(\beta[a,y_2]) = \langle b,c \rangle_a$, then $d(x_2,y_2) \leq \rho + e_\gamma \leq \rho + h$. In both situations we have $l(\beta[y_1,y_2]) \leq l(\gamma[x_1,x_2]) \leq 2\rho'$.

We will now show that the assumption

$$d(x, \alpha \cup \beta) \geq \delta' \tag{2.8}$$

leads to a contradiction.

Let $\omega_1 : x_1 \curvearrowright y_1$ be h-short. In the case $x_1 = y_1 = a$ we choose ω_1 to be the constant curve. We show first that $\omega_1 \cap B(x,\delta') = \emptyset$.

For if $x_1 = a = y_1$ then

$$d(x,\omega_1) = d(x,a) \geq d(x, \alpha \cup \beta) \geq \delta'$$

If $x_1 \neq y_1$, then $d(x,x_1) \geq \rho' - h$ hence

$$d(x,\omega_1) \geq d(x,x_1) - l(\omega_1) \geq \rho' - h - \rho - h = 5\delta + 10h - 4\delta - 9h = \delta + h \geq \delta'$$

In the same way, using $d(x,x_2) \geq \rho' - h$ and $d(x_2,y_2) \leq \rho + h$, one sees that

$$d(x,\omega_2) \geq \delta \geq \delta',$$

and thus $\omega_2 \cap B(x,\delta') = \emptyset$ for $\omega_2 : x_2 \curvearrowright y_2$ a h-short arc. Furthermore we clearly have

$$d(x, \beta[y_1,y_2]) \geq d(x,\beta) \geq \delta'$$

We thus have a path ϕ with endpoints x_1, x_2 and image $\omega_1 \cup \beta[y_1,y_2] \cup \omega_2$ and

$$l(\phi) \leq \rho + h + 2\rho' + \rho + 2h = 18\delta + 37h$$

Since ϕ is a δ'-detour of $\gamma[x_1, x_2]$, then by our assumptions we have

$$27\delta + 56h < G_X^1(\delta') \leq l(\phi) \leq 18\delta + 37h,$$

which is a contradiction.

Case 2: Assume $l(\gamma[a, x_2]) \geq \langle b, c\rangle_a + e_\gamma$.

By (2.7), we see that $l(\gamma[x_2, b]) \leq \langle a, c\rangle_b$. As before, we have a point $y_1 \in \beta$ with $l(\beta[a, y_1]) = l(\gamma[a, x_1])$ and $d(x_1, y_1) \leq \rho$ and now another point $y_2 \in \alpha$ with $l(\alpha[b, y_2]) = l(\gamma[x_2, b])$ and $d(x_2, y_2) \leq \rho$. Let $z_1 \in \beta$ and $z_2 \in \alpha$ be points such that $l(\beta[z_1, c]) = l(\alpha[z_2, c]) = \langle a, b\rangle_c$, then by the Tripod Lemma $d(z_1, z_2) \leq \rho$.

We claim that

$$l(\beta[y_1, z_1]) \leq 2\rho' + h, \tag{2.9}$$
$$l(\alpha[y_2, z_2]) \leq \rho' + h. \tag{2.10}$$

Let us first prove (2.9). By the β analogue of (2.7),

$$\begin{aligned} l(\beta[a, z_1]) &= l(\beta) - l(\beta[z_1, c]) \\ &= \langle a, b\rangle_c + \langle c, b\rangle_a + e_\beta - \langle a, b\rangle_c = \langle c, b\rangle_a + e_\beta. \end{aligned}$$

and so

$$l(\beta[a, z_1]) - \langle b, c\rangle_a \in [0, h]. \tag{2.11}$$

On the other hand,

$$l(\beta[a, y_1]) = l(\gamma[a, x_1]) \leq l(\gamma[a, x]) \leq \langle b, c\rangle_a + e_\gamma \leq \langle b, c\rangle_a + h,$$

and

$$l(\beta[a, y_1]) = l(\gamma[a, x_1]) \geq l(\gamma[a, x_2]) - 2\rho' \geq \langle b, c\rangle_a + e_\gamma - 2\rho' \geq \langle b, c\rangle_a - 2\rho'.$$

These last two estimates imply that

$$l(\beta[a, y_1]) - \langle b, c\rangle_a \in [-2\rho', h]. \tag{2.12}$$

Putting together (2.11) and (2.12), we deduce (2.9).

The proof of (2.10) is similar. In particular, we see as before that

$$l(\alpha[b, z_2]) - \langle a, c\rangle_b \in [0, h]. \tag{2.13}$$

Also $l(\alpha[b, y_2]) = l([x_2, b]) \leq \langle a, c\rangle_b$, while

$$\begin{aligned} l(\alpha[b, y_2]) &= l(\gamma[x_2, b]) \geq l(\gamma[x, b]) - \rho' = l(\gamma) - l(\gamma[a, x]) - \rho' \\ &\geq \langle b, c\rangle_a + \langle a, c\rangle_b + e_\gamma - (\langle b, c\rangle_a + e_\gamma) - \rho' \\ &= \langle a, c\rangle_b - \rho'. \end{aligned}$$

These last two estimates imply that

$$l(\alpha[b,y_2]) - \langle a,c\rangle_b \in [-\rho', h]. \tag{2.14}$$

Putting together (2.13) and (2.14), we deduce (2.10).

Now let $\omega_1 : x_1 \curvearrowright y_1$, $\omega_2 : x_2 \curvearrowright y_2$ and $\omega_3 : z_1 \curvearrowright z_2$ be h-short arcs, which we choose to be constant curves in case the endpoints equal. Then as in Case 1, but with the possibility that $x_2 = b$, the assumption (2.8) implies that

$$\big(\omega_1 \cup \beta[y_1,z_1] \cup \alpha[y_2,z_2] \cup \omega_2\big) \cap B(x,\delta') = \emptyset$$

Now let ϕ be a path with image

$$\omega_1 \cup \beta[y_1,z_1] \cup \omega_3 \cup \alpha[z_2,y_2] \cup \omega_2,$$

and

$$l(\phi) \le \rho + h + \rho' + h + \rho + h + 2\rho' + h + \rho + h = 27\delta + 56h$$

However, if $B(x,\delta') \cap \omega_3 = \emptyset$, then ϕ is a δ'-detour. Thus

$$27\delta + 56h < G_X^h(\delta') \le l(\phi) \le 27\delta + 56h,$$

which is a contradiction, so there must be a point $w \in \omega_3$ with $d(x,w) < \delta'$. Now we show that $d(w,\alpha\cup\beta) \le \delta'$.

Assume the opposite, that $d(w,\alpha\cup\beta) > \delta'$. Then choose $w_1 \in \beta[z_1,c]$, $w_2 \in \alpha[z_2,c]$ such that

$$l(\alpha[z_1,w_1]) = l(\alpha[z_2,w_2]) = \rho'' = 9\delta + 17h$$

if $\langle a,b\rangle_c \ge \rho''$, and let $w_1 = w_2 = c$ otherwise. Then by the Tripod Lemma, we have $d(w_1,w_2) \le \rho$. Let $\omega_4 : w_1 \curvearrowright w_2$ be h-short; take ω_4 to be the constant curve if $w_1 = w_2 = c$. Consider the path ψ with image:

$$\beta[z_1,w_1] \cup \omega_4 \cup \alpha[z_2,w_2],$$

which has length

$$l(\psi) \le \rho'' + \rho + h + \rho'' = 22\delta + 42h$$

If $w_1 = w_2 = c$ we have $\psi \subseteq \alpha\cup\beta$, and thus by assumption

$$\psi \cap B(w,\delta') = \emptyset$$

In the other case, $w_1, w_2 \ne c$, we have $d(z_1,w_1) \ge \rho'' - h$, hence

$$d(w,\omega_4) \ge d(z_1,w_1) - \big(l(\omega_3[z_1,w]) + l(\omega_4)\big) \ge \rho'' - 2\rho - 3h = \delta \ge \delta'$$

This shows that ψ is a δ'-detour of ω_3, so

$$27\delta + 56h < G_X^h(\delta') \le l(\psi) \le 22\delta + 42h,$$

a contradiction. Therefore we must have $d(w,\alpha\cup\beta) \le \delta'$, and hence

$$d(x,\alpha\cup\beta) \le d(x,w) + d(w,\alpha\cup\beta) \le 2\delta',$$

which finishes the proof. □

The next proposition corresponds to Proposition 2.3 in [B].

Proposition 2.15. *If* (X,d) *is a length space with* $\lim_{t\to\infty} \frac{G_X^h(t)}{t} = \infty$*, for some* $h > 0$*, then* X *is Gromov hyperbolic.*

Proof. By [V1, 2.34], we need to show that every h-short triangle is ρ-slim, for some fixed $\rho \geq 0$.

As in [B, Lemma 2.1], we can choose a function $f : (0,\infty) \to (0,\infty)$ with

$$\lim_{t\to\infty} \frac{f(t)}{t} = 0 \text{ and } \lim_{t\to\infty} \frac{G_X^h(f(t))}{t} = \infty.$$

By the properties of the functions G_X^h and f, there is a number $\rho > 0$ with $27t+56h < G_X^h(f(t))$ and $f(t) \leq \frac{1}{4}t$ for $t \geq \rho$.

We will show that every h-short triangle Δ is ρ-slim. Certainly such a triangle Δ is δ_1-slim, where $\delta_1 < \infty$ is the diameter of Δ. Now for $\mathbb{N} \ni k > 1$ define $\delta_{k+1} = 2f(\delta_k)$. If $\delta_k \geq \rho$ we have $\delta_{k+1} \leq \frac{1}{2}\delta_k$, hence $A := \{k \in \mathbb{N} | \delta_k \leq \rho\} \neq \emptyset$. Fix $k_0 = \min A$. Now either $k_0 = 1$, i.e. $\delta_1 \leq \rho$, or else for $k \in \{1,\ldots,k_0-1\}$, $\delta = \delta_k$ and $\delta' = \frac{1}{2}\delta_{k+1} = f(\delta_k)$ satisfies the requirements of Lemma 2.6. Hence a repeated use of this shows that Δ is δ_{k_0}-slim. But $\delta_{k_0} \leq \rho$, and we are done. ☐

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