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We present a hypothetical learning trajectory for a sequence of tasks designed for a calculus module. The purpose of the tasks was to give undergraduates opportunities to use technology to experiment and make conjectures while developing their understanding of the effects of translations on graphs. We consider data from task-based interviews with two students. The hypothetical learning trajectory for this sequence of tasks is compared with the actual learning trajectory of the students, and we conclude there was some evidence that our learning goals were achieved.

Keywords: task design, undergraduate mathematics; conjecturing.

Introduction

The types of tasks that students work on can influence the reasoning and learning processes in which they engage (Jonsson, Norqvist, Liljekvist, & Lithner, 2014). In Ireland, recent studies have highlighted that the majority of tasks both in secondary school textbooks and in undergraduate calculus modules could be solved with imitative reasoning, that is by memorization or following a familiar algorithm (O'Sullivan, 2017; Mac an Bhaird, Nolan, Pfeiffer, & O'Shea, 2017). In this context it is important to design tasks which give students opportunities to develop higher-order mathematical thinking skills, such as conjecturing and generalizing, to move them away from rote-learning. The first and last authors (Breen & O'Shea, 2018) designed a framework of task types for undergraduate calculus modules with the aim of developing mathematical thinking skills such as those suggested by Mason & Johnston-Wilder (2004, p. 109). Subsequently, interactive versions of some of the tasks were developed using the dynamic geometry software GeoGebra. In this paper we will consider a hypothetical learning trajectory (Simon, 1994) for a set of two conjecturing tasks designed using Geogebra on the topic of graph transformations and present data from task-based interviews to explore the actual learning trajectories engendered by this sequence of tasks.

Theoretical Framework

Task Design

The framework of mathematical task types used here has six task types: evaluating mathematical statements; generating examples; analysing reasoning; visualizing; using definitions; conjecturing and generalizing (Breen & O'Shea, 2018). We will focus here on the last of these; we will first review the literature to present a rationale for this task type and for using technology in the design.

The acts of conjecturing and generalizing are well-known to be part of the tools of a professional mathematician (Bass, 2015); indeed, Bass describes the progress of most mathematical work as starting with exploration and discovery, then moving on to conjecture, and finally culminating in proof. He identifies two phases of reasoning here: *reasoning of inquiry* (incorporating exploring and conjecturing) and *reasoning of justification* (rooted in proof). The acts of conjecturing, generalising, experimenting and visualising are included in the list of processes which aid mathematical thinking given by Mason and Johnston-Wilder (2004); these authors also discuss ‘natural’ powers that learners possess such as the ability ‘to imagine and detect patterns,..., to make conjectures, to modify these conjectures in order to try to convince themselves and others’ (Mason & Johnston-Wilder, 2004, p. 34). They stress the importance of creating a ‘conjecturing atmosphere’, so that students can participate in inquiry and develop their mathematical thinking skills.

Dreyfus (2002) defines generalizing as “to derive or induce from particulars, to identify commonalities, to expand domains of validity” (p. 35). He acknowledges the important role played by generalizing in the process of abstraction, in moving from a particular instance to a generality, and notes the difficulty that many students have with generalization. Swan (2008) explains that the process of identifying general properties of a concept from particular cases is one with which a student must be able to engage in order to come to truly understand a concept.

Breda and Dos Santos (2016) examined how GeoGebra tools can enable students to conjecture and provide mathematical proof, and recommended such tools be used to support the study of complex functions. The use of technology has a number of advantages: for instance, information can be gathered and processed quickly so that teachers and students can make decisions efficiently to exploit learning opportunities; moreover, the burden of computation can be removed or reduced to allow students to explore and experiment. Borwein (2005) described specific benefits of the use of technology to mathematicians, including: to gain insight and intuition, to discover new patterns and relationships, to expose mathematical principles through graphs, to test and falsify conjectures, to explore a possible result to see if it merits formal proof, to do lengthy computations. All of these have an important role to play in responding to a conjecturing/generalizing task.

Hypothetical Learning Trajectory Construct

Simon (1995) introduced the notion of a hypothetical learning trajectory (HLT) as part of a model of mathematics teaching. The HLT is made up of three parts: learning goals (as set by the instructor); learning activities (designed or selected by the instructor); and the hypothetical learning process (the instructor’s prediction of how student thinking will develop during the learning activities). Simon describes the symbiotic relationship between the learning activities and the hypothetical learning processes – the ideas which underlie the learning activities are based on the instructor’s beliefs about student learning, and these in turn are influenced by what is observed during the learning activities. Thus theory informs practice and vice versa.

Simon and Tzur (2004) advocate the use of HLTs in task and curriculum design (especially for ‘problematic’ topics) as a mechanism to ensure that adequate thought is given to how student learning might evolve during activities, and as a means to study the success of learning activities. The HLT construct has been used to study teaching tasks and sequences of tasks in a variety of settings including undergraduate mathematics courses. Andrews-Larson, Wawro and Zandieh (2017) note that HLTs are useful ways of tying theory to practice, and use the notion of HLT to outline how certain tasks could lead to undergraduate students developing new understanding in Linear Algebra. Stylianides and Stylianides (2009) compared HLTs and actual learning trajectories to provide evidence that an instructional sequence of tasks had achieved the desired goals.

Our Task Sequence and Hypothetical Learning Trajectory

We will consider the HLT for our task sequence and the actual learning progression of two students.

Learning Goals

Eisenberg and Dreyfus (1994) discuss the fundamental importance of developing 'function sense' with undergraduate students. They describe facets of this as including dependence, variation, co-variation and the effects of operations on functions. One of the most important components of function sense is the flexibility to move between multiple representations of a function. The key to solving many problems is to think of them visually, using a graph - including problems encompassing the main facets of functions mentioned above. However, Eisenberg and Dreyfus report that many students (even those more advanced mathematically) are reluctant to do so.

The function operations that we focus on in this paper are graph transformations, specifically vertical and horizontal translations of the graph of functions from $\boldsymbol{R}$ to $\boldsymbol{R}$. The learning goals are:

1. observing and articulating the effects of translations on graphs: in particular, describing the relationship between the graphs of $y = F(x)$ and those of $y = F_i(x)$ for $i = 1,2$ (where $F_1(x) = F(x) + a$ and $F_2(x) = F(x + a)$);
2. observing that, in general, the functions $F_1(x)$ and $F_2(x)$ are different when $a \neq 0$;
3. making conjectures, in particular generalizing data from examples observed;
4. using the technology to undertake experiments.

Note that we have both local (1 and 2) and more global (3 and 4) learning goals for student reasoning and skill development arising from our task sequence. Goal 1 targets the flexibility to move between representations of a function. The goals of the task sequence do not include students providing proofs for their conjectures since our tasks deal with Bass's (2015) *reasoning of inquiry* rather than *reasoning of justification*.

The Task Sequence

For the last number of years, we have been developing a bank of tasks using our framework. Originally these tasks were paper-based and aimed to give students opportunities to explore, spot patterns, and make conjectures based on their observations. We noticed that some students had difficulties drawing the graphs of the functions mentioned and so were not able to generalize or make a conjecture. In 2016, we redesigned these tasks using GeoGebra; we will refer to these as Tasks A and B. (Task A is shown in Figure 1 and Task B is similar except with $f(x) = (x + a)^3$ etc.). The computational burden was thus removed from the students and we hoped that this would allow them more freedom to experiment and conjecture appropriately. In contrast with the paper-based tasks, the use of GeoGebra allowed us to enable students to quickly see graphs of the form $y = F(x) + a$ (**Task A**), and $y = F(x + a)$ (**Task B**), for values of a ranging over an interval. Both tasks, and others from this project, can be found at http://mathslr.teachingandlearning.ie/GeoGebra/.

Hypothetical Learning Progression

As students engage with the task sequence we expect the following activity and learning from them:

- experimenting with the sliders;

- noticing how the graph of the function f changes as a changes, in particular noticing the difference in behaviour for positive and negative values of a;
- remarking in the case of Task A (respectively Task B) on the vertical (respectively horizontal) shift of the graph and expressing the relationship between f and f_1(respectively f_2) mathematically;
- noticing analogous relationships in the cases of g and h to identify a pattern;
- using the data from f, g, and h to make a conjecture about the effects of transformations of the types in Task A and B on graphs;
- amalgamating the relationships observed to realise that, in general, the functions $f_1(x)$ and $f_2(x)$ are different when $a \neq 0$.

Task A: (GeoGebra Task) Use the slider on each graph to change the values of a in the functions $f(x) = x^3 + a$, $g\ (x) = \frac{1}{x^2} + a$, $h(x) = 3^x + a$.

i. What is the relationship between the pair of graphs $y = f(x)$ and $y = f_1(x)$ below?
[The graphs of $y = f(x)$ (with a initially set at 1) and $y = f_1(x) = x^3$ are shown as well as a slider which allows a to range from -5 to 5. When the value of a changes the graph $y = f(x)$ changes accordingly.]

ii. What is the relationship between the pair of graphs $y = g(x)$ and $y = g_1(x)$ below?
[The graphs of $y = g(x)$ (with a initially set at 1) and $y = g_1(x) = \frac{1}{x^2}$ are shown as well as a slider which allows a to range from -5 to 5.]

iii. What is the relationship between the pair of graphs $y = h(x)$ and $y = h_1(x)$ below?
[The graphs of $y = h(x)$ (with a initially set at 1) and $y = h_1(x) = 3^x$ are shown as well as a slider which allows a to range from -5 to 5.]

iv. Can you make a general conjecture about the relationship between the graphs of $y = F(x)$ and $y = F(x) + a$ from your observations about the graphs of the pairs of functions above? What happens when $a > 0$? What happens when $a < 0$? What happens when $a = 0$?

Figure 1: Task A

Data collection and analysis

In order to investigate the use and effectiveness of the tasks, the second author carried out a series of task-based interviews with a sample of students from a first-year calculus module in which some GeoGebra tasks were trialed. Four students were asked to think aloud while completing a selection of the tasks. Pre and post-tests, each consisting of the same four questions (where one question had two parts), were used at the beginning and end of the interviews in order to help determine if the students' mathematical thinking had changed as a result of completing the tasks. The interviews, which lasted about an hour, used purpose built software to record video, audio, screen and mouse movements. Each student completed between four and seven GeoGebra tasks depending on how quickly they moved through them. The interviews were transcribed by the second author using the audio recording to which she added a summary of what was happening onscreen at that time. The transcriptions were analysed for significant incidents by two of the three authors, and their results were compared and agreed on.

Results

We will consider in some detail the responses of two of the three students who completed the pre-test, then worked on Tasks A and B in the task-based interviews, and subsequently completed the post-test (see Table 1 below). One question which appeared on the pre-test and post-test was the following:

> Q4(ii) Suppose *f(x)* is a function defined for all real values of *x*. Decide if the statement is true or false. Explain your answer. *If a is any real number then f(x+a) = f(x)+a for all values of x.*

All three students answered Q4(ii) correctly on the post-test but two of them (given pseudonyms Áine and Seán) gave incorrect answers on the pre-test.

Student	**Q4(ii) on pre-test**	**Task A**	**Task B**	**Q4(ii) on post-test**
Áine	Says '*It's a function*' and writes 'true'. Later she says she thought that the question was asking whether the expression f(x+a)=f(x)+a describes a function.	Is able to verbalise the relationship between the graphs and is able to make the expected conjecture.	Is able to verbalise the relationship between the graphs and is able to make the expected conjecture.	Writes: if a=0 but not for other values of a. Explains by referring to the GeoGebra tasks.
Seán	Writes 'True' and gives example with f(x)=x, a=1, x=1. Later when asked what he thought this question was asking: *I took it as a set [particular] function rather than an arbitrary function.*	Notices that the y-intercepts of the graphs of f and h depend on the choice of a. For g, he notices that the horizontal asymptotes depend on a, (but does not use correct terminology). Is able to make the expected conjecture.	Notices that the x-intercept varies according to choice of a. (He calls it the origin). Is able to make the expected conjecture.	Says the statement is false and explains by referring to graph transformations.

Table 1: Student responses in task-based interview

Both Áine and Seán were able to use the sliders to obtain graphs of the different translations of the functions in question. Áine worked quickly on both tasks (she spent 2-3 minutes on each of them), she spotted the pattern and was able to verbalise it using mathematical language. For the first pair of functions on Task A she said:

> So as I can see here as I am taking values away from x^3 the graph shifts down the y-axis and as I add values it shifts up the y-axis.

She did the same for the graphs of the translations of g, predicted what would happen with h, and was able to conjecture:

> when a >0 the graph of f moves up the y-axis and when it's less than zero it moves down.

Similarly on Task B, she was able to use the slider to generate vertical translations of the three functions, and she made a conjecture generalizing the pattern she observed.

Seán spent about 6 minutes working with the three graphs on Task A. He used the slider to examine how the functions changed for the range of values of a. He also was able to spot a pattern but focused on certain features of the graphs instead of the whole graph; for the translations of f and h he spoke about their y-intercepts (but used the term *origin*), and for the translations of g he noticed that the horizontal asymptotes depend on a (but did not use this term). For the general conjecture he scrolled back to the first pair of graphs, moved the slider, then looked at the other pairs of graphs:

> All the graphs shift upwards in the y direction by the value whichever value a is from the original position of $y = f(x)$. When $a<0$ all the graphs shift down in the *y* direction by whatever value a is in the… by whatever value a is from wherever $y = f(x)$ happens to be.

Seán worked through Task B in a similar manner for about 6 minutes. However, when asked to give a general conjecture this time Seán gave an appropriate response immediately without having to scroll back up through the three functions, as he did for the general conjecture in Task A.

Discussion

We have only presented evidence from two students who worked on a pair of conjecturing/generalizing tasks. However, from these case studies, we can draw some conclusions for these students' learning. Our data suggests that both students achieved learning goals 1, 3 and 4 as they were able to use the technology to experiment with the translations, they spotted patterns and were able to verbalise them, and they were able to make conjectures based on their experiments. The students' responses to question 4(ii) on the pre- and post-tests give us cause to believe that the students have also achieved learning goal 2 during the task sequence. Both students gave an incorrect answer in the pre-test but in the post-test both revised their answers. Áine recognized that the statement is true for $a = 0$ but not otherwise, and used her experiences on Tasks A and B to explain her reasoning. In the pre-test Seán considered one numeric example in order to explain his response to question 4(ii). When Seán completed the post-test question 4(ii) he immediately stated that his original response was incorrect and referred to the vertical and horizontal translations from Tasks A and B. Finally, at the end of the interview, Seán was asked if he considered any of the tasks helped him respond to the post-test questions, and he said:

> [Tasks A and B] helped me to see and distinguish the differences in changing the values of a because I didn't fully grasp what it was in the beginning.

It is clear from the task-based interviews that GeoGebra took away the burden of computation; if we had asked students to draw the graphs of the three pairs of functions in Task A by hand, then it would probably have taken them a long time and they may have made mistakes. The use of the sliders in GeoGebra, allowed the students to watch how the graphs changed as the values of a changed, and they were then able to spot the pattern and then make a conjecture. Eisenberg and Dreyfus (1994) found that the students involved in their teaching experiment seemed to view function transformations as a sequence of two static states (the initial and final graphs) rather than as a dynamic process. They concluded that this may have been as a result of the graphing software available to them in which there was no means to see the continuous transformation developing before the students' eyes. The advent of dynamic geometry software, such as GeoGebra, means that this is no longer an issue: the students we interviewed for our study seemed to have developed an understanding of function transformation as a dynamic process.

The use of software like GeoGebra makes visualization more immediate for students and we posit that this can help with engagement. We saw, probably because of the ease of visualization in Tasks A and B, that both students felt comfortable in making a conjecture. This corresponds with Borwein's (2005) description of how mathematicians use technology in their own work, and we suggest that giving students the opportunity to use technology in this manner might encourage them to develop mathematical thinking skills (Mason & Johnston-Wilder, 2004).

Furthermore, in the pre-test Seán seemed to see Q4(ii) as referring to a single function, but in the post-test he immediately recognizes that it is a general statement. We suggest that it is his experience of working on Tasks A and B that accounts for this change in perspective, although it may be that he is recalling earlier understanding rather than developing it during the task sequence. We note that the ability to appreciate the distinction between an instance and a generality is crucial in the development of mathematical thinking (Dreyfus 2002). Seán's response could also be interpreted as a move towards seeing functions as objects rather than simply actions.

Eisenberg and Dreyfus (1994) suggested that students found transformations in the horizontal direction (e.g.$f(x) \to f(x+a)$) more difficult than those in the vertical direction (e.g.$f(x) \to f(x) + a$). They contended that one reason for this may simply be that more is involved in visually processing $f(x+a)$ than $f(x)+a$. However, we found no evidence of this in the think-aloud interviews with our students. In fact, the students were quicker to conjecture, and more articulate in their description of, a general relationship between the graphs of $f(x)$ and $f(x+a)$ (Task B) than between $f(x)$ and $f(x)+a$ (Task A) which we supposed was due to the order in which the tasks were presented.

It may be that our students would have had more difficulties justifying their conjecture in Task B rather than Task A, but such justifications were not part of our task sequence. One might criticize our task sequence as not being cognitively challenging, but we wanted to focus on conjecturing rather than proving. The tasks could easily be modified to allow students to input other functions in order to check their hypothesis, and could be expanded to ask for justifications or proofs.

We agree with Simon and Tzur (2004) that the HLT construct is useful in task design as it highlights the importance of having clear learning goals and an informed view of how learning might take place at all stages of the design process. We feel that it can help when designing new versions of tasks if the original learning goals are not met, and furthermore provides a way of evaluating tasks by comparing the hypothetical learning process with actual learning.

We have found some evidence that conjecturing tasks can encourage students to experiment and explore. We note that Bass (2015) sees this exploration as the first step in most mathematical work, and therefore it is a necessary skill that students should develop to improve their mathematical thinking. With this aim in mind, we hope to continue to design and evaluate tasks of this type.

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References

Andrews-Larson, C., Wawro, M., & Zandieh, M. (2017). A hypothetical learning trajectory for conceptualizing matrices as linear transformations. *International Journal of Mathematical Education in Science and Technology, 48*(6), 809-829.

Bass, H. (2015). Mathematics and teaching. *Notices of the American Mathematical Society, 62*(6), 630-636.

Borwein, J.M. (2005). The experimental mathematician: The pleasure of discovery and the role of proof. *International Journal of Computer for Mathematical Learning, 10*, 75-108.

Breda, A.M., & Santos, J.M.D.S.D., (2016). Complex functions with GeoGebra. *Teaching Mathematics and its Applications, 35*(2), 102-110,

Breen, S., & O'Shea, A. (in press). Designing mathematical thinking tasks. *PRIMUS*.

Dreyfus, T. (2002). Advanced mathematical thinking processes. In Tall, D (eds) *Advanced mathematical thinking* (pp. 25-41). Mathematical Education Library, vol 11. Springer, Dordrecht.

Eisenberg, T., & Dreyfus, T. (1994). On understanding how students learn to visualize function transformations. *Research in Collegiate Mathematics Education*, *1*, 45-68.

Jonsson, B., Norqvist, M., Liljekvist, Y., & Lithner, J. (2014) Learning mathematics through algorithmic and creative reasoning. *The Journal of Mathematical Behavior*, *36*, 20-32.

Mac an Bhaird, C., Nolan, B. C., O'Shea, A., & Pfeiffer, K. (2017). A study of creative reasoning opportunities in assessments in undergraduate calculus courses. *Research in Mathematics Education*, *19*(2), 147-162.

Mason, J., & Johnston-Wilder, S. (2004) *Designing and using mathematical tasks*. St Alban's: Tarquin Press.

O'Sullivan, B. (2017). *An analysis of mathematical tasks used at second-level in Ireland* (Unpublished doctoral dissertation), Dublin City University, Dublin, Ireland.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114-145.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, *6*(2), 91-104.

Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education*, *40*(3), 314-352.

Swan, M. (2008). Designing a multiple representation learning experience in secondary algebra. *Educational Designer*, *1*(1), 1-17.