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Yurii Rogozhin's Contributions to the Field of Small Universal Turing Machines

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In the field of small universal Turing machines, Yurii Rogozhin holds a special prize: he was first to close off an infinite number of open questions by drawing a closed curve that separates the infinite set of Turing machines that are universal from a finite set of small machines for which we don't yet know. Rogozhin did this by finding the smallest known universal Turing machines at the time, both in terms of number of states and number of symbols. This brief note summarises this and a few of Yurii's other contributions to the field, including his work with Manfred Kudlek on small circular Post machines.

1. Introduction

The topic of small universal Turing machines is concerned with putting upper and lower bounds on the program size of universal Turing machines [28, 41]. In other words, finding the shortest programs that are capable of universal computation and showing that below this threshold such complicated behaviour is impossible. More generally, the computational complexity of small programs includes investigating the time, space and encoding complexity costs we pay by having tiny general-purpose programs. Simulation

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of small universal Turing machines, or other simple universal models such as Post's tag systems [32] and the cellular automaton Rule 110 [5], is by now a standard way to prove that a large number of other models of computation, including a variety of physically-inspired systems, are computationally universal.

Historically much of the research into small universal Turing machines is concerned with the deterministic one-dimensional single-tape model with the usual notion of blank symbol. We often refer to this as the *standard model*. The size of a program is defined as the number of its states and symbols, written as a (state, symbol) pair. Their product gives an upper bound on the number of instructions in the Turing machine program.

In 1956 Shannon [43] considered the question of finding the smallest possible universal Turing machine. In the early Sixties, Minsky and Watanabe had a running competition to see who could find the smallest universal Turing machine [22, 23, 24, 44, 45]. In 1962, Minsky [23] found a small 7-state, 4-symbol, or $(7, 4)$, universal machine. Minsky's machine worked by simulating 2-tag systems, which were shown to be universal by Cocke and Minsky [4, 23].

The halting problem has been shown to be decidable for all standard Turing machines with the following state-symbol pairs: $(2, 2)$ by Pavlotskaya [30] and later also by Kudlek [10], $(3, 2)$ by Pavlotskaya [31], $(2, 3)$ by Pavlotskaya (unpublished), $(1, n)$ by Hermann [9] and $(n, 1)$ (trivial) for $n \geqslant 1$. Thus there are no standard universal machines of this size that have the property of halting if and only if the simulated Turing machine halts. These results are plotted as the *non-universal*[1] curve in Figure 1.

2. Rogozhin's work on small universal Turing machines

As noted above, the size of a Turing machine program is upper bounded by the product of the number of states and the number of symbols. Rogozhin decided to minimise program size over both of these parameters. Over a sequence of papers [13, 36, 37, 38, 39, 40, 41, 42], describing work that began during his PhD, he gave a significant number of small universal Turing machines. Rogozhin's small machines used Minsky's technique of simulation of 2-tag systems. These results are shown as black disks in Figure 1.

Taken together with the negative results plotted as the non-universal curve on Figure 1, Rogozhin's work left a finite set of (state, symbol) pairs for which it is an open question whether there exists a universal program. By closing off a finite region of the infinite states-symbol quadrant in Figure 1, Rogozhin gave a clear way to frame questions in the field. Thinking about Rogozhin's universality curve (Figure 1), and what interesting beasts lie under it, has strongly influenced much of the work that followed.

Over the years quite a bit of effort has been put into trying to improve upon Rogozhin's universal program-size results. His small universal machines were improved upon by Robinson [35], Kudlek and Rogozhin [13], Baiocchi [3], and Neary and Woods [26]. These results are plotted in Figure 1 as triangles and a hollow circle.

Rogozhin's universal Turing machine with 7 states and 4 symbols used only 26 instructions, which improved on Minsky's $(7, 4)$ that used 27. Also, it is quite impressive that another of Rogozhin's machines, his $(4, 6)$, uses only 22 instructions, and although this number has been since matched by one

[1] It is worth noting that this is one very concrete way in which these machines are not universal, of course this may not hold for other definitions besides the standard model.

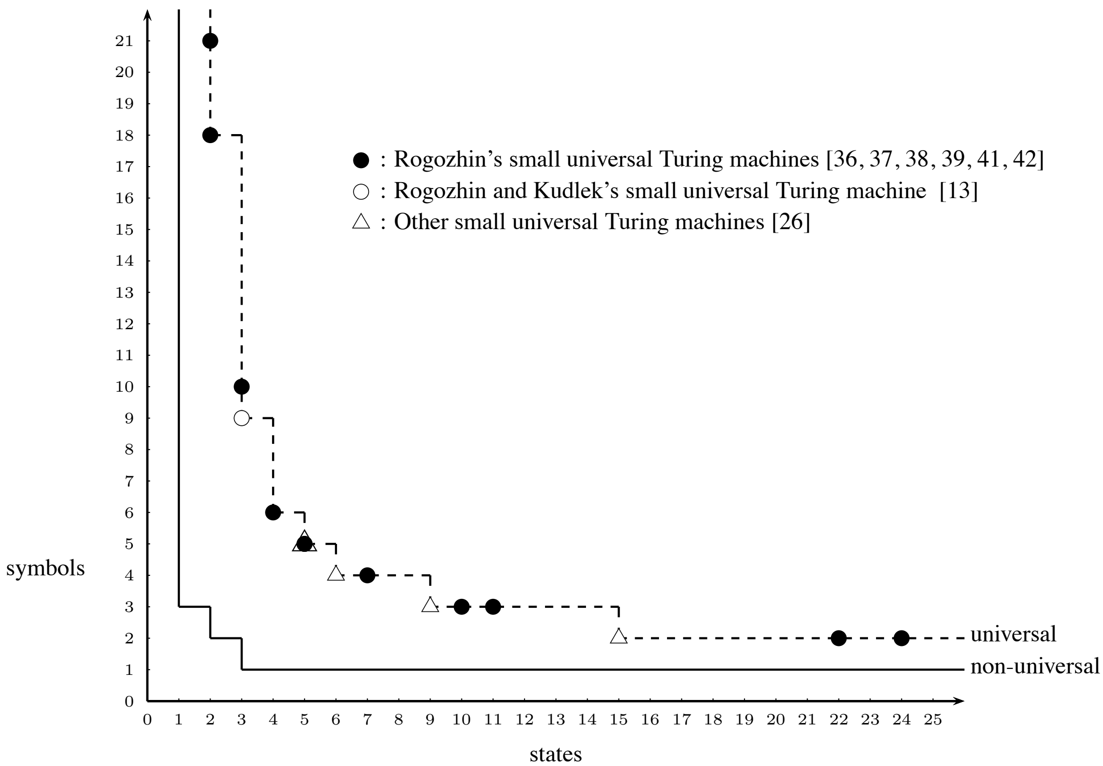

Figure 1. State-symbol plot of the smallest known universal Turing machines for the standard model. All of Yurii Rogozhin's published machines are shown as black disks, and Rogozhin and Kudlek's as hollow circles. Rogozhin was the first to minimise universal program size in terms of *both* states and symbols, and the first to close off a finite portion of the states-symbols quadrant leaving a finite number of open questions. A number of other small machines have been published; the smallest machines for the standard model are shown as triangles. Machines for other Turing machine models besides the standard model are omitted and can be found in [28]. The non-universal curve shows Turing machines that are known to have a decidable halting problem.

other machine—the $(5, 5)$ machine in [26]—in terms of number of instructions no smaller machine has been found since Rogozhin's. His skill[2] can not be understated, and to this day Yurii Rogozhin holds an impressive number of records!

3. After Rogozhin's results: small universal Turing machines

Since Rogozhin's results there have been a number of interesting developments in the field, many of which were at least in part inspired by his work. It is worth putting these developments in context.

We have already noted that a number of authors worked to improve upon Rogozhin's results in the standard model, indeed there are 39 state-symbols pairs for which we still don't know whether there are

[2]Yurii commented to the authors that his work on writing small universal Turing machine programs was good practice ahead of writing, and proving correct, concise programs that were sent into orbit as part of his work with the Soviet space program.

universal Turing machines for the standard model. Another direction involves peering into the gaping hole between the universal and non-universal curves in Figure 1. Two kinds of lens have been used: generalising the standard model to find smaller universal programs, and finding small machines that have seemingly-complicated behaviour.

Some authors generalised the standard model by allowing more complicated, although arguably reasonable and well-motivated, initial configurations and data encodings. For example, in the 1960's Watanabe [45, 46] gave $(5, 4)$ and $(7, 3)$ machines that had an infinitely repeated constant word on one side of the input. Much later, Woods and Neary [48] gave universal $(2, 13)$, $(3, 7)$ and $(4, 5)$ machines for the same model. Generalising the model in this way allowed for smaller universal machines than those standard machines of Rogozhin and others. Indeed, Cook [5] gave even smaller universal machines for a more general model that allows the initial configuration to have the (variable) input along with an infinitely repeated constant word to the left, and another to the right. These machines, later improved by Neary and Woods [27] to $(2, 4)$, $(3, 3)$, and $(6, 2)$, simulate iterations of the simple cellular automaton Rule 110 [5]. So, as one generalises the model definition, smaller and smaller machines can be found. However, it is worth pointing out that all of the machines discussed here have a P-complete prediction problem [8, 25, 47]: in this complexity-theoretic time-bounded sense there is little difference in standard, semi-weakly and weakly universal machines.

Margenstern [17, 18] found another interesting way to explore this space of open questions below the universal curve in Figure 1. He constructed machines with state-symbol pairs of $(11, 2)$, $(5, 3)$, $(4, 4)$, $(3, 6)$ and $(2, 10)$ that simulate iterations of the Collatz function.[3] Baiocchi [2] reduced the size of some of these Collatz simulators to give Turing machines with state-symbol pairs of $(10, 2)$, $(5, 3)$, $(4, 4)$, $(3, 5)$ and $(2, 8)$, and Michel [29] further improved two of these results to give a $(3, 4)$ machine (as well as some larger Collatz simulators that always halt after looping). This suggests that proving decidability of nontrivial statements about the long-term dynamics of these machines could be at least as hard as resolving the Collatz problem. More concretely, for the class of machines with program size equal (or greater than) these machines the following problem is at least as difficult as resolving the Collatz conjecture: give an algorithm that takes as input one of these machines, an initial configuration and a target configuration, and decides whether or not the machine reaches the target from the initial configuration.

Furthermore, Michel [19, 20, 21] showed that there are Turing machines that simulate iterations of Collatz-like functions with state-symbol pairs of $(2, 4)$, $(3, 3)$, and $(5, 2)$. Kudlek [10] has given a $(4, 4)$ machine that accepts a context-sensitive language.

So, even if one cares only about universality results for the standard model,[4] the existence of the beasts described in this section suggests that proving negative results on these machine classes for the standard model will indeed be rather challenging.

Rogozhin's proved his machines universal via an efficient (polynomial time) simulation of 2-tag systems. Unfortunately, Cocke and Minsky's [4] simulation of Turing machines with 2-tag system was exponentially slow and so the machines of Rogozhin and others also suffered from an exponential slowdown. Recent work shows that tag systems, Rogozhin's machines, and all other known small universal Turing machines, are efficient polynomial time simulators of Turing machines [25, 47]. It remains to be

[3]The Collatz function is: $f(x) = 3x + 1$ if x odd, or $f(x) = x/2$ if x even. The following question is open, and many conjecture that the answer is "yes". Start from any $x \in \mathbb{N}$, apply the Collatz function, and iterate: do we eventually get a loop on the set $\{4, 2, 1\}$?

[4]On one occasion, Yurii made it clear that his preference lay with the standard Turing machine model!

seen what kind of simulation time trade-offs, if any, can be shown for Rogozhin's or other small universal Turing machines. See the survey [28] for discussions of these and other results relevant to the field.

4. Rogozhin and Kudlek's work on circular Post machines

Kudlek and Rogozhin [12] introduced a new model which they called a circular Post machine. While the name assigned to this model would seem to associate it with Post machines [33], it would have been just as appropriate to name it a circular Turing machine. In fact, this model could be described as a Turing machine with a single circular tape whose tape head moves in one direction only and has the ability to insert or remove tape cells.[5]

Taking a similar approach to that of Rogozhin for small universal Turing machines, Kudlek and Rogozhin [12] immediately established a universal curve by giving universal circular Post machines for the (state, symbol) pairs $(7,7)$, $(8,6)$, $(11,5)$, and $(13,4)$. Later, Kudlek and Rogozhin [11] improved and extended these results to give universal circular Post machines of size $(4,18)$, $(5,11)$, $(6,8)$, $(9,5)$, $(12,4)$ and $(18,3)$. Alhazov [1] joined them to further improve and extend these results giving universal machines for the pairs $(2,46)$, $(3,22)$, $(4,11)$, $(5,8)$, $(6,6)$, $(8,5)$, $(11,4)$, $(16,3)$ and $(34,2)$. Like many of the smallest known universal Turing machines, all of the above circular Post machines simulate 2-tag systems efficiently and so following the result in [47] all of the above circular Post machines simulate Turing machine with merely a polynomial time simulation slowdown.

Kudlek and Rogozhin [11] complemented their positive universality results by showing that there exist no universal machines for all pairs $(1,m)$ and $(m,1)$, for all $m \geq 1$. They [12] also explored the class of machines between their above mentioned upper and lower bounds on universal program size by encoding open problems in very small machines. They gave a $(6,2)$ circular Post machine that implements Post's tag system (i.e., a 3-tag system with the rules $0 \rightarrow 00, 1 \rightarrow 1101$) from 1921 [34]. It follows that the reachability problem (i.e. is one specified configuration reachable from another?) for $(6,2)$ circular post machines is at least as difficult as the halting problem for Post's tag system, which remains open to this day [7]. In addition, they gave a $(5,3)$ machine that simulates iterations of the Collatz function [14, 15, 16]. This suggests that proving the halting problem decidable for $(5,3)$ machines could be at least as hard as resolving the Collatz problem.

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