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Optimal LQG control over continuous fading channels ⋆

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Abstract: This paper studies the optimal control of linear systems over continuous valued fading channels. The sensor measurements are sent over a fading wireless channel to a remote controller using the analog amplify and forward technique. The controller then computes a control signal, which is transmitted over another fading channel to the actuator. Under the assumption of full channel state information (CSI) for both wireless links, we derive the optimal LQG control law. In the case where full channel state information is not available but channel statistics are available, we present the optimal linear static estimator and controller. Numerical comparisons are made between the full CSI and statistical CSI solutions.

Keywords: Fading channels, LQG control, sensor networks, stability

1. INTRODUCTION

In recent years there has been increasing growth in the use of wireless technologies in diverse applications such as home automation, telecommunications, and industrial monitoring and control. The challenges posed by wireless channels due to its time-varying nature are considerable, and much effort has been devoted to modelling and overcoming these effects.

One way to model the wireless channel is to regard it as a channel where packets can be received if the channel is of sufficiently good quality, and dropped if the channel is poor quality. The Kalman filtering problem with Bernoulli packet losses has been studied in Sinopoli et al. (2004), who showed the existence of a threshold, such that if the packet arrival rate is below this threshold then the expected error covariance becomes unbounded. This work has been extended in various directions such as e.g. Huang and Dey (2007); Epstein et al. (2008); Xu and Hespanha (2005); Schenato (2006). The problem of control over such packet dropping links has been studied in e.g. Sinopoli et al. (2005, 2006); Imer et al. (2006); Gupta et al. (2007), and conditions on the packet arrival rates for stability of the closed loop system has been derived. See also Schenato et al. (2007) and the references therein for a review of related work.

Another way in which one can view the wireless channel is to regard it as a continuous valued channel with time-varying channel gains, with commonly used channel models such as Rayleigh or Nakagami. Kalman filtering with continuous faded measurements has been studied in e.g. Mostofi and Murray (2005); Dey et al. (2009), which showed that under certain conditions on the fading distribution the expected error covariance will always remain bounded.

In this paper we extend the work of Dey et al. (2009) to LQG control over channels with continuous faded measurements. We assume fading channels between both the sensor and controller, and between the controller and actuator/plant. Under the assumption of full channel state information (CSI) we derive the optimal control law and show that the separation principle holds. We consider both the finite horizon and infinite horizon problems. In the case where channel state information is not available but channel statistics are available, the optimal linear estimator/controller will be presented, using results from De Koning (1992).

2. SYSTEM MODEL

A block diagram of the model we study in this paper can be found in Fig. 1. We consider a plant

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$. We have a sensor with measurements

$$y_k = Cx_k + v_k \tag{2}$$

where $y_k \in \mathbb{R}^l$. The noise processes $\{w_k\}$ and $\{v_k\}$ are i.i.d. zero mean Gaussian with covariances $\Sigma_w > 0$ and $\Sigma_v > 0$ respectively. [1]

The sensor transmits this measurement over a fading channel to a remote controller using the analog amplify and forwarding technique (Gastpar and Vetterli (2003)), i.e. the sensor simply amplifies and forwards its measurement to the controller. We assume that all the measurement components are sent separately via orthogonal channels

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[1] For a matrix X, we say that $X > 0$ if X is positive definite, and $X \geq 0$ if X is positive semi-definite.

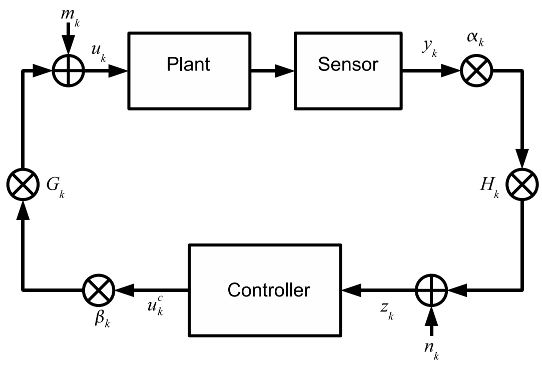

Fig. 1. System model

(Cui et al. (2007)) within the measurement time interval. The controller thus receives

$$z_k = H_k \alpha_k y_k + n_k \tag{3}$$

where $H_k = \text{diag}(h_k^1, \ldots, h_k^l)$, with $h_k^i \geq 0, i = 1, \ldots, l$, are the channel gains, $\alpha_k = \text{diag}(\alpha_k^1, \ldots, \alpha_k^l)$ are the amplification factors in the analog forwarding technique, and n_k is additive noise that represents the channel noise in the communication channel between the sensor and controller. The controller computes a control signal $u_k^c \in \mathbb{R}^m$, which is then sent over another fading channel to the actuator/plant, again using the analog forwarding technique. The control input to the plant is thus

$$u_k = G_k \beta_k u_k^c + m_k \tag{4}$$

where $G_k = \text{diag}(g_k^1, \ldots, g_k^m)$, with $g_k^j \geq 0, j = 1, \ldots, m$, are the channel gains, $\beta_k = \text{diag}(\beta_k^1, \ldots, \beta_k^m)$ are the amplification factors and m_k is the channel noise between the controller and plant. The noise processes $\{n_k\}$ and $\{m_k\}$ are i.i.d. zero mean Gaussian with covariances Σ_n and Σ_m respectively. In this paper we assume the block fading model (see e.g. Caire et al. (1999)), such that the channels stay constant within each fading block represented by the time index k, but are independent from block to block. We also allow the fading processes G_k and H_k to have continuous distributions in general. The noise processes w_k, v_k, n_k, m_k and fading processes G_k, H_k are assumed to be mutually independent.

The system (1)-(4) above can be rewritten as

$$\begin{aligned} x_{k+1} &= A x_k + \bar{B}_k u_k^c + \bar{w}_k \\ z_k &= \bar{C}_k x_k + \bar{v}_k \end{aligned} \tag{5}$$

if we define $\bar{B}_k = B G_k \beta_k,\ \bar{C}_k = H_k \alpha_k C,\ \bar{w}_k = B m_k + w_k,\ \bar{v}_k = H_k \alpha_k v_k + n_k$. The noise processes $\{\bar{w}_k\}$ and $\{\bar{v}_k\}$ have covariances $\Sigma_{\bar{w}} = B \Sigma_m B^T + \Sigma_w$ and $\Sigma_{\bar{v}_k} = H_k \alpha_k \Sigma_v \alpha_k H_k + \Sigma_n$ respectively.

3. OPTIMAL LQG CONTROL UNDER FULL CSI

We first consider the case where we have full CSI (so full knowledge of G_k and H_k is available to the controller at time k). The amplification factors α_k and β_k are usually chosen to satisfy power constraints at the sensor transmitter and in the transmission of the control signals. Here α_k and β_k are taken to be either constant or known functions of time.

3.1 Finite horizon

In the finite horizon case, we have a cost

$$J_N = \mathbb{E}\left[x_N^T Q_N x_N + \sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^{c^T} R_k u_k^c) \right] \tag{6}$$

where $Q_k \geq 0, \forall k$ and $R_k > 0, \forall k$. With full CSI, the information set available to the controller at time k is

$$I_k = \{z_0, \ldots, z_k, u_0^c, \ldots, u_{k-1}^c, H_0, \ldots, H_k, G_0, \ldots, G_k\} \tag{7}$$

Our objective is to minimize J_N for system (5), where the minimization is over $\{u_k^c\}$, with u_k^c being a function of the information set I_k at each time k.

Lemma 1. The optimal control u_k^{c*} that minimizes J_N in (6), subject to u_k^c being a function of the information set I_k in (7), is

$$u_k^{c*} = -(\bar{B}_k^T K_{k+1} \bar{B}_k + R_k)^{-1} \bar{B}_k^T K_{k+1} A \hat{x}_k \tag{8}$$

where $\hat{x}_k = \mathbb{E}[x_k | I_k]$, $\bar{B}_k = B G_k \beta_k$, and $\{K_k\}$ are given recursively by

$$\begin{aligned} K_N &= Q_N, \\ K_k &= \mathbb{E}[A^T (K_{k+1} - K_{k+1} \bar{B}_k \\ &\quad \times (R_k + \bar{B}_k^T K_{k+1} \bar{B}_k)^{-1} \bar{B}_k^T K_{k+1}) A] + Q_k. \end{aligned} \tag{9}$$

The expectation in (9) is with respect to G_k (since $\bar{B}_k = B G_k \beta_k$).

Proof The proof uses dynamic programming and is along similar lines to e.g. Bertsekas (2000), see also Imer et al. (2006). Define

$$\begin{aligned} V_N(I_N) &= \mathbb{E}[x_N^T Q_N x_N | I_N], \\ V_k(I_k) &= \min_{u_k^c} \mathbb{E}[x_k^T Q_k x_k + u_k^{c^T} R_k u_k^c + V_{k+1}(I_{k+1}) | I_k] \end{aligned}$$

We have

$$\begin{aligned} &V_{N-1}(I_{N-1}) \\ &= \min_{u_{N-1}^c} \mathbb{E}\{x_{N-1}^T Q_{N-1} x_{N-1} + u_{N-1}^{c^T} R_{N-1} u_{N-1}^c \\ &\quad + (A x_{N-1} + \bar{B}_{N-1} u_{N-1}^c + \bar{w}_{N-1})^T Q_N \\ &\quad \times (A x_{N-1} + \bar{B}_{N-1} u_{N-1}^c + \bar{w}_{N-1} | I_{N-1}\} \\ &= \mathbb{E}\{x_{N-1}^T (A^T Q_N A + Q_{N-1}) x_{N-1} | I_{N-1}\} \\ &\quad + \mathbb{E}\{\bar{w}_{N-1}^T Q_N \bar{w}_{N-1}\} \\ &\quad + \min_{u_{N-1}^c} \{u_{N-1}^{c^T} (\bar{B}_{N-1}^T Q_N \bar{B}_{N-1} + R_{N-1}) u_{N-1}^c \\ &\quad + 2\mathbb{E}[x_{N-1}^T | I_{N-1}] A^T Q_N \bar{B}_{N-1} u_{N-1}^c\} \end{aligned}$$

which gives

$$u_{N-1}^{c*} = -(\bar{B}_{N-1}^T Q_N \bar{B}_{N-1} + R_{N-1})^{-1} \bar{B}_{N-1}^T Q_N A \hat{x}_{N-1}$$

Substituting back into the expression for $V_{N-1}(I_{N-1})$, we obtain

$$\begin{aligned} V_{N-1}(I_{N-1}) &= \mathbb{E}\{\bar{w}_{N-1}^T Q_N \bar{w}_{N-1}\} \\ &\quad + \mathbb{E}\{(x_{N-1} - \hat{x}_{N-1})^T \tilde{P}_{N-1} (x_{N-1} - \hat{x}_{N-1}) | I_{N-1}\} \\ &\quad + \mathbb{E}\{x_{N-1}^T \tilde{K}_{N-1} x_{N-1} | I_{N-1}\} \end{aligned}$$

where

$$\begin{aligned} \tilde{P}_{N-1} &= A^T Q_N \bar{B}_{N-1} (R_{N-1} + \bar{B}_{N-1}^T Q_N \bar{B}_{N-1})^{-1} \bar{B}_{N-1}^T Q_N A, \\ \tilde{K}_{N-1} &= A^T Q_N A + Q_{N-1} - \tilde{P}_{N-1} \end{aligned}$$

For period $N-2$ we have

$$
\begin{aligned}
&V_{N-2}(I_{N-2})\\
&= \min_{u^c_{N-2}} \mathbb{E}\{x^T_{N-2}Q_{N-2}x_{N-2}\\
&\quad + u^{c^T}_{N-2}R_{N-2}u^c_{N-2} + V_{N-1}(I_{N-1})|I_{N-2}\}\\
&= \mathbb{E}\{x^T_{N-2}Q_{N-2}x_{N-2}|I_{N-2}\}\\
&\quad + \min_{u^c_{N-2}} [u^{c^T}_{N-2}R_{N-2}u^c_{N-2} + \mathbb{E}\{x^T_{N-1}\tilde{K}_{N-1}x_{N-1}|I_{N-2}\}]\\
&\quad + \mathbb{E}\{(x_{N-1} - \hat{x}_{N-1})^T P_{N-1}(x_{N-1} - \hat{x}_{N-1})|I_{N-2}\}\\
&\quad + \mathbb{E}\{\bar{w}^T_{N-1}Q_N\bar{w}_{N-1}\}
\end{aligned}
$$

Now

$$
\begin{aligned}
&\min_{u^c_{N-2}} [u^{c^T}_{N-2}R_{N-2}u^c_{N-2} + \mathbb{E}\{x^T_{N-1}\tilde{K}_{N-1}x_{N-1}|I_{N-2}\}]\\
&= \min_{u^c_{N-2}} [u^{c^T}_{N-2}R_{N-2}u^c_{N-2}\\
&\quad + \mathbb{E}\{(Ax_{N-2} + \bar{B}_{N-2}u^c_{N-2} + \bar{w}_{N-2})^T\tilde{K}_{N-1}\\
&\quad \times (Ax_{N-2} + \bar{B}_{N-2}u^c_{N-2} + \bar{w}_{N-2})|I_{N-2}\}]\\
&= \min_{u^c_{N-2}} \{u^{c^T}_{N-2}R_{N-2}u^c_{N-2} + u^{c^T}_{N-2}\bar{B}^T_{N-2}\mathbb{E}[\tilde{K}_{N-1}]\bar{B}_{N-2}u^c_{N-2}\\
&\quad + 2u^{c^T}_{N-2}\bar{B}^T_{N-2}\mathbb{E}[\tilde{K}_{N-1}]A\hat{x}_{N-2}\}\\
&\quad + \mathbb{E}[x^T_{N-2}A^T\tilde{K}_{N-1}Ax_{N-2}|I_{N-2}] + \mathbb{E}[\bar{w}^T_{N-2}\tilde{K}_{N-1}\bar{w}_{N-2}]
\end{aligned}
$$

and so

$$
\begin{aligned}
u^{c*}_{N-2} &= -(\bar{B}^T_{N-2}\mathbb{E}[\tilde{K}_{N-1}]\bar{B}_{N-2} + R_{N-2})^{-1}\\
&\quad \times \bar{B}^T_{N-2}\mathbb{E}[\tilde{K}_{N-1}]A\hat{x}_{N-2}
\end{aligned}
$$

Continuing on, we will obtain

$$u^{c*}_k = -(\bar{B}^T_k K_{k+1}\bar{B}_k + R_k)^{-1}\bar{B}^T_k K_{k+1}A\hat{x}_k$$

where K_k are given by

$$
\begin{aligned}
K_N &= Q_N,\\
K_k &= \mathbb{E}[A^T(K_{k+1} - K_{k+1}\bar{B}_k\\
&\quad \times (R_k + \bar{B}^T_k K_{k+1}\bar{B}_k)^{-1}\bar{B}^T_k K_{k+1})A] + Q_k.
\end{aligned}
$$

■

The optimal control u^{c*}_k in (8) is a linear function of $\hat{x}_k = \mathbb{E}[x_k|I_k]$, which can be computed with the standard time-varying Kalman filter. Hence a separation principle holds for this problem. Further define

$$
\begin{aligned}
\hat{x}_{k+1|k} &= \mathbb{E}[x_{k+1}|I_k]\\
P_{k+1|k} &= \mathbb{E}[(x_{k+1} - \hat{x}_{k+1|k})(x_{k+1} - \hat{x}_{k+1|k})^T|I_k]\\
P_{k|k} &= \mathbb{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T|I_k]
\end{aligned}
$$

Below, we summarise the Kalman filtering equations:

$$
\begin{aligned}
\hat{x}_k &= \hat{x}_{k|k-1} + S_k(z_k - \bar{C}_k\hat{x}_{k|k-1})\\
\hat{x}_{k|k-1} &= A\hat{x}_{k-1} + \bar{B}_{k-1}u^c_{k-1}\\
S_k &= P_{k|k-1}\bar{C}^T_k(\bar{C}_kP_{k|k-1}\bar{C}^T_k + \Sigma_{\bar{v}_k})^{-1}\\
P_{k|k} &= P_{k|k-1} - S_k\bar{C}_kP_{k|k-1}\\
P_{k|k-1} &= AP_{k-1|k-1}A^T + \Sigma_{\bar{w}}
\end{aligned}
$$

where $\bar{B}_k = BG_k\beta_k$, $\bar{C}_k = H_k\alpha_kC$, $\Sigma_{\bar{w}} = B\Sigma_mB^T + \Sigma_w$ and $\Sigma_{\bar{v}_k} = H_k\alpha_k\Sigma_v\alpha_kH_k + \Sigma_n$.

Remark: The expectation in (9) is in general difficult to compute analytically, due to the difficulty in explicitly evaluating the expectation of the nonlinear term. However in cases such as scalar control signals, closed form expressions can be obtained for specific fading distributions. See Section 5 for an example with Rayleigh fading.

Remark: If the fading process $\{G_k\}$ is discrete (i.e. components of G_k take on discrete values), then the optimal controller can also be derived by using results on optimal control of jump linear systems, see Chizeck and Ji (1988).

Remark: System (5) is a time-varying linear system. If we attempt to apply the standard solution of the time-varying LQG problem to (5) directly, we obtain

$$u^c_k = -(R_k + \bar{B}^T_kK_{k+1}\bar{B}_k)^{-1}\bar{B}^T_kK_{k+1}A\hat{x}_{k|k} \tag{10}$$

where K_k are given by

$$
\begin{aligned}
K_N &= Q_N,\\
K_k &= A^T(K_{k+1} - K_{k+1}\bar{B}_k \qquad (11)\\
&\quad \times (R_k + \bar{B}^T_kK_{k+1}\bar{B}_k)^{-1}\bar{B}^T_kK_{k+1})A + Q_k.
\end{aligned}
$$

However, the recursions given by (11) are non-causal since K_k requires knowledge of $G_{k+j}, j = 1, 2, \ldots$, and so u^c_k given by (10) is not a function of the information set I_k in (7). We will however use this non-causal solution in numerical comparisons with the optimal causal solution in Section 5.

3.2 Infinite horizon

In the infinite horizon case we take $\alpha_k = \alpha, \beta_k = \beta, Q_k = Q > 0, R_k = R > 0, \forall k$.[2] We will assume that the pairs (A, B) and $(A, \Sigma_w^{1/2})$ are stabilizable, and the pairs (A, C) and $(A, Q^{1/2})$ are detectable. We now have a cost function

$$J_\infty = \lim_{N\to\infty}\frac{J_N}{N} = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\sum_{k=0}^{N-1}(x^T_kQx_k + u^{c^T}_kRu^c_k)\right] \tag{12}$$

We have the following result:

Lemma 2. Assume that α and β are invertible, the components of G_k have continuous distributions such that $\Pr(g^i_k > 0) = 1, \forall k, i$, and the components of H_k have continuous distributions such that $\Pr(h^j_k > 0) = 1, \forall k, j$. Furthermore assume that A is invertible and that $\max(0, \log||H_0C||)$ is integrable. Then
(i) The expected error covariance $\mathbb{E}[P_{k|k}]$ remains bounded as $k \to \infty$.
(ii) The optimal cost J^*_∞ is finite, and the optimal control u^{c*}_k that minimizes J_∞ in (12), subject to u^c_k being a function of the information set I_k in (7), is

$$u^{c*}_k = -(\bar{B}^T_kK\bar{B}_k + R)^{-1}\bar{B}^T_kKA\hat{x}_k \tag{13}$$

where $\hat{x}_k = \mathbb{E}[x_k|I_k]$, and K is the unique solution of the fixed point equation

$$K = \mathbb{E}[A^T(K - K\bar{B}_k(R + \bar{B}^T_kK\bar{B}_k)^{-1}\bar{B}^T_kK)A] + Q \tag{14}$$

Proof

(i) Under the assumptions of Lemma 2, the boundedness of the expected error covariance $\mathbb{E}[P_{k+1|k}]$ (and hence $\mathbb{E}[P_{k|k}]$) for Kalman filtering with faded measurements (and no control) has previously been shown in Dey et al. (2009). Noting that the Kalman filtering recursions for $P_{k|k}$ do not depend on the control signals u^c_k (Anderson and Moore, 1979, p.110), the result follows.
(ii) We first show that the optimal control takes the form (13). From the finite horizon recursions (9) and reversing

[2] For instance, the case $\alpha_k = 1, \beta_k = 1$ would correspond to direct forwarding of the sensor measurements and control signals without any scaling.

the time index as in Imer et al. (2006), we have the recursion

$$\acute{K}_{k+1} = \mathbb{E}[A^T(\acute{K}_k - \acute{K}_k BG\beta(R + \beta GB^T \acute{K}_k BG\beta)^{-1} \times \beta GB^T \acute{K}_k)A] + Q$$

Under our stabilizability and detectability assumptions, it can be shown that as $k \to \infty$, $\acute{K}_k$ converges to the unique fixed point of the equation

$$K = \mathbb{E}[A^T(K - KBG\beta(R + \beta GB^T KBG\beta)^{-1}\beta GB^T K)A] + Q$$

by using a similar proof to Theorem 3.3 of Dey et al. (2009). Taking the limit $N \to \infty$ of the solution to the finite horizon problem then gives the desired result.

We now show that the optimal cost J^*_∞ is finite. Let us call $L_k = -(\bar{B}_k^T K \bar{B}_k + R)^{-1}\bar{B}_k^T KA$ and $e_k = x_k - \hat{x}_k$. Noting that $\mathbb{E}[\hat{x}_k e_k^T] = \mathbb{E}[e_k \hat{x}_k^T] = 0$, we have

$$\begin{aligned}
&\frac{1}{N}\mathbb{E}\left[\sum_{k=0}^{N-1}(x_k^T Q x_k + u_k^{c*^T} R u_k^{c*})\right] \\
&= \frac{1}{N}\mathbb{E}\left[\sum_{k=0}^{N-1}(x_k^T Q x_k + \hat{x}_k^T L_k^T R L_k \hat{x}_k)\right] \\
&= \frac{1}{N}\mathbb{E}\left[\sum_{k=0}^{N-1}(x_k^T (Q + L_k^T R L_k) x_k - e_k^T L_k^T R L_k e_k)\right] \\
&= \frac{1}{N}\mathbb{E}\left[\sum_{k=0}^{N-1}\left[\mathrm{Tr}((Q + L_k^T R L_k) x_k x_k^T) - \mathrm{Tr}(L_k^T R L_k e_k e_k^T)\right]\right] \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\mathrm{Tr}(\mathbb{E}(Q + L_k^T R L_k)\mathbb{E}(x_k x_k^T)) \\
&\quad - \frac{1}{N}\sum_{k=0}^{N-1}\mathrm{Tr}(\mathbb{E}(L_k^T R L_k)\mathbb{E}(P_{k|k}))
\end{aligned}$$

where the last line holds since x_k does not depend on G_k (and hence L_k). The second term above remains bounded as $N \to \infty$ by part (i). The first term above will also be bounded as $N \to \infty$ if we can show $\mathbb{E}(x_k x_k^T)$ is bounded for the system

$$x_{k+1} = Ax_k + \bar{B}_k L_k \hat{x}_k + \bar{w}_k = (A + \bar{B}_k L_k)x_k - \bar{B}_k L_k e_k + \bar{w}_k$$

By similar arguments as in Imer et al. (2006), this is true if and only if the system

$$\xi_{k+1} = (A + \bar{B}_k L_k)\xi_k$$

is mean square stable. We can verify that

$$\begin{aligned}
&(A + \bar{B}_k L_k)^T K (A + \bar{B}_k L_k) + L_k^T R L_k + Q \\
&\quad = A^T KA - A^T K^T \bar{B}_k(\bar{B}_k^T K \bar{B}_k + R)^{-1}\bar{B}_k^T KA + Q
\end{aligned}$$

and so

$$K = \mathbb{E}[(A + \bar{B}_k L_k)^T K (A + \bar{B}_k L_k) + L_k^T R L_k + Q]$$

Hence

$$\begin{aligned}
&\mathbb{E}[\xi_{k+1}^T K \xi_{k+1} - \xi_k K \xi_k] \\
&\quad = \mathbb{E}[\xi_k^T((A + \bar{B}_k L_k)^T K (A + \bar{B}_k L_k) - K)\xi_k] \\
&\quad = -\mathbb{E}[\xi_k^T \mathbb{E}(L_k^T R L_k + Q)\xi_k]
\end{aligned}$$

where the last line holds since ξ_k does not depend on G_k. Therefore

$$\mathbb{E}[\xi_{k+1}^T K \xi_{k+1}] = \mathbb{E}[\xi_0^T K \xi_0] - \sum_{i=0}^{k}\mathbb{E}[\xi_i^T \mathbb{E}(L_i^T R L_i + Q)\xi_i]$$

Using similar arguments to Imer et al. (2006) (see also Bertsekas (2000)), we can then show that $\mathbb{E}[\xi_k^T \xi_k] \to 0$ as $k \to \infty$. ■

Thus for any fading processes G_k and H_k satisfying the conditions of Lemma 2, the problem of minimizing (12) is well defined, and the minimum J_∞ will be finite.

4. OPTIMAL LINEAR CONTROL WITH STATISTICAL CSI

In this section we consider the case where we don't have knowledge of the values G_k and H_k (e.g. either because it is too difficult or requires too many resources to obtain), but know their channel statistics. By similar arguments to Schenato et al. (2007), the optimal controller can be shown to be generally nonlinear and difficult to derive. One alternative is to derive the optimal *linear* controller and estimator. Here we will use a static linear estimator and controller of the form

$$\begin{aligned}
\hat{x}_{k+1} &= F\hat{x}_k + Kz_k \\
u_k^c &= -L\hat{x}_k
\end{aligned} \tag{15}$$

We again consider the infinite horizon case where we take $\alpha_k = \alpha, \beta_k = \beta, Q_k = Q > 0, R_k = R > 0, \forall k$, and assume that the pairs (A, B) and $(A, \Sigma_w^{1/2})$ are stabilizable, and the pairs (A, C) and (A, Q) are detectable. The cost function that we wish to minimize is the infinite horizon cost

$$J_\infty = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\sum_{k=0}^{N-1}(x_k^T Q x_k + u_k^{c^T} R u_k^c)\right] \tag{16}$$

where the minimization is over (F, K, L). The situation above falls within the framework of systems with white parameters. In De Koning (1992) necessary and sufficient conditions for minimizing (16) subject to the estimator and controller being of the form (15) is derived, using techniques such as the matrix minimum principle. A method for computing the optimal F, K and L is then also given. Below we will present the method of De Koning (1992) adapted to our situation. [3]

Define the following recursions:

$$\begin{aligned}
X_{1,k+1} &= A^T X_{1,k} A - L_k^T(\mathbb{E}[\bar{B}_k^T X_{1,k}\bar{B}_k] + R \\
&\quad + \mathbb{E}[\bar{B}_k^T X_{2,k}\bar{B}_k] - \mathbb{E}[\bar{B}_k^T]X_{2,k}\mathbb{E}[\bar{B}_k])L_k \\
&\quad + Q + \mathbb{E}[\bar{C}_k^T K_k^T X_{2,k} K_k \bar{C}_k] - \mathbb{E}[\bar{C}_k^T]K_k^T X_{2,k} K_k \mathbb{E}[\bar{C}_k] \\
X_{2,k+1} &= (A - K_k\mathbb{E}[\bar{C}_k])^T X_{2,k}(A - K_k\mathbb{E}[\bar{C}_k]) \\
&\quad + L_k^T(\mathbb{E}[\bar{B}_k^T X_{1,k}\bar{B}_k] + R \\
&\quad + \mathbb{E}[\bar{B}_k^T X_{2,k}\bar{B}_k] - \mathbb{E}[\bar{B}_k^T]X_{2,k}\mathbb{E}[\bar{B}_k])L_k \\
X_{3,k+1} &= AX_{3,k}A^T - K_k(\mathbb{E}[\bar{C}_k X_{3,k}\bar{C}_k^T] + \mathbb{E}[\Sigma_{\bar{v}_k}] \\
&\quad + \mathbb{E}[\bar{C}_k X_{4,k}\bar{C}_k^T] - \mathbb{E}[\bar{C}_k]X_{4,k}\mathbb{E}[\bar{C}_k^T])K_k^T \\
&\quad + \Sigma_{\bar{w}} + \mathbb{E}[\bar{B}_k L_k X_{4,k} L_k^T \bar{B}_k^T] - \mathbb{E}[\bar{B}_k]L_k X_{4,k} L_k^T \mathbb{E}[\bar{B}_k^T] \\
X_{4,k+1} &= (A - \mathbb{E}[\bar{B}_k]L_k)X_{4,k}(A - \mathbb{E}[\bar{B}_k]L_k)^T \\
&\quad + K_k(\mathbb{E}[\bar{C}_k X_{3,k}\bar{C}_k^T] + \mathbb{E}[\Sigma_{\bar{v}_k}] + \mathbb{E}[\bar{C}_k X_{4,k}\bar{C}_k^T] \\
&\quad - \mathbb{E}[\bar{C}_k]X_{4,k}\mathbb{E}[\bar{C}_k^T])K_k^T
\end{aligned} \tag{17}$$

[3] Note that in our situation $\Sigma_{\bar{v}_k}$ is time-varying, however it can be verified that the role of W in De Koning (1992) can be replaced by $\mathbb{E}[\Sigma_{\bar{v}_k}]$ here.

$$
\begin{aligned}
F_{k+1} &= A - \mathbb{E}[\bar{B}_k]L_k - K_k\mathbb{E}[\bar{C}_k] \\
K_{k+1} &= AX_{3,k}\mathbb{E}[\bar{C}_k^T](\mathbb{E}[\bar{C}_k X_{3,k}\bar{C}_k^T] + \mathbb{E}[\Sigma_{\bar{v}_k}] \\
&\quad + \mathbb{E}[\bar{C}_k X_{4,k}\bar{C}_k^T] - \mathbb{E}[\bar{C}_k]X_{4,k}\mathbb{E}[\bar{C}_k^T])^\dagger \\
L_{k+1} &= (\mathbb{E}[\bar{B}_k^T X_{1,k}\bar{B}_k] + R + \mathbb{E}[\bar{B}_k^T X_{2,k}\bar{B}_k] \\
&\quad - \mathbb{E}[\bar{B}_k^T]X_{2,k}\mathbb{E}[\bar{B}_k])^\dagger\mathbb{E}[\bar{B}_k^T]X_{1,k}A
\end{aligned}
\tag{18}
$$

where † represents the Moore-Penrose inverse. We also have the concept of mean square compensatability introduced in De Koning (1992).

Definition: We say that $(A, \bar{B}_k, \bar{C}_k)$ is mean square compensatable if there exist F, K, L such that the system

$$x'_{k+1} = \Phi'_k x'_k$$

has $\mathbb{E}[||x'_k||^2] \to 0$ as $k \to \infty$, where

$$\Phi'_k = \begin{bmatrix} A & -\bar{B}_k L \\ K\bar{C}_k & F \end{bmatrix}.$$

We then have the following:

Lemma 3. (i) Assume that $(A, \bar{B}_k, \bar{C}_k)$ is mean square compensatable. Then starting from $X_{1,0} = 0, X_{2,0} = 0, X_{3,0} = 0, X_{4,0} = 0$ (where 0 here represents the zero matrix), the recursions (17)-(18) converge to limiting values as $k \to \infty$. The optimal F^*, K^*, L^* for (15) are given by the limiting values of the F_k, K_k, L_k recursions respectively.
(ii) $(A, \bar{B}_k, \bar{C}_k)$ is mean square compensatable if and only if the recursions (17)-(18) converge to limiting values as $k \to \infty$.

Proof (i) This is essentially Theorem 3 of De Koning (1992).
(ii) See Theorem 4 of De Koning (1992).
■

The expectations involved in (17)-(18) can usually be computed without difficulty. For instance, we have $\mathbb{E}[\bar{B}_k] = B\mathbb{E}[G_k]\beta = B\mathrm{diag}(\mathbb{E}[g_1], \ldots, \mathbb{E}[g_m])\beta$, and $\mathbb{E}[\bar{C}_k] = \mathrm{diag}(\mathbb{E}[h_1], \ldots, \mathbb{E}[h_l])\alpha C$. Next call

$$\Gamma = \begin{bmatrix} \mathbb{E}[g_1^2] & \mathbb{E}[g_1]\mathbb{E}[g_2] & \ldots & \mathbb{E}[g_1]\mathbb{E}[g_m] \\ \mathbb{E}[g_2]\mathbb{E}[g_1] & \mathbb{E}[g_2^2] & \ldots & \mathbb{E}[g_2]\mathbb{E}[g_m] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[g_m]\mathbb{E}[g_1] & \mathbb{E}[g_m]\mathbb{E}[g_2] & \ldots & \mathbb{E}[g_m^2] \end{bmatrix}$$

Then note that $\mathbb{E}[GXG] = \Gamma \circ X$, where $\circ$ is the Hadamard or element-wise product (Horn and Johnson (1991)). Hence $\mathbb{E}[\bar{B}_k X \bar{B}_k^T] = B\mathbb{E}[G\beta X\beta G]B^T = B(\Gamma \circ (\beta X \beta))B^T$ and $\mathbb{E}[\bar{B}_k^T X \bar{B}_k] = \beta(\Gamma \circ (B^T X B))\beta$. Similarly, if we call

$$\Lambda = \begin{bmatrix} \mathbb{E}[h_1^2] & \mathbb{E}[h_1]\mathbb{E}[h_2] & \ldots & \mathbb{E}[h_1]\mathbb{E}[h_l] \\ \mathbb{E}[h_2]\mathbb{E}[h_1] & \mathbb{E}[h_2^2] & \ldots & \mathbb{E}[h_2]\mathbb{E}[h_l] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[h_l]\mathbb{E}[h_1] & \mathbb{E}[h_l]\mathbb{E}[h_2] & \ldots & \mathbb{E}[h_l^2] \end{bmatrix},$$

then $\mathbb{E}[\bar{C}_k X \bar{C}_k^T] = \Lambda \circ (\alpha C X C^T \alpha)$ and $\mathbb{E}[\bar{C}_k^T X \bar{C}_k] = C^T\alpha(\Lambda \circ X)\alpha C$. We also have $\mathbb{E}[\Sigma_{\bar{v}_k}] = \Lambda \circ (\alpha\Sigma_v\alpha) + \Sigma_n$.

Remark: The stability criteria of Lemma 3 (ii) involves checking if $(X_{1,k}, X_{2,k}, X_{3,k}, X_{4,k})$ converges as $k \to \infty$ in the recursion (17)-(18). Determining whether the recursions converge can be achieved via numerical computation as described above, however analytical criteria seem to be more complicated to obtain.

5. NUMERICAL EXAMPLE

We consider a scalar system, with g_k and h_k both Rayleigh distributed, so that g_k^2 and h_k^2 are exponentially distributed with means $1/\lambda_g$ and $1/\lambda_h$ respectively.

The optimal control in the case of full CSI is then

$$u_k^{c*} = \frac{-g_k a K_{k+1}\beta_k b}{g_k^2\beta_k^2 b^2 K_{k+1} + R_k}\hat{x}_k$$

with

$$
\begin{aligned}
K_N &= Q_N \\
K_k &= \mathbb{E}\left[\frac{a^2 K_{k+1} R_k}{g_k^2\beta_k^2 b^2 K_{k+1} + R_k}\right] + Q_k \\
&= \frac{\lambda_g a^2 R_k}{\beta_k^2 b^2}\exp\left(\frac{\lambda_g R_k}{\beta_k^2 b^2 K_{k+1}}\right) E_1\left(\frac{\lambda_g R_k}{\beta_k^2 b^2 K_{k+1}}\right) + Q_k
\end{aligned}
$$

where $E_1(x)$ is the exponential integral.

In the computation of the optimal linear controller in the case with statistical CSI, the terms in the recursions simplify to $\mathbb{E}[\bar{B}_k] = \beta b\sqrt{\frac{\pi}{4\lambda_g}}, \mathbb{E}[\bar{C}_k] = \alpha c\sqrt{\frac{\pi}{4\lambda_h}}$, $\Sigma_{\bar{w}} = b^2\sigma_m^2 + \sigma_w^2, \mathbb{E}[\Sigma_{\bar{v}_k}] = \frac{\alpha^2\sigma_v^2}{\lambda_h} + \sigma_n^2$, $\mathbb{E}[\bar{B}_k^T X \bar{B}_k] = \mathbb{E}[\bar{B}_k X \bar{B}_k^T] = \frac{\beta^2 b^2}{\lambda_g}X$, $\mathbb{E}[\bar{C}_k^T X \bar{C}_k] = \mathbb{E}[\bar{C}_k X \bar{C}_k^T] = \frac{\alpha^2 c^2}{\lambda_h}X$.

We will consider a case with $b = c = 1$, $\sigma_w^2 = \sigma_v^2 = \sigma_n^2 = \sigma_m^2 = 1$, $\alpha = \beta = 1$, $Q = R = 1$, $\lambda_g = 2, \lambda_h = 5$. In Figure 2 we plot the finite horizon expected cost J_N for horizon $N = 10$, and various values of a. We compare between the causal control given by (8)-(9), and the non-causal control given by (10)-(11). The causal solution can be seen to perform quite closely to the non-causal solution.

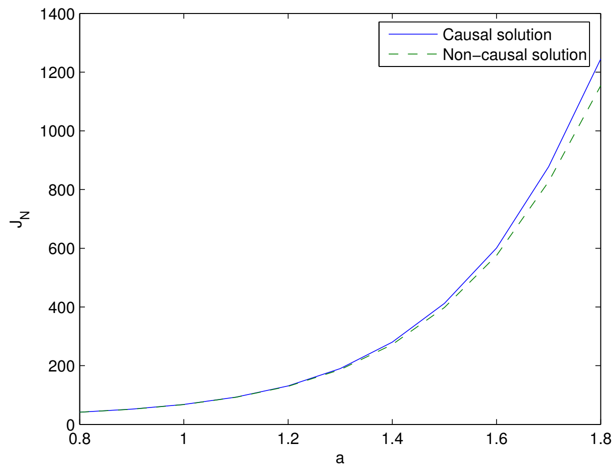

Fig. 2. Scalar system, finite horizon. Comparison between causal and non-causal control.

In Figure 3 we plot the infinite horizon cost J_∞ for the infinite horizon solution with full CSI (13)-(14), and infinite horizon solution with statistical CSI of Section 4, for different values of a. From numerical computation, we find that for values of a greater than around 1.35, the stability criteria is no longer satisfied in the case with statistical CSI and the infinite horizon cost diverges.

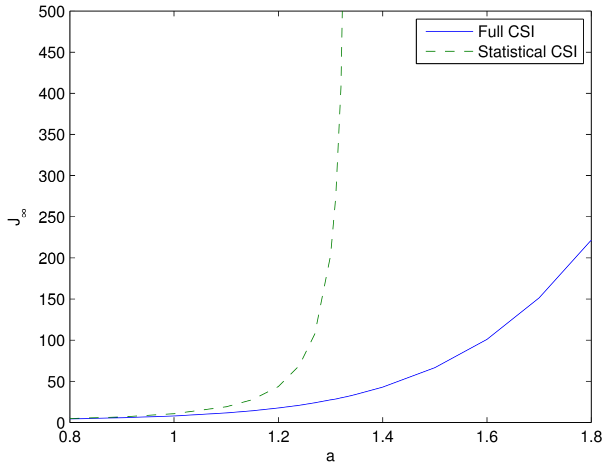

Fig. 3. Scalar system, infinite horizon. Comparison between full CSI and statistical CSI solutions

6. CONCLUSION

In this paper we have considered the optimal control of a system where there are continuous valued fading channels between the sensor and controller, and between the controller and actuator. We have derived the optimal LQG controller under full CSI and statistical CSI assumptions. Future work will include jointly optimizing the powers used in transmitting the sensor measurements and control signals over the fading channels, by optimizing the choices of the amplification factors α_k and β_k.

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