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Finitely-Generated Algebras of Smooth Functions, in One Dimension

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We characterise the closure in $C^\infty(\mathbb{R}, \mathbb{R})$ of the algebra generated by an arbitrary finite point-separating set of C^∞ functions. The description is local, involving Taylor series. More precisely, a function $f \in C^\infty$ belongs to the closure of the algebra generated by $\psi_1, ..., \psi_r$ as soon as it has the "right kind" of Taylor series at each point a such that $\psi_1'(a) = \cdots = \psi_r'(a) = 0$. The "right kind" is of the form $q \circ (T_a^\infty \psi_1 - \psi_1(a), ..., T_a^\infty \psi_r - \psi_r(a))$, where q is a power series in r variables, and $T_a^\infty \psi_i$ denotes the Taylor series of ψ_i about a. © 1998 Academic Press

1. INTRODUCTION AND NOTATION

By $C^\infty(\mathbb{R}^d, \mathbb{R}^r)$ we mean the Fréchet space of infinitely-differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}^r$. The usual topology on $C^\infty(\mathbb{R}^d, \mathbb{R}^r)$ is metrisable, and a sequence f_n converges to a function f in this topology if and only if the partial derivatives $\partial^i f_n \to \partial^i f$ uniformly on compact subsets of $\mathbb{R}^d$, for each multi-index i. We abbreviate $C^\infty(\mathbb{R}^d, \mathbb{R})$ to $C^\infty(\mathbb{R}^d)$, or just C^∞, when the value of d is clear from the context.

Suppose we take r functions $\psi_1, ..., \psi_r \in C^\infty(\mathbb{R}^d)$ and consider the real algebra $\mathbb{R}[\psi_1, ..., \psi_r]$ that they generate. It is of interest to describe the closure of the algebra in $C^\infty(\mathbb{R}^d)$. This problem was posed by I. Segal, about 1949 [11, p. 311]. The purpose of this paper is to describe the closure in the case when $d = 1$ and the functions $\psi_1, ..., \psi_r$ together separate points. The description we give is local, involving the Taylor series of the functions.

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We denote the algebras of polynomials and of formal power series in r variables by $\mathbb{R}[x_1, ..., x_r]$ and $\mathbb{R}[[x_1, ..., x_r]]$, respectively. For each $a \in \mathbb{R}^d$, the Taylor series map

$$T_a^\infty : C^\infty(\mathbb{R}^d, \mathbb{R}^r) \to \mathbb{R}[[x_1, ..., x_d]]^r$$

is continuous when $\mathbb{R}[[x_1, ..., x_d]]$ is given the usual projective limit topology, and is an algebra homomorphism when $r = 1$. For each $k \in \mathbb{Z}_+$, the Taylor polynomial map

$$T_a^k : C^\infty(\mathbb{R}^d, \mathbb{R}^r) \to \mathbb{R}[x_1, ..., x_d]_k^r,$$

where $\mathbb{R}[x_1, ..., x_d]_k$ denotes the space of polynomials of degree at most k, is also continuous with respect to the usual topology on $\mathbb{R}[x_1, ..., x_d]_k$. We abbreviate T_0^k to T^k, and we also use T^k for the truncation map on power series:

$$T^k : \begin{cases} \mathbb{R}[[x_1, ..., x_d]] \to \mathbb{R}[x_1, ..., x_d]_k, \\ \sum_{|i| \geqslant 0} \alpha_i x^i \mapsto \sum_{0 \leqslant |i| \leqslant k} \alpha_i x^i. \end{cases}$$

By a classical theorem of Émile Borel, T_a^∞ is surjective, i.e., each formal power series is the Taylor series of some C^∞ function.

If $p_1, ..., p_r \in \mathbb{R}[[x_1, ..., x_d]]$ have $p_i(0) = 0$, for all i, and if $q \in \mathbb{R}[[x_1, ..., x_r]]$, then we may form the composition $q \circ (p_1, ..., p_r)$. We denote the set of power series so obtained, with $p_1, ..., p_r$ fixed and q ranging over all of $\mathbb{R}[[x_1, ..., x_r]]$, by $\mathbb{R}[[p_1, ..., p_r]]$.

We observe that if $f \in C^\infty(\mathbb{R}^d, \mathbb{R}^m)$, $g \in C^\infty(\mathbb{R}^m, \mathbb{R})$, and $a \in \mathbb{R}^d$, then

$$T_a^\infty(g \circ f) = (T^\infty{}_{f(a)} g) \circ (T_a^\infty f - f(a)).$$

This could be described as the higher order version of the Chain Rule.

We can now state the main result.

THEOREM. *Suppose* $\Psi = (\psi_1, ..., \psi_r) \in C^\infty(\mathbb{R}, \mathbb{R}^r)$ *is injective. Let* $f \in C^\infty(\mathbb{R}, \mathbb{R})$. *Then the following are equivalent*:

(1) $f \in \mathrm{clos}_{C^\infty(\mathbb{R})} \mathbb{R}[\psi_1, ..., \psi_r]$;

(2) $T_a^k f \in T^k \mathbb{R}[T_a^k \Psi]$ *whenever* $a \in \mathbb{R}$ *and* $k \in \mathbb{N}$;

(3) $T_a^\infty f \in \mathbb{R}[[T_a^\infty \Psi - \Psi(a)]]$, $\forall a \in \mathbb{R}$;

(4) $T_a^k f \in T^k \mathbb{R}[T_a^k \Psi]$ *whenever* $\Psi'(a) = 0$ *and* $k \in \mathbb{N}$;

(5) $T_a^\infty f \in \mathbb{R}[[T_a^\infty \Psi - \Psi(a)]]$ *whenever* $\Psi'(a) = 0$.

To illustrate the result, we mention a few simple consequences. These examples are all well-known and classical, and indeed more can be said about them, as we shall explain below. Some more elaborate applications are given in the Corollaries at the end of the paper.

EXAMPLES. 1. The closure of $\mathbb{R}[x^3]$ is precisely the set of those $f \in C^\infty(\mathbb{R})$ such that $f^{(i)}(0) = 0$ unless 3 divides i.

2. The closure of $\mathbb{R}[x^2, x^3]$ is the same as the closure of $\mathbb{R}[\cos x, x^3]$, and consists of all functions with $f'(0) = 0$.

3. The closure of $\mathbb{R}[x^3, x^5]$ is the set of f with $f'(0) = f''(0) = f^{(iv)}(0) = f^{(vii)}(0) = 0$.

4. The closure of $\mathbb{R}[x^3 + x^7, x^5]$ is the set of f with $f'(0) = f''(0) = f^{(iv)}(0) = f^{(vii)}(0) - \binom{7}{3} f'''(0) = 0$.

Remarks. 1. In case Ψ has no critical points, the result is a special case of Nachbin's theorem [10], which characterises the maximal closed subalgebras of $C^\infty(M)$, for arbitrary smooth manifolds M. The Whitney spectral theorem [9, 13] provides a description of the closed ideals in $C^\infty(M)$, and hence of those closed algebras of the form $\mathbb{R}1 + I$, where I is a closed ideal. Apart from these results, both pre-1950, the main previous result about closed subalgebras of $C^\infty(M)$ was Tougeron's 1971 spectral theorem [12]. When applied to $M = \mathbb{R}$, Tougeron's theorem yields the special case of our theorem in which all the critical points of Ψ are isolated and of finite order. Most of the work of the present paper involves the detailed analysis of the set of accumulation points of the critical set of Ψ.

2. Tougeron's theorem is sufficiently general to give a full and satisfactory description of the closure of the algebra generated by any finite collection of real-analytic functions on $\mathbb{R}^d$, for any natural number d. In the particular case of real-analytic Ψ, a good deal more is known. Consider the following four function spaces associated to a $\Psi: \mathbb{R} \to \mathbb{R}^r$:

$$A = \{g \circ \Psi : g \in C^\infty(\mathbb{R}^r)\},$$

$$B = \mathrm{clos}\, \mathbb{R}[\Psi],$$

$$C = \mathrm{clos}\, A,$$

$$D = \{f \in C^\infty(\mathbb{R}) : T_a^\infty f \in \mathbb{R}[[T^\infty \Psi - \Psi(a)]], \forall a \in \mathbb{R}\}.$$

By (the classical) Lemma 8 below, $A \subset B$, so

$$A \subset B = C \subset D.$$

In the present paper, we are focussed only on the approximation question: When is $B = D$ (or, equivalently, $C = D$)? Evidently, a sufficient condition would be that $A=D$. This condition is not necessary, as was noted already by Glaeser [6] (see the first example after Corollary 9 below). The problem of deciding when $A=D$ has received a great deal of study. This began with the paper of Whitney [15] on characterizing the even functions as those of the form $f(x^2)$, involved significant progress by Glaeser [6] and Tougeron [14], and culminated in the penetrating result of Bierstone and Milman [3] which relates $A=D$ to semi-coherence of the image of Ψ. The result applies to proper real-analytic Ψ, and extends to higher dimensions. See also [2, 4], and forthcoming work of Bierstone and Milman in the *Annals of Mathematics*. These results show, for example, that $A=D$ holds in examples 1 to 4 given above. As far as the problem of deciding when $B=C$ is concerned, these results do not advance on Tougeron's.

The problem of deciding whether $A=D$ for a given general (not necessarily analytic) smooth, injective, proper Ψ has received little attention. The referee of this paper remarks that $A=D$ is probably true for those $\Psi\colon \mathbb{R}\to\mathbb{R}^r$ that are proper, injective, and have only critical points of finite order. This is a reasonable conjecture, and could probably be approached by using the methods that work for analytic functions.

3. A result similar to our theorem holds (with essentially the same proof) for finitely generated subalgebras of C^∞ functions on the other 1-dimensional manifold, the circle. The C^k analogue also works ($1\leqslant k<\infty$), and is somewhat easier.

2. NOTATION AND DEFINITIONS

We use $\mathbb{N}$ for the set of natural numbers and $\mathbb{Z}_+$ for the set of non-negative integers, $\mathbb{N}\cup\{0\}$.

For a propositional function $P(x)$, we say that $P(x)$ holds for x *near* A if $\{x : P(x)\}$ is a neighbourhood of A.

E^d denotes the set of accumulation points, or derived set, of E.

Let $f\in C^\infty(\mathbb{R},\mathbb{R})$. Then spt f denotes the *support* of f, i.e., $\mathbb{R}\sim \operatorname{int} f^{-1}(0)$. We say that f is *flat* at a point $a\in\mathbb{R}$ if all derivatives $(d^if/dx^i)(a)=0$ $(i\geqslant 1)$. Note that it does *not* entail $f(a)=0$. We say that f is *locally-constant near* a set $E\subset\mathbb{R}$ if $\forall a\in E\ \exists r>0$ such that f is constant on $(a-r,a+r)$.

If $p(x)=\sum_{i=0}^{+\infty}\lambda_i x^i\in\mathbb{R}[[x]]$ is a power series, then ord p, the *order* of p, is $\inf\{i:\lambda_i\neq 0\}$.

3. TOOLS

We gather here the lemmata we shall use to prove the theorem. The first is easy to prove, and well-known.

LEMMA 1. *If S is a semigroup of non-negative integers under addition, $g=\text{g.c.d.}(S)$ and $g>0$, then $\exists N\in\mathbb{N}$ such that $kg\in S$ whenever $k\in\mathbb{N}$ and $kg>N$.* ∎

LEMMA 2. *Let $p_1,\dots,p_r\in\mathbb{R}[[x]]$ and $p_i(0)=0$, $\forall i$. Then the subalgebra*

$$A=\mathbb{R}[[p_1,\dots,p_r]]$$

is closed in $\mathbb{R}[[x]]$.

(This lemma holds in the more general situation where the p_i are power series in many variables, and it may be proved by a short inductive argument, or by appealing to [5, Section II, Lemma 7]. We include the following argument for the one-dimensional case because it has a constructive character, and the method is useful in working examples.)

Proof. We may assume that p_1 has minimal order, say g, among the p_i. If $g=+\infty$, then A has only constants and the result is trivial, so we may assume $g\neq+\infty$.

Let $S=\{\text{ord } t : t\in A\}$. Then S is a sub-semigroup of $(\mathbb{Z}_+,+)$. Let $d=\text{g.c.d.}(S)$, and let $w=g/d$. By Lemma 1, there exists $T\in S$ such that $T+kd\in S$, $\forall k\in\mathbb{N}$. Choose $u_1,\dots,u_r\in\mathbb{R}[[x_1,\dots,x_r]]$ such that

$$u_i\circ(p_1,\dots,p_r)=x^{T+id}+\text{higher terms}.$$

For each $k\in\mathbb{Z}_+$, let $A_k=\{T^kt : t\in A\}$. Then A_k is a linear subspace of the finite-dimensional vector space $\mathbb{R}[x]_k$ of all polynomials of degree at most k. It is therefore closed with respect to the usual topology on $\mathbb{R}[x]_k$. Note also that if power series $t_n\to t$ in $\mathbb{R}[[x]]$, then the truncations $T^kt_n\to T^kt$ in $\mathbb{R}[x]_k$.

Suppose $\{q_n\}_{n=1}^{+\infty}\subset\mathbb{R}[[x_1,\dots,x_r]]$ and $q_n\circ(p_1,\dots,p_r)\to p$ as $n\uparrow+\infty$, for some $p\in\mathbb{R}[[x]]$. We have to show that $\exists f\in\mathbb{R}[[x_1,\dots,x_r]]$ such that $p=f\circ(p_1,\dots,p_r)$.

For each $k\in\mathbb{Z}_+$, we have $T^k(q_n\circ(p_1,\dots,p_r))\to T^kp$, hence $T^kp\in A_k$. Thus $\exists f_k\in\mathbb{R}[[x_1,\dots,x_r]]$ such that $T^kp=T^k(f_k\circ(p_1,\dots,p_r))$. Typically, f_k is highly non-unique. Fix $K=T+g$, and pick some f_K, as above. Then

$$p=f_K\circ(p_1,\dots,p_r)+\beta_{K+1}x^{K+1}+\text{higher-order terms}.$$

We proceed inductively to pick $f_{K+1},f_{K+2},\dots$, in a specific way.

Suppose f_k has been chosen for some $k\geqslant K$, with

$$p=f_k\circ(p_1,\dots,p_r)+\beta_{k+1}x^{k+1}+\text{higher-order terms}.$$

There are two possibilities.

Case 1^0. $\beta_{k+1}=0$. In this case, we take $f_{k+1}=f_k$.

Case 2^0. $\beta_{k+1}\neq 0$. In this case, $k+1$ belongs to the semigroup S, because there exists some $f'_{k+1}\in\mathbb{R}[[x_1,...,x_r]]$ such that

$$T^{k+1}p=T^{k+1}(f'_{k+1}\circ(p_1,...,p_r)),$$

hence

$$(f'_{k+1}-f_k)\circ(p_1,...,p_r)=\beta_{k+1}x^{k+1}+\cdots.$$

Thus we may choose $h\in\mathbb{N}$ such that

$$k+1-hg=T+id$$

for some $i\in\{1,...,w\}$. We then choose

$$f_{k+1}=f_k+\beta_{k+1}x_1^h u_i.$$

Then

$$\begin{aligned} f_{k+1}\circ(p_1,...,p_r)&=f_k\circ(p_1,...,p_r)+\beta_{k+1}(x^g+\cdots)^h(x^{T+id}+\cdots)\\ &=f_k\circ(p_1,...,p_r)+\beta_{k+1}x^{k+1}+\cdots,\end{aligned}$$

so

$$T^{k+1}(f_{k+1}\circ(p_1,...,p_r))=T^{k+1}p,$$

as required.

The key feature of this construction is that in either case f_{k+1} is produced from f_k by adding terms of order at least h, and

$$h\geqslant\frac{k+1-T-g}{g}\uparrow+\infty,$$

as $k\uparrow+\infty$. Thus, given $j\in\mathbb{N}$ there exists $J=J(j)$ such that

$$T^jf_k=T^jf_J \qquad \forall k\geqslant J.$$

Consequently, $\{f_k\}_{k=1}^{\infty}$ converges in $\mathbb{R}[[x_1,...,x_r]]$ to a limit f, and for each $k\in\mathbb{N}$

$$\begin{aligned} T^k(f\circ(p_1,...,p_r))&=T^k((T^kf)\circ(p_1,...,p_r))\\ &=T^k((T^kf_{J(k)})\circ(p_1,...,p_r))=T^kp,\end{aligned}$$

hence

$$f\circ(p_1,...,p_r)=p. \quad ∎$$

COROLLARY 3. *Let* $p_1,...,p_r\in\mathbb{R}[[x]]$ *and* $p_i(0)=0\ \forall i$. *Let* $f\in\mathbb{R}[[x]]$. *Then the following are equivalent*:

(i) $f\in\mathbb{R}[[p_1,...,p_r]]$;

(ii) $T^kf\in T^k\mathbb{R}[p_1,...,p_r],\ \forall k\in\mathbb{N}$;

(iii) $T^kf\in T^k\mathbb{R}[T^kp_1,...,T^kp_r],\ \forall k\in\mathbb{N}$.

LEMMA 4. *Suppose that* $f\in C^\infty(\mathbb{R},\mathbb{R})$, $0<\eta<\delta$, $a\in\mathbb{R}$, *f is flat at a*,

$$\operatorname{dist}(x,f'^{-1}(0))<\eta, \qquad \forall x\in(a,a+\delta),$$

$k\in\mathbb{N}$, *and*

$$M=\max\{|f^{(k+1)}(x)|;a\leqslant x\leqslant a+\delta\}.$$

Then for each $x\in[a,a+\delta]$, *we have*

$$|f(x)-f(a)|\leqslant\frac{k^kM\eta^k\delta}{k!}, \qquad \text{and}$$

$$|f^{(i)}(x)|\leqslant\frac{k^{k+1-i}M\eta^{k+1-i}}{(k+1-i)!} \qquad (\text{for } 1\leqslant i\leqslant k).$$

Proof. If $\delta\leqslant k\eta$, then we apply Taylor's theorem with Lagrange's form of the remainder. Since f is flat at a, we get (for $x\in[a,a+\delta]$ and suitable ξ_i):

$$|f(x)-f(a)|=\left|\frac{f^{(k+1)}(\xi_0)(x-a)^{k+1}}{(k+1)!}\right|\leqslant\frac{M\,\delta^{k+1}}{(k+1)!},$$

and, for $1\leqslant i\leqslant k$,

$$|f^{(i)}(x)|=\left|\frac{f^{(k+1)}(\xi_i)(x-a)^{k+1-i}}{(k+1-i)!}\right|\leqslant\frac{M\delta^{k+1-i}}{(k+1-i)!},$$

so

$$|f(x)-f(a)|\leqslant\frac{Mk^k\eta^k\delta}{(k+1)!},$$

and

$$|f^{(i)}(x)| \leqslant \frac{Mk^{k+1-i}\eta^{k+1-i}}{(k+1-i)!}.$$

These easily yield the desired estimates, in this case.

So suppose $k\eta < \delta$. Then we may choose k distinct points $\xi_1, ..., \xi_k$, in the interval

$$I = (x - k\eta, x + k\eta) \cap (a, a + \delta),$$

at each of which $f' = 0$. By Newton's interpolation formula,

$$f'(x) = (x - \xi_1) \cdots (x - \xi_k)\, f'[\xi_1, ..., \xi_k, x]$$
$$= \frac{(x - \xi_1) \cdots (x - \xi_k)\, f^{(k+1)}(\xi)}{k!},$$

(cf. [7, p. 47]) so

$$|f'(x)| \leqslant \frac{(k\eta)^k M}{k!}.$$

By applying Rolle's theorem, we see that for $2 \leqslant i \leqslant k$, $f^{(i)}$ has $k+1-i$ zeros in I, and the same argument shows that

$$|f^{(i)}(x)| \leqslant \frac{(k\eta)^{k+1-i} M}{(k+1-i)!}.$$

Finally,

$$|f(x) - f(a)| = \left| \int_a^x f'(t)\, dt \right|$$
$$\leqslant \frac{k^k M \eta^k \delta}{k!}.$$

Thus we have the desired estimates in this case also, and the proof is complete. ▮

The next lemma is well-known. For instance, functions of the required type may be obtained by integrating the bump-functions Φ_ε that appear in [13, Chapter IV, Lemme 3.3, p. 77].

LEMMA 5. *There are universal constants $c_k > 0$ with the following property. Given $\delta > 0$ there exists $\phi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $0 \leqslant \phi \leqslant 1$, $\phi = 0$ near $(-\infty, 0]$, $\phi = 1$ near $[\delta, +\infty)$, and*

$$\left|\frac{d^k \phi}{dx^k}\right| \leqslant \frac{c_k}{\delta^k} \qquad \forall k \geqslant 1.$$

LEMMA 6. *Let $E \subset \mathbb{R}$ be closed and $f \in C^\infty(\mathbb{R}, \mathbb{R})$. Suppose each point of E is a critical point of f. Let F be the set of points of E at which f is flat. Then f belongs to the closure in C^∞ of the set of functions $g \in C^\infty$ such that*

$$g \text{ is locally constant near } F, \tag{1}$$

and

$$\text{for each } a \in E, \text{ we have that } g \text{ is flat at } a, \text{ or } T_a^\infty g = T_a^\infty f. \tag{2}$$

Proof. Observe that f is flat on E^d, so $E^d \subset F$.

Fix $k \in \mathbb{N}$ and $R > 0$. We will show that given $\varepsilon > 0$ there exists $g \in C^\infty$ having properties (1) and (2) and such that

$$\max_{0 \leqslant i \leqslant k} \sup_{-R \leqslant x \leqslant R} |g^{(i)}(x) - f^{(i)}(x)| < \varepsilon.$$

This will suffice.

Since modifications to f off $[-R, R]$ are of no consequence, we may alter it so that it is locally-constant near each point of $E \sim [-R, R]$. In fact, if f is flat at $-R$, we may assume $f(x) = f(-R)$ for all $x < -R$, whereas if f is not flat at $-R$, then $\exists \alpha > 0$ such that $E \cap (-R - \alpha, -R) = \varnothing$, and we may modify f to have $f(x) = f(-R - \alpha)$ for all $x < -R - \alpha$. Similar modifications may be made on $[R, +\infty)$.

Let $F_R = F \cap [-R, R]$. Let

$$M = \max_{0 \leqslant i \leqslant k+1} \sup_{-R-1 \leqslant x \leqslant R+1} |f^{(i)}(x)|.$$

Fix $\delta \in (0, \frac{1}{4})$.

Each connected component of F_R is a singleton or a closed interval of positive length. For each such component $C = [a, b]$ consider the open interval $I = (a - \delta/2, b + \delta/2)$. Select a finite number $I_1, ..., I_n$ of these intervals, corresponding to components $C_1, ..., C_n$ of F_R, covering F_R. We may suppose that no I_j is contained in the union of the rest, and that they are ordered so that, with $I_j = (c_j, d_j)$, we have $c_j \leqslant c_{j+1}$.

We now carry out a process to "disjointify" the I_j.

Suppose $c_{j+1} \leqslant d_j$ for some j.

If $(c_{j+1}, d_j) \not\subset E$, pick points $d'_j < c'_{j+1}$ belonging to the same connected component of $(c_{j+1}, d_j) \sim E$, and replace I_j by (c_j, d'_j) and I_{j+1} by (c'_{j+1}, d_{j+1}).

If $c_{j+1} < d_j$ and $(c_{j+1}, d_j) \subset E$, then there is a connected component A of E^d containing $[c_{j+1}, d_j]$, and we must have $A \neq C_j$ (since $d_j \notin C_j$). Since d_j is no more than distance $\delta/2$ from C_j, we see that $(d_j - \delta/2, d_j) \sim E$ is nonempty. Pick $d'_j < c'_{j+1}$ belonging to the same component of $(d_j - \delta/2, d_j) \sim E$, and replace I_j by (c_j, d'_j) and I_{j+1} by (c'_{j+1}, d_{j+1}).

If $c_{j+1} = d_j$, then it belongs to $[-R, R]$, and either it is not a point of E, or it is an isolated point of E, since the I_j's together cover $E^d \cap [-R, R]$. In either case we may pick points $c'_{j+1} < d'_j$ in a single component of $(d_j - \delta/2, d_j) \sim E$ and proceed as in the previous case.

The effect of this modification is to produce a covering $\{I_j\}$ of F_R such that the sets clos I_j are pairwise disjoint, I_j contains a component C_j of F_R, and no point of I_j is more than δ away from C_j. Also, $c_j \notin E$ for $j > 1$ and $d_j \notin E$ for $j < n$.

Let

$$\alpha_j = \inf F_R \cap I_j, \qquad \gamma_j = f(\alpha_j),$$

$$\beta_j = \sup F_R \cap I_j, \qquad \delta_j = f(\beta_j).$$

In what follows, I_1 and I_n may need special treatment, so assume for the moment that $j \neq 1$, $j \neq n$. Then

$$c_j < \alpha_j \leqslant \beta_j < d_j.$$

We consider in turn the sets $(c_j, \alpha_j) \sim E$, $(\alpha_j, \beta_j) \sim E$, and $(\beta_j, d_j) \sim E$.

The open set $(c_j, \alpha_j) \sim E$ is nonempty, so the supremum of the lengths of its component intervals is positive. Denote this supremum by η_j^-, and select an interval $(r_j^-, s_j^-) \subset (c_j, \alpha_j) \sim E$ with $s_j^- - r_j^- = \eta_j^-$. Let $\delta_j^- = \alpha_j - c_j$. Applying Lemma 4, we see that

$$|f(x) - \gamma_j| \leqslant \frac{k^k M(\eta_j^-)^k \, \delta_j^-}{k!}, \tag{3}$$

$$|f^{(i)}(x)| \leqslant \frac{k^{k+1-i} M(\eta_j^-)^{k+1-i}}{(k+1-i)!}, \qquad (1 \leqslant i \leqslant k), \tag{4}$$

whenever $x \in (c_j, \alpha_j)$.

Similarly, in the nonempty open set $(\beta_j, d_j) \sim E$, we select an open interval (r_j^+, s_j^+) whose length is the supremum η_j^+ of the lengths of such intervals, and we let $\delta_j^+ = d_j - \beta_j$. Then we have

$$|f(x) - \delta_j| \leqslant \frac{k^k M(\eta_j^+)^k \, \delta_j^+}{k!},$$

$$|f^{(i)}(x)| \leqslant \frac{k^{k+1-i} M(\eta_j^+)^{k+1-i}}{(k+1-i)!}, \qquad (1 \leqslant i \leqslant k),$$

whenever $x \in (\beta_j, d_j)$.

Now it may happen that α_j and β_j belong to the same component of F_R. This occurs precisely when $C_j = [\alpha_j, \beta_j]$ and I_j has no points of F_R other than the points of C_j. We call this the "two-interval case," and otherwise we say we have the "three-interval case."

In the three-interval case, $(\alpha_j, \beta_j) \sim E$ is a nonempty open set. Let η_j^0 be the supremum of the lengths of its components, and select $(r_j^0, s_j^0) \subset (\alpha_j, \beta_j) \sim E$ with $s_j^0 - r_j^0 = \eta_j^0$. Let $\delta_j^0 = \beta_j - \alpha_j$. Then

$$|f(x) - \gamma_j| \leqslant \frac{k^k M(\eta_j^0)^k \, \delta_j^0}{k!},$$

$$|f(x) - \delta_j| \leqslant \frac{k^k M(\eta_j^0)^k \, \delta_j^0}{k!},$$

$$|f^{(i)}(x)| \leqslant \frac{k^{k+1-i} M(\eta_j^0)^{k+1-i}}{(k+1-i)!}, \qquad (1 \leqslant i \leqslant k),$$

whenever $x \in (\alpha_j, \beta_j)$.

By Lemma 5 we may select $\phi_j^- \in C^\infty$ such that

$$0 \leqslant \phi_j^- \leqslant 1,$$

$$\phi_j^- = 0 \text{ near}(-\infty, r_j^-],$$

$$\phi_j^- = 1 \text{ near}[s_j^-, +\infty),$$

$$|(\phi_j^-)^{(i)}| \leqslant \frac{c_i}{(\eta_j^-)^i}, \qquad \forall i \geqslant 0.$$

Similarly, we select functions ϕ_j^+, ϕ_j^0 which go from 0 to 1 across (r_j^+, s_j^+) and (r_j^0, s_j^0), and have bounds

$$|(\phi_j^+)^{(i)}| \leqslant \frac{c_i}{(\eta_j^+)^i}, \qquad |(\phi_j^0)^{(i)}| \leqslant \frac{c_i}{(\eta_j^0)^i}.$$

Now consider $j=1$. It is possible that $(c_1, \alpha_1) \subset E$. This occurs precisely when $c_1 < -R$ and $[c_1, -R] \subset E$. If this is the case, then construct ϕ_1^+ and (if necessary) ϕ_1^0 exactly as before, but take $\phi_1^- \equiv 1$. If, on the other hand, $(c_1, \alpha_1) \sim E \neq \varnothing$, then no special treatment is needed: just choose $\phi_1^\pm$ and (if necessary) ϕ_1^0 in the usual way.

Finally consider $j=n$. If $(\beta_n, d_n) \not\subset E$, then proceed as usual. Otherwise, choose ϕ_n^- and (if necessary) ϕ_n^0 as usual, but take $\phi_n^+ \equiv 1$.

In the two-interval case, let

$$h_j = \phi_j^-(1-\phi_j^+)(f-\gamma_j).$$

In the three interval case, let

$$h_j = \phi_j^-(1-\phi_j^0)(f-\gamma_j) + \phi_j^0(1-\phi_j^+)(f-\delta_j).$$

Let

$$g_\delta = f - \sum_{j=1}^n h_j.$$

Then $g_\delta \in C^\infty$. Each point $a \in F_R$ belongs to some $[\alpha_j, \beta_j]$. Now $h_r = 0$ on I_j, $\forall r \neq j$. If the two-interval case obtains, then $h_j = f - \gamma_j$ near $[\alpha_j, \beta_j]$, and hence $g_\delta = \gamma_j$ is constant near a. In the three-interval case, $h_j = f - \gamma_j$ near $[\alpha_j, r_j^0]$, $h_j = f - \delta_j$ near $[s_j^0, \beta_j]$, and $a \in [\alpha_j, \beta_j] \sim (r_j^0, s_j^0)$, so near a we have either $g_\delta = \gamma_j$ or $g_\delta = \delta_j$. Thus g_δ is locally constant near F_R.

Now consider a point $a \in E \sim F_R$. Carefully examining all the possible cases, we note that each function $\phi_j^-, \phi_j^+, \phi_j^0$ is identically 0 or identically 1 on a neighbourhood N of a, and hence, on N, h_j equals one of $0, f-\gamma_j$ or $f-\delta_j$. Moreover, the h_j have pairwise-disjoint supports, so g_δ equals one of $f, \gamma_1, \delta_1, ..., \gamma_n, \delta_n$, identically on N. Thus $T_a^\infty g_\delta = g_\delta(a)$ or $T_a^\infty f$.

It remains to estimate $|f^{(i)} - g_\delta^{(i)}|$, for $0 \leqslant i \leqslant k$. Fix $x \in [-R, R]$. We have

$$|f^{(i)}(x) - g_\delta^{(i)}(x)| = \max_{1 \leqslant j \leqslant n} |h_j^{(i)}(x)| = \max_{1 \leqslant j \leqslant n} \{A_j^-, A_j^0, A_j^+\}$$

where

$$A_j^- = \sup_{(r_j^-, s_j^-)} \left| \frac{d^i}{dx^i} \phi_j^-(f-\gamma_j) \right|,$$

$$A_j^0 = \sup_{(r_j^0, s_j^0)} \left| \frac{d^i}{dx^i} \phi_j^0(f-\gamma_j) \right| + \left| \frac{d^i}{dx^i} \phi_j^0(f-\delta_j) \right|,$$

$$A_j^+ = \sup_{(r_j^+, s_j^+)} \left| \frac{d^i}{dx^i} \phi_j^+(f-\gamma_j) \right|.$$

The three estimates are similar, so we discuss only the first. As is well-known, $\sup |f|$ and $\sup |f^{(k)}|$ together control the intermediate $\sup |f^{(i)}|$, so we need only consider $i=0$ and $i=k$. The estimate (3) trivially yields

$$|\phi_j^-(f-\gamma_j)| \leqslant \text{const} \cdot \delta,$$

since $\delta_j^- \leqslant \delta$. By Leibnitz' formula

$$\frac{d^k}{dx^k} \phi_j^-(f-\gamma_j) = \sum_{i=0}^k \binom{k}{i} (f-\gamma_j)^{(i)} (\phi_j^-)^{(k-i)}.$$

By (3), we have, where $x \in (r_j^-, s_j^-)$

$$|(f(x)-\gamma_j)(\phi_j^-)^{(k)}(x)| \leqslant \text{const} \cdot \delta,$$

and for $1 \leqslant i \leqslant k$,

$$|f^{(i)}(x)(\phi_j^-)^{(k-i)}(x)| \leqslant \text{const} \cdot \eta_j^- \leqslant \text{const} \cdot \delta.$$

Thus

$$|A_j^-| \leqslant \text{const} \cdot \delta.$$

We conclude that

$$\max_{0 \leqslant i \leqslant k} \sup_{-R \leqslant x \leqslant R} |f^{(i)}(x) - g_\delta^{(i)}(x)| \leqslant \text{const} \cdot \delta$$

where the constant depends on R and k, but not on δ. Thus we obtain the desired estimate by taking δ sufficiently small. ∎

LEMMA 7 (Factorisation Lemma). *Let $\Psi: \mathbb{R}^d \to \mathbb{R}^r$ be C^∞ and injective. Suppose $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$ is locally-constant near the critical set of Ψ. Let $K \subset \mathbb{R}^d$ be compact. Then there exists $\phi \in C^\infty(\mathbb{R}^r, \mathbb{R})$ such that $f = \phi \circ \Psi$ on K.*

Proof. Let U be an open ball in $\mathbb{R}^d$, containing K. The map Ψ is a homeomorphism of U onto $V = \Psi(U)$. Let E be the critical set of Ψ. Then Ψ is a diffeomorphism of $U \sim E$ onto the smooth imbedded d-dimensional submanifold $V \sim \Psi(E) \subset \mathbb{R}^r$. For $y \in \Psi(U)$, let $x \in U$ have $\Psi(x) = y$ and define $\phi(y) = f(x)$. Then ϕ is a C^∞ function on $V \sim \Psi(E)$ and is locally-constant on a relative neighbourhood of $\Psi(E \cap U)$ in V. The existence of a C^∞ extension of ϕ to $\mathbb{R}^r$ is a local question, so it is clear that ϕ has such an extension (since smooth functions extend from submanifolds, and constants are easy to extend). This is enough. ∎

The last lemma is a well-known consequence of de la Vallée Poussin's xtension of Weierstrass' polynomial approximation theorem to C^k pproximation.

LEMMA 8. *Let* $\Psi=(\psi_1, ..., \psi_r) \in C^\infty(\mathbb{R}^d, \mathbb{R}^r)$ *and* $\phi \in C^\infty(\mathbb{R}^r, \mathbb{R})$. *Then* $\circ \Psi$ *belongs to the closure of* $\mathbb{R}[\psi_1, ..., \psi_r]$ *in* $C^\infty(\mathbb{R}^d, \mathbb{R})$.

4. PROOF OF THEOREM

Let $\Psi=(\psi_1, ..., \psi_r)\colon \mathbb{R} \to \mathbb{R}^r$ be injective. Fix $f \in C^\infty(\mathbb{R}, \mathbb{R})$.

$(1) \Rightarrow (2)$: This is immediate from the continuity of the map $f \mapsto T_a^k f$ and the fact that $T^k\mathbb{R}[T_a^k\Psi]$ is closed in $\mathbb{R}[x]_k$.

$(2) \Rightarrow (3)$ and $(4) \Rightarrow (5)$ follow from Corollary 3.

$(2) \Rightarrow (4)$ and $(3) \Rightarrow (5)$ are obvious.

It remains to prove that $(5) \Rightarrow (1)$. (See Fig. 1.)

Suppose f has

$$T_a^\infty f \in \mathbb{R}[[T_a^\infty \Psi - \Psi(a)]]$$

whenever $\Psi'(a)=0$.

Let E denote the set $\{a \in \mathbb{R}\colon \Psi'(a)=0\}$ of critical points of Ψ. Then f is flat on E^d. By Lemma 6, we may approximate f in C^∞ by functions g that are locally-constant near E^d, and still have $T_a^\infty g \in \mathbb{R}[[T_a^\infty \Psi - \Psi(a)]]$, $\forall a \in E$. So it suffices to show that we can approximate such a function g by elements of $\mathbb{R}[\psi_1, ..., \psi_r]]$. Fix such a g.

Fix $R>0$. Since g is locally-constant near E^d, we may pick $\eta>0$ such that g is constant on $(a-\eta, a+\eta)$ for each $a \in E^d \cap [-R, R]$. Let

$$N = \bigcup_{a \in E^d \cap [-R,R]} (a-\eta, a+\eta).$$

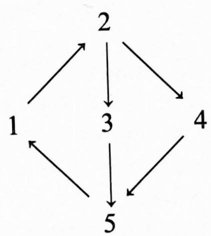

FIG. 1. The pattern of proof.

Then N is a finite union of open intervals, on each of which g is constant, and $E \cap [-R, R] \sim N$ is discrete, and hence finite. Let the open intervals be $J_1, ..., J_m$. If any endpoint of a J_i belongs to E, then we may shrink J_i by at most $\eta/2$ to avoid this. In this way we obtain

$$E \cap [-R, R] = \{a_1, ..., a_t\} \cup (N \cap E)$$

where $C = \operatorname{clos} N$ is a compact set that contains the $\eta/2$ neighbourhood of $E^d \cap [-R, R]$, g is locally-constant on N, and $E \cap \operatorname{bdy} N = \varnothing$.

For each i, pick $p_i \in \mathbb{R}[[x_1, ..., x_r]]$ such that

$$T_{a_i}^\infty g = p_i \circ (T_{a_i}^\infty \Psi - \Psi(a_i)).$$

By Borel's theorem, we may choose $\phi_i \in C^\infty(\mathbb{R}^r, \mathbb{R})$ such that $T_{\Psi(a_i)}^\infty \phi_i = p_i$.

The points $\Psi(a_i), ..., \Psi(a_t)$ are distinct, and lie outside the compact set $\Psi(C)$, so we may choose $\chi_i \in C^\infty(\mathbb{R}^r, \mathbb{R})$ such that $\chi_i = 1$ near $\Psi(a_i)$ and $\chi_i = 0$ near $\Psi(C) \cup \{\Psi(a_j)\colon j \neq i\}$. Replacing ϕ by $\chi_i \phi_i$, if need be, we may assume that

$$\operatorname{spt} \phi_i \cap \operatorname{spt} \phi_j = \varnothing, \qquad \text{whenever} \quad i \neq j,$$

and

$$\operatorname{spt} \phi_i \cap \Psi(E^d) = \varnothing, \forall i.$$

Now let $h = g - \sum_{i=1}^t \phi_i \circ \Psi$. Then $h \in C^\infty(\mathbb{R})$, h is locally-constant on N, and h is zero and flat at each point of $(E \cap [-R, R]) \sim N$. Applying Lemma 6 with E replaced by $E \cap [-R, R]$, we see that h may be approximated in C^∞ by a sequence h_n of functions that are locally-constant near $E \cap [-R, R]$. By the Factorisation Lemma, $h_n = \rho_n \circ \Psi$ near $[-R, R]$, where $\rho_n \in C^\infty(\mathbb{R}^n, \mathbb{R})$. By Lemma 8, $\rho_n \circ \Psi$ may be approximated in C^∞ by polynomials in $(\psi_1, ..., \psi_r)$, hence h can be so approximated on $[-R, R]$. Another application of Lemma 8 to $\phi_i \circ \Psi$ then yields the result. ∎

The following corollary is worth noting.

COROLLARY 9. *If* $\Psi=(\psi_1, ..., \psi_r) \in C^\infty(\mathbb{R}, \mathbb{R}^r)$ *is injective and is flat on the critical set E of* Ψ, *then* $\mathbb{R}[\psi_1, ..., \psi_r]$ *is dense in the set* $\{f \in C^\infty(\mathbb{R}) : f$ *is flat on* $E\}$.

For instance, taking

$$\psi(x) = \begin{cases} \operatorname{sgn}(x) \exp\left(\dfrac{-1}{|x|}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

we observe that $\sqrt{|\psi|}$ belongs to the closure in C^∞ of $\mathbb{R}[\psi]$. This shows that, even in the point separating case, the set $\{\phi \circ (\psi_1, ..., \psi_r): \phi \in C^\infty(\mathbb{R}^r, \mathbb{R})\}$ may be a proper subset of $\mathrm{clos}_{C^\infty} \mathbb{R}[\psi_1, ..., \psi_r]$. A very similar example (not injective) was already noted by Glaeser [6].

To give an example having a substantial critical set, we could take any injective C^∞ function $\psi: \mathbb{R} \to \mathbb{R}$ that is flat precisely on the classical Cantor set, C. Such a function may be obtained, for instance, by taking any function $\phi: \mathbb{R} \to [0, +\infty)$, smooth off C and vanishing on C, and satisfying a Hölder condition with some positive exponent, and then letting

$$\rho(x) = \begin{cases} \exp(-\phi(x)), & x \notin C, \\ 0, & x \in C, \end{cases}$$

$$\psi(x) = \int_0^x \rho(t)\, dt.$$

The corollary then says that each function flat on C belongs to the closure in C^∞ of $\mathbb{R}[\psi]$.

Finally, we record a regularity result for these algebras.

COROLLARY 10. *Suppose* $\Psi = (\psi_1, ..., \psi_r) \in C^\infty(\mathbb{R}, \mathbb{R}^r)$ *is injective. Let* A *and* B *be disjoint closed subsets of* $\mathbb{R}$. *Then* $\exists f \in \mathrm{clos}_{C^\infty} \mathbb{R}[\psi_1, ..., \psi_r]$ *such that* $f = 0$ *on* A *and* $f = 1$ *on* B.

(This result is trivial to prove if we add the hypothesis that Ψ be proper.)

Proof. Let E be the critical set of Ψ. Then E is closed and nowhere dense. It is not difficult to construct a function $f \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $f = 0$ on A, $f = 1$ on B, and for each $a \in E$ there exists $r > 0$ such that $f = 0$ on $(a - r, a + r)$ or $f = 1$ on $(a - r, a + r)$. By Corollary 9, f belongs to the closure of $\mathbb{R}[\psi_1, ..., \psi_r]$ in C^∞. ∎

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REFERENCES

1. E. Bierstone, Differentiable functions, *Bol. Soc. Brasil. Mat.* **11** (1980), 139–190.
2. E. Bierstone and P. D. Milman, Composite differentiable functions, *Ann. of Math.* **116** (1982), 541–558.
3. E. Bierstone and P. D. Milman, Geometric and differential properties of subanalytic sets, *Bull. Amer. Math. Soc. (2)* **25** (1991), 385–393.

4. E. Bierstone, P. D. Milman, and W. Pawlucki, Composite differentiable functions, *Duke Math. J.* **83** (1996), 607–620.
5. C. Chevalley, On the theory of local rings, *Ann. of Math.* **44** (1943), 690–708.
6. G. Glaeser, Fonctions composées différentiables, *Ann. of Math.* **77** (1963), 193–209.
7. F. B. Hildebrand, "Introduction to Numerical Analysis," 2nd edition, McGraw–Hill, New York, 1974.
8. J. G. Llavona, "Approximation of Continuously Differentiable Functions," North-Holland, Amsterdam, 1986.
9. B. Malgrange, "Ideals of Differentiable Functions," Tata Institute of Fundamental Research, Bombay, 1966.
10. L. Nachbin, Sur les algèbres dense de fonctions différentiables sur une variété, *C.R. Acad. Sci. Paris* **228** (1949), 1549–1551.
11. L. Nachbin, A look at approximation theory, *in* "Approximation Theory and Functional Analysis" (J. B. Prolla, Ed.), North-Holland, Amsterdam, 1979.
12. J. C. Tougeron, An extension of Whitney's spectral theorem, *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), 139–148.
13. J. C. Tougeron, "Idéaux de fonctions différentiables," Springer-Verlag, Berlin/New York, 1972.
14. J. C. Tougeron, Fonctions composées différentiables: cas algébrique, *Ann. Inst. Fourier (Grenoble)* **30** (1980), 51–74.
15. H. Whitney, Differentiable even functions, *Duke Math. J.* **10** (1943), 159–160.

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