Reduced-complexity estimation for Poisson processes modulated by nearly completely decomposable Markov chains

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Abstract — **In this paper, we address the problem of complexity reduction in state estimation of Poisson processes modulated by continuous-time nearly completely decomposable Markov chains.**

I. Introduction

Nearly completely decomposable Markov chains (NCDMC) are usually large scale, and show strong interactions within groups (called "superstates") and weak interactions between the groups. The problem of reduced-complexity estimation for partially observed discrete-time NCDMCs was first addressed in [1]. It was shown that for sufficiently small ϵ, (where ϵ is the parameter that signifies the weak coupling between the superstates) one can obtain $O(\epsilon^2)$ (and under some special cases, $O(\epsilon^3)$) approximations to the aggregate as well as the full-order conditional probability estimates. The number of computations per unit time for this algorithm is $O(M^2)$ instead of $O(S^2)$ where S is the total number of states and M is the number of "superstates" within which the NCDMC shows strong interactions. Typically $M \ll S$, which implies the significant computational savings achievable through this algorithm. Following similar ideas, we address the problem of reduced-complexity state estimation of Poisson processes modulated by continuous-time NCDMCs. Applications of such processes have been identified in studying Markov modulated traffic in queueing networks with bursty traffic (e.g, ATM networks) [2], due to 1) inherent multiple-scale nature of specialized traffic like variable bit rate video traffic and 2) multiplexing of heterogeneous traffic, featuring sources with very different time constants.

II. State Estimation

Consider a probability space $(\Omega, \mathcal{F}, P)$. Let $X_t, t \geq 0$ be a continuous-time Markov chain defined on this space with state space $\{e_1, e_2, \ldots, e_S\}$ where $e_i \in \mathbb{R}^S$ is the unit vector with 1 in the i-th position. Let the infinitesimal generator or transition rate matrix be denoted by A where $\sum_{j=1}^{S} a_{ij} = 0, \ \forall i \in \{1, 2, \ldots, S\}$. Define $P(X_t = i) = p_t^i, \ i \in \{1, 2, \ldots, S\}$. The probability distribution $p_t = (p_t^1 \, p_t^2 \ldots p_t^S)'$ satisfies the forward equation $\frac{dp_t}{dt} = A' p_t$ where $'$ denotes the transpose operation. The Markov chain X_t modulates an L-variate integrable Poisson process $N_t = (N_t^{(1)} \, N_t^{(2)} \ldots N_t^{(L)})'$ as follows:

$$dN_t^{(l)} = \langle X_t, g^{(l)} \rangle dt + dm_t^{(l)}, \ l = 1, 2, \ldots, L \tag{1}$$

where $N_t^{(l)}$ denotes the number of events (e.g, in the case of teletraffic, number of packets arrivals) with mark l that occur during the interval $[0, t]$, and $g^{(l)} = (g_1^{(l)} \, g_2^{(l)} \ldots g_S^{(l)})'$ is the vector of intensities of the l-th component of the process N_t. $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^S$ and $m_t^{(l)}$ is a $\mathcal{F}_t$ martingale where $\mathcal{F}_t$ denotes $\sigma(X_s, N_s; s \leq t)$. Denote the observation history as $\mathcal{N}_t^{(l)} = \sigma(N_s^{(l)} : s \leq t)$ and $\mathcal{N}_t = \bigvee_{l=1}^{L} \mathcal{N}_t^{(l)}$. Note also that for $L = 1$, we drop the superscript l in the appropriate notations. Defining an unnormalized measure of $E[X_t | \mathcal{N}_t]$ as q_t, we have the following Zakai filter

$$dq_t = A' q_t dt + \sum_{l=1}^{L} (B^{(l)} - I) q_t dn_t^{(l)} \tag{2}$$

where $B^{(l)} = diag[g^{(l)}]$ and $n_t^{(l)} = N_t^{(l)} - t$.

Considering the univariate case ($L = 1$), a robust time-discretized approximation to q_t (by sampling at regular time intervals separated by h), can be written as (following similar techniques as in [3])

$$\alpha_{n+1} = \alpha_n (I + Ah) C_{n+1} \tag{3}$$

where $\alpha_n = q_n'$ and C_{n+1} is a diagonal matrix with the i-th diagonal entry being $\exp[-(g_i - 1)h] g_i^{\Delta N_{(n+1)}}$, $\Delta N_{(n+1)} = N_{(n+1)h} - N_{nh}$.

Note that $(I + Ah)$ is a stochastic matrix and the normalized conditional probability estimate $\hat{\alpha}_n$ can be obtained as $\hat{\alpha}_n = \frac{\alpha_n}{<\alpha_n, \underline{1}>}$ where $\underline{1} \in \mathbb{R}^S$ is a column vector of all 1's.

III. Reduced-complexity estimation with nearly completely decomposable Markov chains

For a nearly completely decomposable structure, the transition probability matrix of the Markov chain is given as $\bar{A} + \epsilon \bar{B}$ where $\bar{A}$ has a block diagonal structure where the i-th block $\bar{A}_{ii} \in \mathbb{R}^{s_i \times s_i}, \ \forall i$, $\sum_i s_i = S$, $\epsilon > 0$ is a small perturbation parameter, and $B \in \mathbb{R}^{S \times S}$. $\bar{A}_{ii}, \forall i$ are also infinitesimal generators and we assume that A and $\bar{A}_{ii}, \forall i$ are irreducible and aperiodic. We also assume that the intensities of the Poisson arrival process only depend on the state partitions, i.e, $g_i = \bar{g}_j, \ \forall X_t = e_i \in \mathcal{S}_j$. Using the same decoupling techniques as in [1], it can then be shown that when the above assumptions hold and ϵ is sufficiently small, one can obtain $O(\epsilon^2)$ approximations to $\alpha_n, \hat{\alpha}_n, \forall n$ with $O(M^2)$ computations rather than $O(S^2)$ computations as demanded by (3). Since typically $M \ll S$, this implies a substantial reduction in computations. For all related details, see [1].

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