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On Multi-Sensor Linear State Estimation Under High Rate Quantization

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***Abstract*—In this paper we consider state estimation of an unstable scalar system using multiple sensors, where the sensors quantize their individual innovations, which are then combined at the fusion center to form a global state estimate. We obtain an asymptotic expression for the error covariance (or mean squared error) that relates the system parameters and bit rates used by the different sensors. Numerical results show close agreement with the true mean squared error for quantization at high rates. An optimal rate allocation problem amongst the different sensors is also considered.**

I. Introduction

Linear state estimation using multiple sensors is a commonly performed task in areas such as radar tracking and industrial monitoring. Nowadays, much of the communication systems used in practice are digital in nature. For instance, analog measurements made by sensors will need to be quantized before transmission to a central processor or fusion center. Characterizing the performance loss due to quantization, for a linear state estimation problem, is the focus of this paper. This can be seen as a first step towards achieving a quantization rate versus state estimation error trade-off (for both single and multi-sensor cases) for linear dynamical systems, which is largely unavailable in the current literature.

We consider an unstable scalar linear system. A number of sensors take measurements, perform some local processing before transmitting a processed signal to a fusion center, that then combines these signals to form a global state estimate. At the sensor level, each sensor will quantize their innovations[1]. This is motivated by the fact that for unstable systems, while the state will become unbounded (leading to possible saturation of the quantizer), the innovations process remains of bounded variance [1]. These quantized innovations are then sent to a fusion center to form a global state estimate, using a modification of the exact decentralized scheme for unquantized Kalman filtering in [2].

The work of [3] gave structural results on optimal coding for state estimation with measurements obtained over a finite rate digital link, though the focus is more on determining minimum bit rates required for stability. For a linear quadratic control problem with quantized state feedback, the performance with high rate quantization has been studied in [4]. The idea of quantizing innovations has also been considered in [5]–[7] with slightly different filtering equations from ours. However [7] only considers the case of a single sensor, while the multi-sensor setup in [6] does not involve a fusion center but instead requires sensors to broadcast their quantized innovations to all other sensors. In [8] quantization of measurements is carried out after performing an optimization of the quantization levels, but their scheme requires feedback of the state estimates from the fusion center back to the sensors. In [9] a filter which involves quantizing the true innovations at the sensor (rather than the approximation to the true innovations considered here and in [5]–[7]) is given, but it is shown that for unstable systems the mean squared error always becomes unbounded with this scheme. Particle filtering schemes are also considered in [9], though such schemes are difficult to analyze theoretically.

The paper is organized as follows. We first briefly consider the single sensor case to motivate our choice of quantization method, filtering equations, and asymptotic analysis techniques for high rate quantization. We then consider the multi-sensor case. We obtain an asymptotic approximation for the error covariance in terms of the bit rates used by the different sensors in quantizing their innovations, as well as the system parameters. Numerical comparisons are made between the asymptotic expression and Monte Carlo simulations of the true mean squared error. While our asymptotic expressions are derived assuming high rate quantization, numerical results suggest that they are quite accurate even for bit rates as low as 3, in agreement with the conventional wisdom [10]. We also solve a rate allocation problem in the multi-sensor case for minimizing the steady-state error covariance at the fusion centre when the total rate across the sensors is limited. Performance of this optimal rate allocation scheme is analyzed both qualitatively and numerically.

II. Single Sensor

A. System model

The system is a scalar linear system

$$x_{k+1} = a x_k + w_k$$

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[1]To be more specific, we quantize an approximation to the true innovations due to the nonlinear effect of quantization

with a single scalar sensor measurement

$$y_k = cx_k + v_k$$

where $w_k \sim N(0, \sigma_w^2)$ and $v_k \sim N(0, \sigma_v^2)$ are i.i.d. in time. We consider unstable systems, i.e. $|a| > 1$, and we assume that $\{w_k\}$ and $\{v_k\}$ are mutually independent. Define the state estimates and error covariances[2]

$$\begin{aligned} \hat{x}_{k|k-1}^{kf} &= \mathbb{E}[x_k | y_0, \ldots, y_{k-1}] \\ \hat{x}_{k|k}^{kf} &= \mathbb{E}[x_k | y_0, \ldots, y_k] \\ P_{k|k-1}^{kf} &= \mathbb{E}[(x_k - \hat{x}_{k|k-1}^{kf})^2 | y_0, \ldots, y_{k-1}] \\ P_{k|k}^{kf} &= \mathbb{E}[(x_k - \hat{x}_{k|k}^{kf})^2 | y_0, \ldots, y_k] \end{aligned}$$

The innovations process is

$$\tilde{y}_k = y_k - \mathbb{E}[y_k | y_0, \ldots, y_{k-1}] = y_k - c\hat{x}_{k|k-1}^{kf}$$

It is well-known (see e.g. [1]) that

$$\tilde{y}_k \sim N(0, c^2 P_{k|k-1}^{kf} + \sigma_v^2)$$

The Kalman filtering equations (without quantization) are:

$$\begin{aligned} \hat{x}_{k|k-1}^{kf} &= a\hat{x}_{k-1|k-1}^{kf} \\ \hat{x}_{k|k}^{kf} &= \hat{x}_{k|k-1}^{kf} + K_k^{kf}(y_k - c\hat{x}_{k|k-1}^{kf}) = \hat{x}_{k|k-1}^{kf} + K_k^{kf}\tilde{y}_k \\ K_k^{kf} &= \frac{P_{k|k-1}^{kf} c}{c^2 P_{k|k-1}^{kf} + \sigma_v^2} \\ P_{k|k-1}^{kf} &= a^2 P_{k-1|k-1}^{kf} + \sigma_w^2 \\ P_{k|k}^{kf} &= P_{k|k-1}^{kf} - K_k^{kf} c P_{k|k-1}^{kf} = \frac{P_{k|k-1}^{kf}\sigma_v^2}{c^2 P_{k|k-1}^{kf} + \sigma_v^2} \end{aligned} \tag{1}$$

For $c \neq 0$, as $k \to \infty$, $P_{k|k-1}^{kf}$ converges to a steady state value:

$$\begin{aligned} P_\infty^{kf} &= \\ &\frac{-(\sigma_v^2 - c^2\sigma_w^2 - a^2\sigma_v^2) + \sqrt{(\sigma_v^2 - c^2\sigma_w^2 - a^2\sigma_v^2)^2 + 4c^2\sigma_v^2\sigma_w^2}}{2c^2} \\ &= \frac{-(1 - \sigma_w^2 s - a^2) + \sqrt{(1 - \sigma_w^2 s - a^2)^2 + 4\sigma_w^2 s}}{2s} \end{aligned} \tag{2}$$

where $s \triangleq \frac{c^2}{\sigma_v^2}$ can be regarded as a sensor signal to noise ratio.

B. Quantized filtering scheme

Here we consider a suboptimal scheme, where we run a slightly modified version of the unquantized filtering equations given in (1):

$$\begin{aligned} \hat{x}_{k|k-1} &= a\hat{x}_{k-1|k-1} \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k q(y_k - c\hat{x}_{k|k-1}) \\ K_k &= \frac{P_{k|k-1} c}{c^2 P_{k|k-1} + \sigma_v^2 + \sigma_n^2} \\ P_{k|k-1} &= a^2 P_{k-1|k-1} + \sigma_w^2 \\ P_{k|k} &= P_{k|k-1} - K_k c P_{k|k-1} = \frac{P_{k|k-1}(\sigma_v^2 + \sigma_n^2)}{c^2 P_{k|k-1} + \sigma_v^2 + \sigma_n^2} \end{aligned} \tag{3}$$

where $q(y_k - c\hat{x}_{k|k-1})$ is the quantization of $y_k - c\hat{x}_{k|k-1}$, and σ_n^2 is a term to account for "quantization noise", similar to e.g. [11]. Note that due to quantization $\hat{x}_k$, P_k, and $y_k - c\hat{x}_{k|k-1}$ are not the true conditional mean, error covariance and innovations respectively, but for high rate quantization the approximations are quite accurate.

We will use a uniform quantizer, motivated by the result that for uniform quantization of certain random variables such as Gaussian random variables, the quantization error at high rates is approximately uncorrelated with the quantizer input [12], so that one can write

$$q(y_k - c\hat{x}_{k|k-1}) \approx y_k - c\hat{x}_{k|k-1} + n_k$$

where n_k is the quantization noise. Under high rate quantization, the distortion is approximated by (see e.g. [10]):

$$D \approx \frac{1}{12N^2} \int \frac{f(x)}{\lambda^2(x)} dx \triangleq \sigma_n^2$$

where N is the number of quantization levels, $f(x)$ is the p.d.f. of the random variable to be quantized, and $\lambda(x)$ is the point density. At high rates, we will also assume that the quantity $y_k - c\hat{x}_{k|k-1}$ is approximately $N(0, c^2 P_\infty + \sigma_v^2)$, where P_∞ is the limit of $P_{k|k-1}$ in (3) as $k \to \infty$. We use a uniform quantizer with N quantization levels, truncating the range to $[-3\sqrt{c^2 P_\infty + \sigma_v^2}, 3\sqrt{c^2 P_\infty + \sigma_v^2}]$, which is motivated by the rule of thumb in statistics that 99.7% of samples lie within 3 standard deviations of the mean. The point density will then be $\lambda(x) = \frac{1}{6\sqrt{c^2 P_\infty + \sigma_v^2}}$, and hence

$$\sigma_n^2 = \frac{1}{12N^2} \int 36(c^2 P_\infty + \sigma_v^2) f(x) dx = \frac{3(c^2 P_\infty + \sigma_v^2)}{N^2}.$$

The value of P_∞ can then be found by solving for

$$P_\infty = \frac{a^2 P_\infty(\sigma_v^2 + \sigma_n^2)}{c^2 P_\infty + \sigma_v^2 + \sigma_n^2} + \sigma_w^2$$

with $\sigma_n^2 = \frac{3(c^2 P_\infty + \sigma_v^2)}{N^2}$. This can be rearranged into a quadratic equation

$$\begin{aligned} &\left[c^2(1 + \frac{3}{N^2}) - \frac{3a^2c^2}{N^2}\right] P_\infty^2 - \sigma_w^2\sigma_v^2(1 + \frac{3}{N^2}) \\ &+ \left[\sigma_v^2(1 + \frac{3}{N^2}) - \sigma_w^2 c^2(1 + \frac{3}{N^2}) - a^2\sigma_v^2(1 + \frac{3}{N^2})\right] P_\infty = 0 \end{aligned}$$

[2]As in [9], we use the superscript "*kf*" to denote the true Kalman filtering quantities.

so that

$$P_\infty = \frac{-(1+\frac{3}{N^2})(1-a^2-\sigma_w^2 s)}{2s(1+\frac{3}{N^2}-\frac{3a^2}{N^2})} + \frac{\sqrt{(1+\frac{3}{N^2})^2(1-a^2-\sigma_w^2 s)^2+4\sigma_w^2 s(1+\frac{3}{N^2}-\frac{3a^2}{N^2})(1+\frac{3}{N^2})}}{2s(1+\frac{3}{N^2}-\frac{3a^2}{N^2})} \quad (4)$$

with $s = c^2/\sigma_v^2$.

C. Asymptotic analysis

Here we determine the asymptotic behaviour of (4) for large N. We have

$$\begin{aligned}&\sqrt{(1+\tfrac{3}{N^2})^2(1-a^2-\sigma_w^2 s)^2+4\sigma_w^2 s(1+\tfrac{3}{N^2}-\tfrac{3a^2}{N^2})(1+\tfrac{3}{N^2})}\\ &= \left[D_1 + \frac{D_2}{N^2} + O(\frac{1}{N^4})\right]^{1/2}\\ &= \sqrt{D_1} + \frac{D_2}{2\sqrt{D_1}}\frac{1}{N^2} + O(\frac{1}{N^4})\end{aligned}$$

where $D_1 \triangleq (1-a^2-\sigma_w^2 s)^2 + 4\sigma_w^2 s$ and $D_2 \triangleq 6(1-a^2-\sigma_w^2 s)^2 + 4\sigma_w^2 s(6-3a^2) = 6D_1 - 12\sigma_w^2 s a^2$. Then

$$\begin{aligned}P_\infty &= \Big(\frac{-(1-a^2-\sigma_w^2 s) - \frac{3(1-a^2-\sigma_w^2 s)}{N^2}}{2s}\\ &\quad + \frac{\sqrt{D_1} + \frac{D_2}{2\sqrt{D_1}}\frac{1}{N^2} + O(\frac{1}{N^4})}{2s}\Big)\Big(1 - \frac{3-3a^2}{N^2} + O(\frac{1}{N^4})\Big)\\ &= P_\infty^{kf} + \frac{\frac{D_2}{2\sqrt{D_1}} + 3(a^2-1)\sqrt{D_1} - 3a^2(1-a^2-\sigma_w^2 s)}{2s}\frac{1}{N^2}\\ &\quad + O(\frac{1}{N^4})\end{aligned}$$

where P_∞^{kf} as given by (2) is the steady state error covariance when there is no quantization. Alternatively, we can express this in terms of the bit rate $R = \log_2(N)$, so that

$$\begin{aligned}P_\infty = P_\infty^{kf} &+ \frac{\frac{D_2}{2\sqrt{D_1}} + 3(a^2-1)\sqrt{D_1} - 3a^2(1-a^2-\sigma_w^2 s)}{2s}\frac{1}{2^{2R}}\\ &+ O(\frac{1}{2^{4R}})\end{aligned}$$

We thus see that P_∞ asymptotically behaves like the unquantized error covariance P_∞^{kf} plus a term that decays to zero at the rate $1/2^{2R}$.

III. Multiple Sensors

The system is again a scalar linear system

$$x_{k+1} = a x_k + w_k,$$

but now with M different sensors taking scalar sensor measurements (using i to denote the sensor index and k the time index):

$$y_{i,k} = c_i x_k + v_{i,k}, i = 1, \ldots, M$$

where $w_k \sim N(0, \sigma_w^2)$ and $v_{i,k} \sim N(0, \sigma_{i,v}^2)$ are i.i.d. in time. We assume that $\{w_k\}$ and $\{v_{i,k}\}, \forall i$ are mutually independent. It is assumed that the individual sensors can perform some local processing, with a fusion center then using an appropriate fusion rule to compute a global estimate of the state x_k. See Fig. 1 for a diagram of the system model.

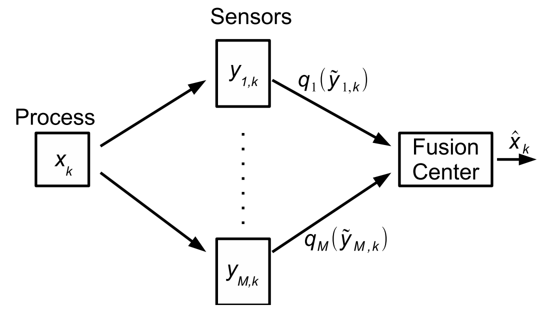

Fig. 1. System model: Multi-sensor

A. Decentralized Kalman filter

In [2], it is shown that in the case where there is no quantization, each sensor can run its own individual Kalman filter to obtain local state estimates, which can then be combined at the fusion center to obtain a global state estimate, with this global estimate being the same as if the fusion center had access to the individual measurements. We summarize the equations below.

Define the local estimates and error covariances:

$$\begin{aligned}\hat{x}_{i,k|k-1}^{kf} &= \mathbb{E}[x_k | y_{i,0}, \ldots, y_{i,k-1}]\\ \hat{x}_{i,k|k}^{kf} &= \mathbb{E}[x_k | y_{i,0}, \ldots, y_{i,k}]\\ P_{i,k|k-1}^{kf} &= \mathbb{E}[(x_k - \hat{x}_{i,k|k-1}^{kf})^2 | y_{i,0}, \ldots, y_{i,k-1}]\\ P_{i,k|k}^{kf} &= \mathbb{E}[(x_k - \hat{x}_{i,k|k}^{kf})^2 | y_{i,0}, \ldots, y_{i,k}]\end{aligned}$$

and the global estimates and error covariances:

$$\begin{aligned}\hat{x}_{k|k-1}^{kf} &= \mathbb{E}[x_k | \mathbf{y}_0, \ldots, \mathbf{y}_{k-1}]\\ \hat{x}_{k|k}^{kf} &= \mathbb{E}[x_k | \mathbf{y}_0, \ldots, \mathbf{y}_k]\\ P_{k|k-1}^{kf} &= \mathbb{E}[(x_k - \hat{x}_{k|k-1}^{kf})^2 | \mathbf{y}_0, \ldots, \mathbf{y}_{k-1}]\\ P_{k|k}^{kf} &= \mathbb{E}[(x_k - \hat{x}_{k|k}^{kf})^2 | \mathbf{y}_0, \ldots, \mathbf{y}_k]\end{aligned}$$

where $\mathbf{y}_k \triangleq (y_{1,k}, \ldots, y_{M,k})$.

The sensors run their individual Kalman filtering equations, for $i = 1, \ldots, M$, whose equations take the form (1) but replacing y_k with $y_{i,k}$, c with c_i, σ_v^2 with $\sigma_{i,v}^2$ etc. The fusion center makes use of the local estimates $\hat{x}_{i,k|k-1}^{kf}$ and $\hat{x}_{i,k|k}^{kf}$ and local error covariances $P_{i,k|k-1}^{kf}$ and $P_{i,k|k}^{kf}$ to compute global state estimates as follows:

$$\begin{aligned}\hat{x}^{kf}_{k|k-1} &= a\hat{x}^{kf}_{k-1|k-1}\\ \hat{x}^{kf}_{k|k} &= P^{kf}_{k|k}\left(\frac{\hat{x}^{kf}_{k|k-1}}{P^{kf}_{k|k-1}} + \sum_{i=1}^{M}\left\{\frac{\hat{x}^{kf}_{i,k|k}}{P^{kf}_{i,k|k}} - \frac{\hat{x}^{kf}_{i,k|k-1}}{P^{kf}_{i,k|k-1}}\right\}\right)\\ P^{kf}_{k|k-1} &= a^2 P^{kf}_{k-1|k-1} + \sigma_w^2\\ P^{kf}_{k|k} &= \frac{P^{kf}_{k|k-1}}{1 + P^{kf}_{k|k-1}\sum_{i=1}^{M}\frac{c_i^2}{\sigma_{i,v}^2}}\end{aligned} \quad (5)$$

Note that instead of the sensors sending their local estimates and error covariances, the local innovations $\tilde{y}_{i,k} = y_{i,k} - c_i\hat{x}^{kf}_{i,k|k-1}$ can be sent to the fusion center instead, since the fusion center can reconstruct $\hat{x}^{kf}_{i,k|k-1}$, $\hat{x}^{kf}_{i,k|k}$, $P^{kf}_{i,k|k-1}$ and $P^{kf}_{i,k|k}$ from $\tilde{y}_{i,k}$ provided it has knowledge of all the sensor parameters c_i and $\sigma_{i,v}^2$, $i = 1, \dots, M$.

For later reference, the local error covariances $P^{kf}_{i,k|k-1}$ have steady state values

$$P^{kf}_{i,\infty} = \frac{-(1 - \sigma_w^2 s_i - a^2)}{2s_i} + \frac{\sqrt{(1 - \sigma_w^2 s_i - a^2)^2 + 4\sigma_w^2 s_i}}{2s_i}$$

while the error covariance $P^{kf}_{k|k-1}$ has steady state value

$$P^{kf}_{\infty} = \frac{-(1 - \sigma_w^2\sum_{i=1}^{M} s_i - a^2)}{2\sum_{i=1}^{M} s_i} + \frac{\sqrt{(1 - \sigma_w^2\sum_{i=1}^{M} s_i - a^2)^2 + 4\sigma_w^2\sum_{i=1}^{M} s_i}}{2\sum_{i=1}^{M} s_i}$$

where $s_i \triangleq \frac{c_i^2}{\sigma_{i,v}^2}$ can be regarded as the individual sensor signal to noise ratios.

B. Quantized filtering scheme

As in the single sensor case, we can consider a suboptimal scheme which is a slightly modified version of the unquantized Kalman filtering equations. The individual sensors run the following equations, for $i = 1, \dots, M$:

$$\begin{aligned}\hat{x}_{i,k|k-1} &= a\hat{x}_{i,k-1|k-1}\\ \hat{x}_{i,k|k} &= \hat{x}_{i,k|k-1} + K_{i,k}q_i(y_{i,k} - c_i\hat{x}_{i,k|k-1})\\ K_{i,k} &= \frac{P_{i,k|k-1}c_i}{c_i^2 P_{i,k|k-1} + \sigma_{i,v}^2 + \sigma_{i,n}^2}\\ P_{i,k|k-1} &= a^2 P_{i,k-1|k-1} + \sigma_w^2\\ P_{i,k|k} &= \frac{P_{i,k|k-1}(\sigma_{i,v}^2 + \sigma_{i,n}^2)}{c_i^2 P_{i,k|k-1} + \sigma_{i,v}^2 + \sigma_{i,n}^2}\end{aligned} \quad (6)$$

while the fusion center runs the following equations:

$$\begin{aligned}\hat{x}_{k|k-1} &= a\hat{x}_{k-1|k-1}\\ \hat{x}_{k|k} &= P_{k|k}\left(\frac{\hat{x}_{k|k-1}}{P_{k|k-1}} + \sum_{i=1}^{M}\left\{\frac{\hat{x}_{i,k|k}}{P_{i,k|k}} - \frac{\hat{x}_{i,k|k-1}}{P_{i,k|k-1}}\right\}\right)\\ P_{k|k-1} &= a^2 P_{k-1|k-1} + \sigma_w^2\\ P_{k|k} &= \frac{P_{k|k-1}}{1 + P_{k|k-1}\sum_{i=1}^{M}\frac{c_i^2}{\sigma_{i,v}^2 + \sigma_{i,n}^2}}\end{aligned} \quad (7)$$

where $q_i(y_{i,k} - c_i\hat{x}_{i,k|k-1})$ is the quantization of $y_{i,k} - c_i\hat{x}_{i,k|k-1}$, with corresponding term $\sigma_{i,n}^2$ to account for the quantization noise. The values $q_i(y_{i,k} - c_i\hat{x}_{i,k|k-1})$ are the quantities that are sent to the fusion center. Similar to the remark in the previous subsection, the fusion center can reconstruct $\hat{x}_{i,k|k-1}$, $\hat{x}_{i,k|k}$, $P_{i,k|k-1}$ and $P_{i,k|k}$ from $q_i(y_{i,k} - c_i\hat{x}_{i,k|k-1})$ and knowledge of the sensor parameters.

We will again use uniform quantization, with N_i quantizer levels for sensor i. At high rates, assuming that $y_{i,k} - c_i\hat{x}_{i,k|k-1}$ is approximately $N(0, c_i^2 P_{i,\infty} + \sigma_{i,v}^2)$, and using a quantization range $[-3\sqrt{c_i^2 P_{i,\infty} + \sigma_{i,v}^2}, 3\sqrt{c_i^2 P_{i,\infty} + \sigma_{i,v}^2}]$, we obtain similar to the single sensor case that

$$\sigma_{i,n}^2 = \frac{3(c_i^2 P_{i,\infty} + \sigma_{i,v}^2)}{N_i^2}$$

where $P_{i,\infty}$ is the steady state value of $P_{i,k|k-1}$ and satisfies

$$P_{i,\infty} = \frac{a^2 P_{i,\infty}(\sigma_{i,v}^2 + \sigma_{i,n}^2)}{c_i^2 P_{i,\infty} + \sigma_{i,v}^2 + \sigma_{i,n}^2} + \sigma_w^2.$$

Then $P_{i,\infty}$ has solution of the form (4), but replacing c with c_i, σ_v^2 with $\sigma_{i,v}^2$, s with s_i, N with N_i etc.

The quantity $P_{k|k-1}$ has steady state value

$$P_{\infty} = \frac{-(1 - \sigma_w^2\sum_{i=1}^{M}\frac{c_i^2}{\sigma_{i,v}^2 + \sigma_{i,n}^2} - a^2)}{2\sum_{i=1}^{M}\frac{c_i^2}{\sigma_{i,v}^2 + \sigma_{i,n}^2}} + \frac{\sqrt{(1 - \sigma_w^2\sum_{i=1}^{M}\frac{c_i^2}{\sigma_{i,v}^2 + \sigma_{i,n}^2} - a^2)^2 + 4\sigma_w^2\sum_{i=1}^{M}\frac{c_i^2}{\sigma_{i,v}^2 + \sigma_{i,n}^2}}}{2\sum_{i=1}^{M}\frac{c_i^2}{\sigma_{i,v}^2 + \sigma_{i,n}^2}} \quad (8)$$

C. Asymptotic analysis

We now determine the asymptotic behaviour of (8) as $N_i \to \infty, \forall i$. From the analysis of the single sensor case, we have

$$P_{i,\infty} = P^{kf}_{i,\infty} + O\left(\frac{1}{N_i^2}\right)$$

and hence

$$\sigma_{i,n}^2 = \frac{3(c_i^2 P^{kf}_{i,\infty} + \sigma_{i,v}^2)}{N_i^2} + O\left(\frac{1}{N_i^4}\right)$$

By similar methods, we can further obtain

$$\sum_{i=1}^{M}\frac{c_i^2}{\sigma_{i,v}^2 + \sigma_{i,n}^2} = \sum_{i=1}^{M}\left(s_i - \frac{3s_i(s_i P^{kf}_{i,\infty} + 1)}{N_i^2} + O\left(\frac{1}{N_i^4}\right)\right)$$

and

$$\sqrt{(1-\sigma_w^2\sum_{i=1}^M \frac{c_i^2}{\sigma_{i,v}^2+\sigma_{i,n}^2}-a^2)^2+4\sigma_w^2\sum_{i=1}^M \frac{c_i^2}{\sigma_{i,v}^2+\sigma_{i,n}^2}}$$
$$=\sqrt{E_1}+\frac{1}{2\sqrt{E_1}}\sum_{i=1}^M \frac{E_{2,i}}{N_i^2}+\sum_{i,j} O\left(\frac{1}{N_i^2 N_j^2}\right)$$

where

$$E_1 \triangleq (1-a^2-\sigma_w^2\sum_{j=1}^M s_j)^2+4\sigma_w^2\sum_{j=1}^M s_j$$

and

$$E_{2,i} \triangleq -6\sigma_w^2\left(1+a^2+\sigma_w^2\sum_{j=1}^M s_j\right)s_i(s_i P_{i,\infty}^{kf}+1)$$

for $i=1,\dots,M$. Finally, after some algebraic manipulation we can obtain

$$\begin{aligned} P_\infty &= P_\infty^{kf}+\sum_{i=1}^M \frac{e_i}{N_i^2}+\sum_{i,j} O\left(\frac{1}{N_i^2 N_j^2}\right) \\ &= P_\infty^{kf}+\sum_{i=1}^M \frac{e_i}{2^{2R_i}}+\sum_{i,j} O\left(\frac{1}{2^{2R_i+2R_j}}\right) \end{aligned} \quad (9)$$

where $R_i=\log_2(N_i)$ and

$$\begin{aligned} e_i \triangleq\ & \frac{3s_i(s_i P_{i,\infty}^{kf}+1)}{2\left(\sum_{j=1}^M s_j\right)^2} \\ & \times\left[a^2-1+\sqrt{E_1}-\frac{\sigma_w^2(1+a^2+\sigma_w^2\sum_{j=1}^M s_j)\sum_{j=1}^M s_j}{\sqrt{E_1}}\right] \end{aligned} \quad (10)$$

It is not too difficult to show that $e_i \geq 0, \forall i$.

Thus in the multi-sensor case, P_∞ asymptotically behaves like the unquantized error covariance P_∞^{kf} plus a sum of M terms, with each term decaying to zero at the rate $1/2^{2R_i}$.

D. A rate allocation problem

Suppose we are given R_{tot}, where R_{tot} is large. We want to determine how this total rate is to be allocated amongst the sensors. Let $R_i=\alpha_i R_{tot}$ where $0\leq\alpha_i\leq 1$. One way to allocate the rates is to minimize the asymptotic expression given by (9), subject to the total rate being less than or equal to R_{tot}, i.e. the problem:

$$\begin{aligned} &\min_{\alpha_1,\dots,\alpha_M} P_\infty^{kf}+\sum_{i=1}^M \frac{e_i}{2^{2\alpha_i R_{tot}}} \\ &\text{s.t.} \sum_{i=1}^M \alpha_i \leq 1, \alpha_i\geq 0 \end{aligned} \quad (11)$$

with e_i given by (10). Note that in problem (11), we are not constraining the rates to be integer values, which allows this problem to be solved analytically.[3] We have the following result:

Lemma 1: The optimization problem (11), where $e_i\geq 0$ are constants, has solution

$$\alpha_i^* = \frac{1}{M}+\frac{1}{2R_{tot}}\log_2\frac{e_i}{\left(\prod_{j=1}^M e_j\right)^{1/M}} \quad (12)$$

Proof: The optimal solution follows from analyzing the Karush-Kuhn-Tucker conditions. The derivation is omitted. ■

From the solution (12) we see that larger values of e_i should be allocated higher rates. From (10) we see that larger values of e_i correspond to larger values of $s_i(s_i P_{i,\infty}^{kf}+1)$. It is not difficult to show that

$$s_i P_{i,\infty}^{kf}=\frac{-(1-\sigma_w^2 s_i-a^2)+\sqrt{(1-\sigma_w^2 s_i-a^2)^2+4\sigma_w^2 s_i}}{2}$$

is an increasing function of s_i, hence $s_i(s_i P_{i,\infty}^{kf}+1)$ is also an increasing function of s_i. Thus larger values of e_i correspond to larger values of the sensor signal to noise ratios $s_i=\frac{c_i^2}{\sigma_{i,v}^2}$.

Another observation that can be made from (12) is that for fixed M, as $R_{tot}\to\infty$, $\alpha_i^*\to\frac{1}{M}$, so for high R_{tot} each sensor should be allocated approximately equal proportions of the total rate.

E. Numerical studies

In Fig. 2 we plot P_∞ given by (8), the asymptotic expression for P_∞ given by (9) and compare these expressions with Monte Carlo simulations of the mean squared error $\mathbb{E}[(x_k-\hat{x}_{k|k-1})^2]$ using the estimator given by the equations (6)-(7). We consider the case with two sensors, with parameters $a=1.2, c_1=1, c_2=1$, $\sigma_w^2=1$, $\sigma_{1,v}^2=0.1, \sigma_{2,v}^2=1$. We set $N_1=N_2=N$. As we can see, the asymptotic approximation becomes more accurate for higher rates. Note that in this example $P_\infty^{kf}=1.1211$.

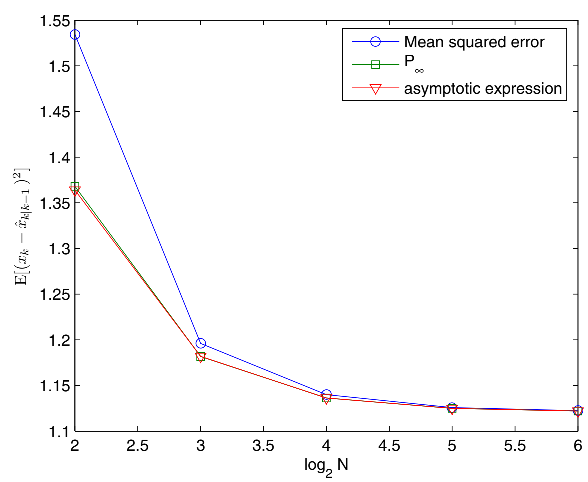

Fig. 2. Mean squared error and asymptotic expression: Multi-sensor

[3]Otherwise we would have an integer programming problem that is difficult to solve efficiently.

TABLE I
MEAN SQUARED ERROR AND ASYMPTOTIC EXPRESSION

$R_1 = \log_2(N_1)$	$R_2 = \log_2(N_2)$	Mean squared error	P_∞ expression (8)	Asymptotic expression (9)
1	7	2.0503	1.7129	2.0699
2	6	1.4507	1.3334	1.3585
3	5	1.1981	1.1790	1.1805
4	4	1.1368	1.1363	1.1363
5	3	1.1278	1.1262	1.1262
6	2	1.1290	1.1262	1.1277
7	1	1.1375	1.1302	1.1441

We will next consider the optimal rate allocation problem (11). For the same parameters $a = 1.2, c_1 = 1, c_2 = 1, \sigma_w^2 = 1, \sigma_{1,v}^2 = 0.1, \sigma_{2,v}^2 = 1$, with $R_{tot} = 8$, we find that the optimal solution is $\alpha_1^* = 0.6682$, $\alpha_2^* = 0.3318$, corresponding to rates $R_1^* = 5.3459$, $R_2^* = 2.6541$. In Table I we tabulate the results for some integer combinations of $R_1 = \log_2(N_1)$ and $R_2 = \log_2(N_2)$, with $R_1 + R_2 = 8$. We see that $R_1 = 5$, $R_2 = 3$ gives the best performance. Thus in this case, the best integer valued rates are given by rounding the solution obtained from (11) to the nearest integers.

IV. CONCLUSION

We have derived an asymptotic approximation to the error covariance for linear state estimation of an unstable scalar system with quantized innovations, valid when the sensors use high rate quantization. Extensions of this work to vector systems is currently under investigation.

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