FROBENIUS-SCHUR INDICATORS OF CHARACTERS IN BLOCKS WITH CYCLIC DEFECT

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Abstract. Let p be an odd prime and let B be a p-block of a finite group which has cyclic defect groups. We show that all exceptional characters in B have the same Frobenius-Schur indicators. Moreover the common indicator can be computed, using the canonical character of B. We also investigate the Frobenius-Schur indicators of the non-exceptional characters in B.

For a finite group which has cyclic Sylow p-subgroups, we show that the number of irreducible characters with Frobenius-Schur indicator -1 is greater than or equal to the number of conjugacy classes of weakly real p-elements in G.

1. Introduction and preliminary results

The Frobenius-Schur (F-S) indicator of an ordinary character χ of a finite group G is

$$\epsilon(\chi) := \frac{1}{|G|}\sum_{g\in G}\chi(g^2).$$

If χ is irreducible then $\epsilon(\chi) = 0, \pm 1$. Moreover $\epsilon(\chi) \neq 0$ if and only if χ is real-valued.

R. Brauer showed how to partition the irreducible characters of G into p-blocks, for each prime p. Each p-block has an associated defect group, which is a p-subgroup of G, unique up to G-conjugacy, which determines much of the structure of the block. If the defect group is trivial, the block contains a unique irreducible character. In the next most complicated case, E. Dade [D] determined the structure of a block which has a cyclic defect group and defined the Brauer tree of the block.

Recall that a p-block is said to be real if it contains the complex conjugates of its characters. We wish to determine the F-S indicators of the irreducible characters in a real p-block which has a cyclic defect group. In [M2, Theorem 1.6] we dealt with the case $p = 2$; there are six possible indicator patterns, and the *extended* defect group of the block determines which occurs. In this paper we consider the case $p \neq 2$.

R. Gow showed [G, 5.1] that a real p-block has a real irreducible character, if $p = 2$. This is false for $p \neq 2$, as was first noticed by H. Blau in the early 1980's, in response to a question posed by Gow. His example was for $p = 5$ and $G = 6.S_6$ (*Atlas* notation). G. Navarro has recently found a solvable example with $p = 3$ and $G = \mathrm{SmallGroup}(144, 131)$ (*GAP* notation). We give examples for blocks with cyclic defect below.

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Now let B be a real p-block which has a cyclic defect group D. The inertial index of B is a certain divisor e of $p-1$. Dade showed that B has e irreducible Brauer characters and $e+\frac{|D|-1}{e}$ ordinary irreducible characters. The latter he divided into $\frac{|D|-1}{e}$ exceptional characters and e non-exceptional characters.

Suppose that $\frac{|D|-1}{e}=1$ (which can only occur when $|D|=p$). Then the choice of exceptional character is arbitrary, and the convention in [F] is to regard B as having no exceptional characters. However, we will see that in this event B has real irreducible characters, all of which have the same F-S indicators. So our convention is to assume that B has a real exceptional character.

The Brauer tree of B is a planar graph which describes the decomposition matrix of B. There is one exceptional vertex, representating all the exceptional characters, and one vertex for each of the non-exceptional characters. Two vertices are connected by an edge if their characters share a modular constituent.

J. Green [Gr] showed that all real objects in the Brauer tree lie on a line segment, now called the real-stem of B. The exceptional vertex belongs to the real-stem (see Lemma 6 below). So it divides the real non-exceptional vertices into two, possibly empty, subsets. We find it convenient to refer to the corresponding real non-exceptional characters as being on the left or the right of the exceptional vertex. Here is our main theorem:

Theorem 1. *Let p be an odd prime and let B be a real p-block which has a cyclic defect group. Then*

(i) *All exceptional characters in B have the same F-S indicators.*
(ii) *On each side of the exceptional vertex, the real non-exceptional characters have the same F-S indicators.*
(iii) *If B has a real exceptional character then all real irreducible characters in B have the same F-S indicators.*
(iv) *Suppose that B has no real exceptional characters, and that there are an odd number of non-exceptional vertices on each side of the exceptional vertex. Then the real non-exceptional characters have F-S indicator $+1$ on one side of the exceptional vertex and -1 on the the other side.*

Note that (i) is not a consequence of Galois conjugacy, as there are at least two Galois conjugacy classes of exceptional characters, when $|D|>p$.

In Proposition 15 we show that the F-S indicators of the exceptional characters in B agree with those of the Brauer corresponding block in the normalizer of a defect group. In Theorem 16 we compute this common indicator using the 'canonical character' of B.

Next recall that an element of G is said to be weakly real if it is conjugate to its inverse in G, but it is not inverted by any involution in G. Here is an application of Theorem 1 whose statement does not refer to blocks or to modular representation theory:

Theorem 2. *Let p be an odd prime and let G be a finite group which has cyclic Sylow p-subgroups. Then the number of irreducible characters of G with F-S indicator -1 is greater than or equal to the number of conjugacy classes of weakly real p-elements in G.*

We use the notation and results of [NT] for group representation theory, and use [D] and [F, VII] for notation specific to blocks with cyclic defect. When referring to the character tables of a finite simple group we use the conventions of the *ATLAS* [A]. For other character tables, we use the notation of the computer algebra system *GAP* [GAP].

2. Examples

We begin with a number of examples which illustrate the possible patterns of F-S indicators in a block which has a cyclic defect group. Throughout G is a finite group and B is a real p-block of G which has a cyclic defect group D. Also N_0 is the normalizer in G of the unique order p subgroup of D and B_0 is the Brauer correspondent of B in N_0.

Example 1: There are many blocks with cyclic defect group whose irreducible characters all have the same F-S indicators. For blocks with all indicators $+1$, choose $n \geq 2$, a prime p with $n/2 \leq p \leq n$ and any p-block of the symmetric group S_n. There are numerous blocks with all indicators -1 among the faithful p-blocks of the double cover $2.A_n$ of an alternating group, with $n/2 \leq p \leq n$ e.g. the four faithful irreducible characters of $2.A_5$ have F-S indicator -1 and constitute a 5-block with a cyclic defect group.

Example 2: If e is odd then B has a real non-exceptional character. Now it follows from [D, Part 2 of Theorem 1 & Corollary 1.9] that B has a Galois conjugacy class consisting of $\frac{p-1}{e}$ exceptional characters. So B has a real exceptional character if $\frac{p-1}{e}$ is odd. Thus B always has a real irreducible character if $p \equiv 3 \pmod 4$.

When e is even and $p \equiv 1 \pmod 4$, B may have no real irreducible characters. For example SmallGroup$(80, 29) = \langle a, b \mid a^{20}, a^{10} = b^4, a^b = a^7 \rangle$ has such a block, for $p = 5$. It consists of the four irreducible characters lying over the non-trivial irreducible character of $\langle a^{10} \rangle$. Here is its character table. The first two rows indicate the 2 and 5 parts of the class centralizers. The third row labels the classes by their element orders:

2	4	4	3	3	4	4	2	3	3	3	3	2	2	2
5	1	1	.	1	.	.	1	.	.	.	.	1	1	1
	$1a$	$2a$	$2b$	$4a$	$4b$	$4c$	$5a$	$8a$	$8b$	$8c$	$8d$	$10a$	$20a$	$20b$
$X.9$	2	-2	.	.	$2i$	$-2i$	2	.	.	.	.	-2	.	.
$X.10$	2	-2	.	.	$-2i$	$2i$	2	.	.	.	.	-2	.	.
$X.13$	4	-4	.	.	.	.	-1	.	.	.	.	1	$\sqrt{-5}$	$-\sqrt{-5}$
$X.14$	4	-4	.	.	.	.	-1	.	.	.	.	1	$-\sqrt{-5}$	$\sqrt{-5}$

Note that SmallGroup$(80, 29)$ has Sylow 2-subgroups isomorphic to SmallGroup$(16, 6) = \langle s, t \mid s^8, t^2, s^t = s^5 \rangle$. This 2-group is sometimes denoted $M_4(2)$.

Example 3: B may have a real non-exceptional character but no real exceptional characters. For example SmallGroup$(60, 7) = \langle a, b \mid a^{15}, b^4, a^b = a^2 \rangle$ has such a block, for $p = 5$. It consists of the four irreducible characters lying over a non-trivial irreducible character of $\langle a^5 \rangle$. This is also an example of part (iv) of Theorem 1; the non-exceptional characters $X.5$ and $X.6$ have F-S indicators -1 and $+1$, respectively. Here is the table of character values, with $\alpha = (1+\sqrt{-15})/2$:

2	2	2	1	2	2	.	1	.	.
3	1	1	1	.	.	1	1	1	1
5	1	.	1	.	.	1	.	1	1
	$1a$	$2a$	$3a$	$4a$	$4b$	$5a$	$6a$	$15a$	$15b$
$X.5$	2	-2	-1	.	.	2	1	-1	-1
$X.6$	2	2	-1	.	.	2	-1	-1	-1
$X.8$	4	.	-2	.	.	-1	.	α	$\overline{\alpha}$
$X.9$	4	.	-2	.	.	-1	.	$\overline{\alpha}$	α

Example 4: There is no apparent relationship between the F-S indicators of the non-exceptional characters in B and in B_0. For example, let B be the 5-block $2.A_8$ with $\mathrm{Irr}(B) = \{\chi_{15}, \chi_{19}, \chi_{21}, \chi_{22}\}$. Then the two non-exceptional characters χ_{15} and χ_{19} have F-S indicator $+1$ and -1, respectively. However B_0 is a real block which has no real irreducible characters.

The character table of B can be found on p22 of *The Atlas*. Now N_0 is isomorphic to $\mathrm{SmallGroup}(120,7) = \langle a, b \mid a^{15}, b^8, a^b = a^2 \rangle$. Here is the table of character values of its 5-block B_0. Again $\alpha = (1+\sqrt{-15})/2$. In order to save space, we have omitted 4 columns of zero values for the four classes of elements of order 8:

2	3	3	2	3	3	1	2	1	2	2	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	.	.	1	1	1	.	.	1	1	1	1
	$1a$	$2a$	$3a$	$4a$	$4b$	$5a$	$6a$	$10a$	$12a$	$12b$	$15a$	$15b$	$30a$	$30b$
$X.11$	2	-2	-1	$2i$	$-2i$	2	1	-2	$-i$	i	-1	-1	1	1
$X.12$	2	-2	-1	$-2i$	$2i$	2	1	-2	i	$-i$	-1	-1	1	1
$X.15$	4	-4	-2	.	.	-1	2	1	.	.	α	$\overline{\alpha}$	$-\alpha$	$-\overline{\alpha}$
$X.16$	4	-4	-2	.	.	-1	2	1	.	.	$\overline{\alpha}$	α	$-\overline{\alpha}$	$-\alpha$

We note that B has 2 irreducible modules and 2 weights, in conformity with Alperin's weight conjecture [Al]. However the irreducible modules are self-dual and the weights are duals of each other. This shows that there is no obvious 'real' version of the weight conjecture for p-blocks, when $p \neq 2$.

Consider the inclusion of groups $N_0 < \mathrm{PSL}_2(11) < M_{11}$, where $N_0 \cong 11:5$. The principal 11-blocks each have 5 non-exceptional characters. It is somewhat surprising that the number of real non-exceptional characters in these blocks is 1, 5 and 3, respectively.

Example 5: Finally B may have a real exceptional character but no real non-exceptional characters. For example let B be the 5-block containing the four faithful irreducible characters of SmallGroup$(20,1) = \langle a, b \mid a^5, b^4, a^b = a^{-1}\rangle$. The two exceptional characters have F-S indicators -1, but neither of the two non-exceptional characters is real. Here is the character table of B, with $\beta = (-1+\sqrt{5})/2$ and $*\beta = (-1-\sqrt{5})/2$:

2	2	2	2	2	1	1	1	1
5	1	1	.	.	1	1	1	1
	$1a$	$2a$	$4a$	$4b$	$5a$	$5b$	$10a$	$10b$
$X.3$	1	-1	i	$-i$	1	1	-1	-1
$X.4$	1	-1	$-i$	i	1	1	-1	-1
$X.5$	2	-2	.	.	β	$*\beta$	$-\beta$	$-*\beta$
$X.6$	2	-2	.	.	$*\beta$	β	$-*\beta$	$-\beta$

3. Miscellaneous results

We need general results from representation theory, some of which are not so well-known. So in this section p is a prime and B is a p-block of a finite group G.

Let χ be an irreducible character in B, let x be a p-element of G and let y be a p-regular element of $\mathrm{C}_G(x)$. Then

$$\chi(xy) = \sum_{\varphi} d^{(x)}_{\chi,\varphi}\varphi(y),$$

where φ ranges over the irreducible Brauer characters in blocks of $\mathrm{C}_G(x)$ which Brauer induce to B, and each $d^{(x)}_{\chi,\varphi}$ is an algebraic integer, called a generalized decomposition number; if $x = 1$, φ is an irreducible Brauer character in B and $d^{(x)}_{\chi,\varphi}$ is simplified to $d_{\chi,\varphi}$. It is an integer called an ordinary decomposition number of B.

Brauer [B, Theorem (4A)] used his Second Main Theorem to prove the following remarkable 'local-to-global' formula for F-S indicators:

$$\sum_{\chi} \epsilon(\chi) d^{(x)}_{\chi,\varphi} = \sum_{\psi} \epsilon(\psi) d^{(x)}_{\psi,\varphi}, \tag{1}$$

where χ ranges over the irreducible characters in B and ψ ranges over the irreducible characters in blocks of $\mathrm{C}_G(x)$ which Brauer induce to B. We have previously used this formula to determine the F-S indicators of the irreducible characters in 2-blocks with a cyclic, Klein-four or dihedral defect group.

Our next result relies on Clifford theory. However it was inspired by (and can be proved using) the notion of a weakly real 2-block, as introduced in [M1]. Suppose that N is a normal subgroup of G and $\phi \in \mathrm{Irr}(N)$, with stabilizer G_ϕ in G. If $G_\phi \subseteq H \subseteq G$, the Clifford correspondence is a bijection $\mathrm{Irr}(G \mid \phi) \leftrightarrow \mathrm{Irr}(H \mid \phi)$ such that $\chi \leftrightarrow \psi$ if and only if $\langle \chi{\downarrow_H}, \phi\rangle \neq 0$ or $\chi = \psi{\uparrow^G}$. The stabilizer of $\{\phi, \overline{\phi}\}$ in G is called the extended stabilizer of ϕ, here denoted by G^*_ϕ. So $|G^*_\phi : G_\phi| \leq 2$, with equality if and only if $\phi \neq \overline{\phi}$ but ϕ

is G-conjugate to $\overline{\phi}$. If $G_\phi^* \subseteq H$ it is easy to see that χ is real if and only if ψ is real. Moreover in this case $\epsilon(\chi) = \epsilon(\psi)$.

We need one other idea. Suppose that T is a degree 2 extension of G. Then the *Gow indicator* [G, 2.1] of a character χ of G with respect to T is defined to be

$$\epsilon_{T/G}(\chi) := \frac{1}{|G|} \sum_{t \in T \setminus G} \chi(t^2).$$

Clearly $\epsilon(\chi{\uparrow}^T) = \epsilon(\chi) + \epsilon_{T/G}(\chi)$. Just like the F-S indicator, $\epsilon_{T/G}(\chi) = 0, \pm 1$, for each $\chi \in \mathrm{Irr}(G)$. Moreover $\epsilon_{T/G}(\chi) \neq 0$ if and only if χ is T-conjugate to $\overline{\chi}$.

Lemma 3. *Let N be a normal odd order subgroup of G and let $\phi \in \mathrm{Irr}(N)$. Suppose that G_ϕ^* does not split over G_ϕ. Then there exists $\chi \in \mathrm{Irr}(G \mid \phi)$ such that $\epsilon(\chi) = -1$.*

Proof. We first show that there exists $\psi \in \mathrm{Irr}(G \mid \phi)$ such that $\epsilon(\psi) = +1$. For let S be a Sylow 2-subgroup of G. As $\phi{\uparrow}^G$ vanishes on the 2-singular elements of G, we have $(\phi{\uparrow}^G){\downarrow}_S = \frac{\phi(1)|G|}{|N||S|}\rho_S$, where ρ_S is the regular character of S. Now $\frac{\phi(1)|G|}{|N||S|}$ is an odd integer. So $\langle(\phi{\uparrow}^G){\downarrow}_S, 1_S\rangle$ is odd. Moreover $\phi{\uparrow}^G$ is a real character of G. So $\langle(\phi{\uparrow}^G), \psi\rangle = \langle(\phi{\uparrow}^G), \overline{\chi}\rangle$, for each $\psi \in \mathrm{Irr}(G)$. Pairing each irreducible character of G with its complex conjugate, we see that there exists a real-valued $\psi \in \mathrm{Irr}(G \mid \phi)$ such that $\langle\psi{\downarrow}_S, 1_S\rangle$ is odd. Then $\epsilon(\psi) = \epsilon(1_S) = +1$.

Following the discussion before the lemma, we may assume that $G = G_\phi^*$. So $|G : G_\phi| = 2$. Next suppose that $g \in G$ and $\phi{\uparrow}^{G_\phi}(g^2) \neq 0$. Write $g = xy = yx$, where x is a 2-element and y is a 2-regular element. Then $g^2 = x^2y^2$. As $\phi{\uparrow}^{G_\phi}$ vanishes off N, we have $x^2 = 1$ and $y^2 \in N$. So $x \in G_\phi$, as G_ϕ contains all involutions in G. Moreover $y \in N$, as y has odd order. Thus $g \in G_\phi$, whence

$$\epsilon_{G/G_\phi}(\phi{\uparrow}^{G_\phi}) = \frac{1}{|G_\phi|} \sum_{g \in G \setminus G_\phi} \phi{\uparrow}^{G_\phi}(g^2) = 0.$$

Now $\mathrm{Irr}(G_\phi \mid \phi)$ contains no real characters, as $\phi \neq \overline{\phi}$. So $\epsilon(\phi{\uparrow}^G) = \epsilon_{G/G_\phi}(\phi{\uparrow}^{G_\phi}) + \epsilon(\phi{\uparrow}^{G_\phi}) = 0$. Equivalently

$$\sum_{\chi \in \mathrm{Irr}(G)} \langle\phi{\uparrow}^G, \chi\rangle\epsilon(\chi) = 0.$$

Together with the fact that $\langle\phi{\uparrow}^G, \psi\rangle\epsilon(\psi) > 0$, this implies that $\langle\phi{\uparrow}^G, \chi\rangle\epsilon(\chi) < 0$, for some $\chi \in \mathrm{Irr}(G)$. Thus $\chi \in \mathrm{Irr}(G \mid \phi)$ and $\epsilon(\chi) = -1$, which completes the proof. □

It is well-known that each G-invariant irreducible character of a normal subgroup of G extends to G, when the quotient group is cyclic.

Lemma 4. *Suppose that N is a normal subgroup of G such that G/N is cyclic and of even order. Let $\varphi \in \mathrm{Irr}(N)$ be real and G-invariant. Then φ has a real extension to G if and only if φ has a real extension to T, where $N \subset T \subseteq G$ and T/N has order 2.*

Proof. The 'only if' part is obvious. So assume that φ has a real extension to T. Then both extensions of φ to T are real. Let ω be a generator of the abelian group $\mathrm{Irr}(G/N)$ and let χ be any extension of φ to G. Then $\omega^i\chi$, $i \geq 0$ give all extensions of φ to G. Here $\omega^i = \omega^j$ if and only if $i \equiv j \pmod{|G/N|}$.

As $\overline{\chi}$ lies over φ, we have $\overline{\chi} = \omega^i\chi$, for some $i \geq 0$. Now $\chi{\downarrow_T}$ is an extension of φ to T and $\overline{\chi}{\downarrow_T} = (\omega^i{\downarrow_T})(\chi{\downarrow_T})$. As $\chi{\downarrow_T}$ is real, it follows that $\omega^i{\downarrow_T}$ is trivial. So $T \subseteq \ker(\omega^i)$, whence $i \equiv 2j \pmod{|G/N|}$, for some $j \geq 0$. Now $\overline{\omega^j\chi} = \omega^{i-j}\chi = \omega^j\chi$. So $\omega^j\chi$ is a real extension of φ to G. □

Notice that in this context φ has a real extension to T if and only if $\epsilon(\varphi) = \epsilon_{T/N}(\varphi)$. When G/N has even order, but is not cyclic, and φ is a real irreducible character of N which extends to G, it is not clear whether there is a sensible sufficient criteria for φ to have a real extension to G.

Finally we need the following consequence of the first orthogonality relation:

Lemma 5. *Let $W \subseteq X \subseteq Y$ be finite abelian groups. Then for $\lambda \in \mathrm{Irr}(Y)$ we have*

$$\sum_{x \in X \setminus W} \lambda(x) = \begin{cases} |X| - |W|, & \text{if } X \subseteq \ker(\lambda). \\ -|W|, & \text{if } W \subseteq \ker(\lambda) \text{ but } X \not\subseteq \ker(\lambda). \\ 0, & \text{if } W \not\subseteq \ker(\lambda). \end{cases}$$

4. The Brauer tree and its real-stem

From now on G is a finite group, p is an odd prime and B is a real p-block of G which has a cyclic defect group. To avoid trivialities we assume that the defect group is non-trivial.

Dade asserts [D, Theorem 1, Part 2] that each decomposition number in B is either 0 or 1. The Brauer tree of B is a planar graph with edges labelled by the irreducible Brauer character in B and with vertices labelled by the irreducible characters in B (the exceptional characters in B label a single 'exceptional' vertex). The edge labelled by an irreducible Brauer character θ meets the vertex labelled by an irreducible character χ if and only if the decomposition number $d_{\chi,\theta}$ is not 0.

When B is real, complex conjugation acts on the Brauer tree of B, and in particular fixes the exceptional vertex. However, as we have seen in Examples 2,3 and 4 above, B may have no real exceptional characters. So we restate [F, VII,9.2] in the following more precise fashion:

Lemma 6. *The subgraph of the Brauer tree of B consisting of the exceptional vertex and those vertices and edges which correspond to real characters and Brauer characters is a straight line segment.*

Feit calls this line segment the real-stem of B. An easy consequence is:

Corollary 7. *The number of real non-exceptional characters in B equals the number of real irreducible Brauer characters in B.*

Proof. Suppose that B has r real irreducible Brauer characters. Then the real-stem of the Brauer tree has r edges and $r+1$ vertices. One of these is the exceptional vertex. So B has r real non-exceptional characters. □

Let θ be a real irreducible p-Brauer character of a finite group G. As p is odd, the G-representation space of θ affords a non-degenerate G-invariant bilinear form which is either symmetric or skew-symmetric. Given the symmetry groups of such forms, we refer to θ as being of orthogonal or symplectic type. Thompson and Willems [W, 2.8] proved that there is a real irreducible character χ of G such that $d_{\chi,\theta}$ is odd. Moreover θ has orthogonal type if $\epsilon(\chi) = +1$ or symplectic type if $\epsilon(\chi) = -1$. This implies that $\epsilon(\psi) = \epsilon(\chi)$, for all real irreducible characters ψ such that $d_{\psi,\theta}$ is odd.

Proof of part (ii) of Theorem 1. Let X and Y be real non-exceptional characters which lie on the same side of the exceptional vertex in the real-stem of B. Then by Lemma 6 there is a sequence $X = X_0, X_1, \ldots, X_n = Y$ of real non-exceptional characters and a sequence $\theta_1, \ldots, \theta_n$ of real irreducible Brauer characters such that $d_{X_{i-1},\theta_i} = 1 = d_{X_i,\theta_i}$, for $i = 1, \ldots, n$. The Thompson-Willems result implies that $\epsilon(X_{i-1}) = \epsilon(X_i)$, for $i = 1, \ldots, n$. So $\epsilon(X) = \epsilon(Y)$. This gives part (ii) of Theorem 1. □

A similar argument gives the following weak form of parts (i) and (iii) of Theorem 1:

Lemma 8. *If B has a real exceptional character and a real non-exceptional character, then all real irreducible characters in B have the same F-S indicators.*

Notice that if B is the principal p-block of a group with a cyclic Sylow p-subgroup, and B has an irreducible character with F-S indicator -1 (e.g. the principal 7-block of $\mathrm{U}(3,3)$) then the lemma implies that B has no real exceptional characters.

5. The exceptional characters

We outline some results from [D] using the language of subpairs. See [NT, Chapter 5.9] for a full description of the theory. We then prove results about the local blocks in B, in Proposition 10, and the exceptional characters in B, in Proposition 11. This allows us to prove parts (i), (iii) and (iv) of Theorem 1.

Recall that B is a p-block with a non-trivial cyclic defect group D. Write $|D| = p^a$, where $a > 0$, and let $1 \subset D_{a-1} \subset D_{a-2} \subset \cdots \subset D_1 \subset D_0 = D$ be the complete list of subgroups of D. So $[D : D_i] = p^i$, for $i = 0, \ldots, a-1$. Set $C_i = C_G(D_i)$ and $N_i = N_G(D_i)$. So $C_0 \subseteq C_1 \subseteq \ldots \subseteq C_{a-1}$, and $N_0 \subseteq N_1 \subseteq \ldots \subseteq N_{a-1}$.

As p is odd, $\mathrm{Aut}(D_i)$ is a cyclic group of order $p^{a-i-1}(p-1)$. So N_i/C_i is a cyclic group whose order divides $p^{a-i-1}(p-1)$. Moroever the centralizer of D_i in $\mathrm{Aut}(D)$ has order p^i. So $C_i \cap N_0/C_0$ is a cyclic p-group. We note that the unique involution in $\mathrm{Aut}(D)$ inverts every element of D.

Fix a Sylow B-subpair (D, b_0). So b_0 is a p-block of C_0 such that $b_0^G = B$ and the pair (D, b_0) is uniquely determined up to G-conjugacy. Set $b_i := b_0^{C_i}$, for $i = 1, \ldots, a-1$. Then by [NT, 5.9.3] the lattice of B-subpairs contained in (D, b_0) is

(2) $$(1, B) \subset (D_{a-1}, b_{a-1}) \subset \cdots \subset (D_1, b_1) \subset (D, b_0).$$

Set $E := \mathrm{N}(D, b_0)$, the stabilizer of b_0 in N_0. Then $e := |E : C_0|$ is called the inertial index of B. Now $p \not| e$, by Brauer's extended first main theorem. So $e \mid (p-1)$. Let $x \in E$. Then $D_i^x = D_i$. As $(D_i, b_i), (D_i, b_i^x) \subseteq (D, b_0)$, it follows from (2) that $b_i^x = b_i$. So $EC_i \subseteq \mathrm{N}(D_i, b_i)$. Conversely let $n \in \mathrm{N}(D_i, b_i)$. As (D, b_0) and $(D, b_0)^n$ are Sylow b_i-subpairs (in the group C_i), there is $c \in C_i$ such that $nc_i \in E$. This shows that $\mathrm{N}(D_i, b_i) \subseteq EC_i$. This recovers Dade's observation that $\mathrm{N}(D_i, b_i) = EC_i$.

Now $E \cap C_i/C_0$ is a subgroup of $C_i \cap N_0/C_0$ and a quotient of the group E/C_0. As $C_i \cap N_0/C_0$ is a p-group and E/C_0 has p'-order, we deduce that $E \cap C_i = C_0$. It follows from this $EC_i/C_i \cong E/C_0$, and in particular $|EC_i : C_i| = e$.

By [D, Theorem 1, Part 1] B has e irreducible Brauer characters, listed as $\chi_1, \ldots, \chi_e$. Each b_i has inertial index 1. So b_i has a unique irreducible Brauer character, denoted φ_i.

From the above discussion there are $|N_i : EC_i| = \frac{|N_i:C_i|}{e}$ distinct blocks of C_i which induce to B, namely b_i^τ as τ ranges over N_i/EC_i. Also there are $\frac{p^{a-i}-p^{a-i-1}}{|N_i:C_i|}$ conjugacy classes of G which contain a generator of D_i. So B has $\frac{p^{a-i}-p^{a-i-1}}{e}$ subsections (x, b), with $D_i = \langle x \rangle$. A consequence of Brauer's second main theorem [NT, 5.4.13(ii)] is that the number of irreducible characters in a block equals the number of columns in the block.

Lemma 9. *A complete set of columns of B is*

$$(1, \chi_1), \ldots, (1, \chi_e), \qquad (x_i^{\sigma_i}, \varphi_i^{n_i}), \quad i = 0, \ldots, a-1.$$

Here x_i is a fixed generator of D_i, σ_i ranges over a set of representatives for the cosets of the image of N_i/C_i in $\mathrm{Aut}(D_{a-i})$ and n_i ranges over a set of representatives for the cosets of EC_i in N_i. In particular $k(B) = e + \frac{p^a-1}{e}$.

Let Λ be a set of representatives for the $\frac{p^a-1}{e}$ orbits of E on $\mathrm{Irr}(D)^\times$. Then

(3) $$\mathrm{Irr}(B) = \{X_1, \ldots, X_e\} \bigcup \{X_\lambda \mid \lambda \in \Lambda\}.$$

Also set $X_0 := \sum_{\lambda \in \Lambda} X_\lambda$. Dade refers to the X_λ as the exceptional characters of B.

Notice that as $\ell(b_i) = 1$, b_i is real if and only if φ_i is real. The next two propositions are relatively elementary.

Proposition 10. *All the blocks $b_0, b_1, \ldots, b_{a-1}$ are real or none of them are real.*

Proof. We have $(b_i^o)^G = B^o = B$. So (D, b_0) and (D, b_o^o) are Sylow B-subpairs, and there is $n \in \mathrm{N}_0$ such that $b_0^o = b_0^n$.

Suppose that b_j is real, for some $j = 0, \ldots, a-1$. As $(D_j, b_j^n), (D_j, b_j^o) \subset (D_0, b_0^o)$, it follows from (2) that $b_j^n = b_j^o = b_j$. So $n \in \mathrm{N}(D_j, b_j) = EC_j$. Write $n = ec$, where $e \in E$ and $c \in C_j$. Then $c = e^{-1}n \in C_j \cap N_0$ and $b_0^c = b_0^n = b_0^o$. So $c^2 \in C_j \cap E = C_0$. But $C_j \cap N_0/C_0$ has odd order, as it is a p-group. So $c \in C_0$, which shows that $n \in E$. As $b_0^n = b_0$, it follows that b_0 is real.

Now let $i = 0, \dots, a-1$. Then $(D_i, b_i), (D_i, b_i^o) \subset (D_0, b_0) = (D_0, b_0^o)$. So $b_i = b_i^o$, for $i = 0, \dots, a-1$, using (2). This shows that all $b_0, \dots, b_{a-1}$ are real. □

We showed in [M3, 1.1] that the number of real irreducible characters in a block equals the number of real columns in the block. Here (x, φ) is real if $x^g = x^{-1}$ and $\varphi^g = \overline{\varphi}$, for some $g \in G$.

Let $i = 0, \dots, a-1$. As b_i has inertial index 1, it has $|D|$ irreducible characters. Modifying [D, p26] we use the notation

$$\mathrm{Irr}(b_i) = \{X'_{i,\lambda} \mid \lambda \in \mathrm{Irr}(D)\}. \tag{4}$$

Here $X'_{i,1}$ is the unique non-exceptional character in b_i, and all characters $X'_{i,\lambda}$ with $\lambda \neq 1$ are exceptional. Suppose that b_i is real. The columns of b_i are (d, φ_i), for $d \in D$. As C_i acts trivially on the columns, the only real column is $(1, \varphi_i)$. So $X'_{i,1}$ is the only real irreducible character in b_i.

We will refine the next result in part (i) of Theorem 1:

Proposition 11. *All exceptional characters in B are real or none are real.*

Proof. It follows from Corollary 7 and Lemma 9 that the number of real exceptional characters in B equals the number of real columns (x, φ) with $x \in D^\times$ and $\varphi \in \mathrm{IBr}(\mathrm{C}_G(x))$.

Suppose that B has a real exceptional character, and let (x, φ) be a real column of B, with $x \in D^\times$. Then $\langle x \rangle = D_i$, for some $i = 0, \dots, a-1$. As N_i/C_i is abelian, the columns $(x', \varphi_i^{n_i})$ are real, for all generators x' of D_i and all $n_i \in N_i$. In particular (x_i, φ_i) is a real column. Choose $n \in N_i$ such that $x_i^n = x_i^{-1}$ and $\varphi_i^n = \overline{\varphi}_i$. We may suppose that $n^2 \in C_i$.

Suppose first that b_i is real. As $\varphi_i = \overline{\varphi}_i$, n fixes φ_i and inverts D_i. So nC_i is an involution in EC_i/C_i. As $EC_i/C_i \cong E/C_0$, we may assume without loss that nC_0 is an involution in E/C_0. Now all the blocks $b_0, \dots, b_{a-1}$ are real. Hence all $\varphi_0, \dots, \varphi_{a-1}$ are real. As n inverts D_j and fixes φ_j, all columns (x_j, φ_j) are real. Thus all columns (x, φ), with $x \in D^\times$, are real. So all exceptional characters in B are real in this case.

Conversely, suppose that b_i is not real. As nC_i is the unique involution in N_i/C_i, but $n \notin EC_i$, it follows that $|EC_i : C_i| = e$ is odd. Now (D, b_0) and (D, b_0^o) are Sylow B-subpairs, but $b_0 \neq b_0^o$. So there is $m \in N_0 \backslash E$ such that $b_0^m = b_0^o$. As $m^2 \in E$ and $|E : C_0|$ is odd, we may choose m so that $m^2 \in C_0$. Then mC_0 is the unique involution in N_0/C_0. In particular m inverts every element of D. Let $j = 0, \dots, a-1$. Then (D_j, b_j^m) and (D_j, b_j^o) are B-subpairs contained in (D, b_0^o). So $b_j^m = b_j^o$ and thus $(d_j, \varphi_j)^m = (d_j^{-1}, \overline{\varphi}_j)$. It follows that all exceptional characters in B are real in this case also. □

Examination of the proof shows that:

Corollary 12. *All exceptional characters in B are real if and only if b_0 is real and e is even, or b_0 is not real and e is odd.*

We need some additional notation. Set $\Lambda_u := \{\lambda \in \Lambda \mid \ker(\lambda) = D_u\}$, for $u = 1, \dots, a$. So $|\Lambda_u| = \frac{p^u - p^{u-1}}{e}$. Now choose $\lambda \in \Lambda_u$ and set

$$\epsilon_u := \epsilon(X_\lambda).$$

Note that X_λ and X_μ are Galois conjugates, for all $\lambda, \mu \in \Lambda_u$ (this follows from [D, part 2 of Theorem 1 and Corollary 1.9]). So ϵ_u does not depend on λ.

Recall our notation (4) for the irreducible characters $X'_{i,\lambda}$ in b_i. As already noted, $X'_{i,1}$ is the only possible real irreducible character in b_i. We set

$$\nu_i := \epsilon(X'_{i,1}), \quad \text{for } i = 0, \dots, a-1.$$

Now let $i = 0, \dots, a-1$ and choose $x \in D_i - D_{i+1}$ and $\rho \in N_i$. According to [D, Theorem 1, Part 3] there are signs $\varepsilon'_0, \varepsilon_0, \varepsilon_1 \dots, \varepsilon_e$ and γ_i such that

$$\begin{aligned} d^{(x)}_{X_\lambda,\varphi_i^\rho} &= \varepsilon_0\gamma_i \sum_{\tau\in EC_i/C_i} \lambda({}^{\rho\tau}x), & d^{(x)}_{X_j,\varphi_i^\rho} &= \varepsilon_j\gamma_i, \quad \text{for } j = 1, \dots, e \\ d^{(x)}_{X'_{i,\lambda},\varphi_i^\rho} &= \varepsilon'_0\gamma_i\lambda({}^{\rho}x), & d^{(x)}_{X'_{i,1},\varphi_i^\rho} &= 1 \end{aligned}$$

Here EC_i/C_i is a set of representatives for the cosets of C_i in EC_i. Note that Feit uses the notation $\delta_0 = -\varepsilon_0$ and $\delta_j = \varepsilon_j$, for $j = 1, \dots, e$. Now let $i = 0, \dots, a-1$ and $x \in D_i - D_{i+1}$. Then it follows from [D, Corollary 1.9] that $X_j(x) = |N_i : EC_i|\varphi_i(1)\delta_j\gamma_i$. So $\delta_j\gamma_i$ is the sign of the integer $X_j(x)$.

There is a nice relationship between the signs $\varepsilon_0, \varepsilon_1 \dots, \varepsilon_e$ and the Brauer tree of B. Suppose that j and k are adjacent vertices in the Brauer tree. Then $X_j + X_k$ is a principal indecomposable character of G. So it vanishes on $D^\times$, and hence $\delta_j + \delta_k = 0$ (see [F, V11, Section 9]). So suppose that there are d_j edges between the vertex j and the exceptional vertex 0 in the Brauer tree. Then $\delta_j = (-1)^{d_j}\delta_0$. So $\varepsilon_j = (-1)^{d_j-1}\varepsilon_0$, for $j = 1, \dots, e$.

We now prove part (i) of our main theorem. But note that this proof does not depend on Propositions 10 and 11:

Proof of part (i) of Theorem 1. Applying (1), with $\rho \in N_i$ and $x \in D_i - D_{i+1}$, we get

$$\sum_{j=1}^{e} \epsilon(X_j)\varepsilon_j\gamma_i + \sum_{\lambda\in\Lambda} \epsilon(X_\lambda)\varepsilon_0\gamma_i \sum_{\tau\in EC_i/C_i} \lambda({}^{\rho\tau}x) = \nu_i.$$

Now set $\sigma := \varepsilon_0 \sum_{j=1}^{e} \epsilon(X_j)\varepsilon_j$. So σ is independent of i, ρ and x. Then the above equality transforms to

$$\sum_{u=1}^{a} \epsilon_u \sum_{\lambda\in\Lambda_u} \sum_{\tau\in EC_i} \lambda({}^{\rho\tau}x) = \varepsilon_0\gamma_i\nu_i - \sigma,$$

where the right hand side is independent of ρ and x. Let ρ range over a set of representatives for the $\frac{|N_i:C_i|}{e}$ cosets of EC_i in N_i and let x range over a set of representatives for the $\frac{p^{a-i}-p^{a-i-1}}{|N_i:C_i|}$ orbits of N_i on the generators of D_i. Then ${}^{\rho\tau}x$ will range over all generators of D_i. Summing the resulting equalities gives

$$\sum_{u=1}^{a} \epsilon_u \sum_{\lambda\in\Lambda_u} \sum_{x\in D_i-D_{i+1}} \lambda(x) = \left(\frac{p^{a-i} - p^{a-i-1}}{e}\right)(\varepsilon_0\gamma_i\nu_i - \sigma).$$

We use $|\Lambda_u| = \frac{p^u - p^{u-1}}{e}$ and Lemma 5 to transform this equality to

$$(p^{a-i} - p^{a-i-1}) \sum_{u=1}^{i} \frac{p^u - p^{u-1}}{e} \epsilon_u - p^{a-i-1} \frac{p^{i+1} - p^i}{e} \epsilon_{i+1} = \frac{p^{a-i} - p^{a-i-1}}{e} (\varepsilon_0 \gamma_i \nu_i - \sigma).$$

After cancelling the factor $\frac{p^{a-i-1}(p-1)}{e}$, we get

$$\sum_{u=1}^{i} (p^u - p^{u-1}) \epsilon_u - p^i \epsilon_{i+1} = \varepsilon_0 \gamma_i \nu_i - \sigma. \tag{5}$$

Here $\sum_{u=1}^{0} (p^u - p^{u-1}) \epsilon_u$ is taken to be 0, when $i = 0$. We write down the equalities (5) for $i = 0, 1, 2, \ldots$ in turn:

$$\begin{aligned} -\epsilon_1 &= \varepsilon_0 \gamma_0 \nu_0 - \sigma \\ (p-1)\epsilon_1 - p\epsilon_2 &= \varepsilon_0 \gamma_1 \nu_1 - \sigma \\ (p-1)\epsilon_1 + (p^2 - p)\epsilon_2 - p^2 \epsilon_3 &= \varepsilon_0 \gamma_2 \nu_2 - \sigma \\ (p-1)\epsilon_1 + (p^2 - p)\epsilon_2 + (p^3 - p^2)\epsilon_3 - p^3 \epsilon_4 &= \varepsilon_0 \gamma_3 \nu_3 - \sigma \\ &\vdots \\ (p-1)\epsilon_1 + (p^2 - p)\epsilon_2 + \cdots + (p^{a-1} - p^{a-2})\epsilon_{a-1} - p^{a-1} \epsilon_a &= \varepsilon_0 \gamma_{a-1} \nu_{a-1} - \sigma \end{aligned} \tag{6}$$

Subtract the first equality from the second to get

$$p(\epsilon_1 - \epsilon_2) = \varepsilon_0 (\gamma_1 \nu_1 - \gamma_0 \nu_0).$$

The left hand side equals $-p, 0$ or p and the right hand equals $-2, 0$ or 2. As p is odd, the common value is 0. So $\epsilon_2 = \epsilon_1$ and $\gamma_1 \nu_1 = \gamma_0 \nu_0$. Substitute these values back into all equations in (6). Now subtract the first from the third equality to get

$$p^2 (\epsilon_1 - \epsilon_3) = \varepsilon_0 (\gamma_2 \nu_2 - \gamma_0 \nu_0).$$

Once again both sides are 0. So $\gamma_2 \nu_2 = \gamma_0 \nu_0$ and $\epsilon_3 = \epsilon_1$. Proceeding in this way, we get

$$\epsilon_1 = \epsilon_2 = \cdots = \epsilon_a, \quad \gamma_0 \nu_0 = \gamma_1 \nu_1 = \cdots = \gamma_{a-1} \nu_{a-1}.$$

□

Following the above proof, and the discussion before the proof, we obtain:

Corollary 13. *Suppose that b_0 is real and let $D = \langle x \rangle$. Then for each $i = 0, \ldots, a-1$ and $j = 0, \ldots, e$, the integer $X_j(x^{p^i}) X_j(x)$ has sign $\epsilon(X'_{i,1}) \epsilon(X'_{0,1})$.*

There is no apparent relationship between the F-S indicators $\nu_0, \ldots, \nu_{a-1}$:

Example: The 2-nilpotent group $G = \langle a, b, c \mid a^4, a^2 = b^2, a^b = a^{-1}, c^9, a^c = b, b^c = ab \rangle$ has isomorphism type $3.\mathrm{SL}(2,3)$. Set $D = \langle c \rangle$. Then D is cyclic of order 9, with $C_0 = D \times \langle a^2 \rangle$ and $C_1 = G$. Let θ be the non-trivial irreducible character of C_0/D, and let b_0 be the 3-block of C_0 which contains θ. Then $\theta = X'_{0,1}$ is the unique non-exceptional character in b_0. So $\nu_0 = \epsilon(X'_{0,1}) = +1$. Set $b_1 = b_0^G$. Then b_1 also has a unique non-exceptional character $X'_{1,1}$. But now $\nu_1 = \epsilon(X'_{1,1}) = -1$, as $X'_{1,1}$ restricts to the non-linear irreducible character of $\langle a, b\rangle \cong Q_8$.

This example arises from the fact that the Glauberman correspondence [NT, 5.12] does not preserve the F-S indicators of characters.

proof of part (iii) of Theorem 1. This is an immediate consequence of Lemma 8 and part (i) of Theorem 1. □

Consider the real-stem of B as a horizontal line segment with s vertices and $s-1$ edges, where $s \geq 1$. We label the vertices using an interval $[-\ell, \ldots, -2, -1, 0, 1, 2, \ldots, r]$ so that 0 labels the exceptional vertex. Thus $s = r + \ell + 1$, and there are ℓ real non-exceptional characters on the left of the exceptional vertex, and r on the right (the choice of left and right is unimportant).

As above, X_0 is the sum of the exceptional characters in B. Now we relabel the non-exceptional characters in B so that X_i is the real non-exceptional character corresponding to vertex i, for $i = -\ell, \ldots, r$ and $i \neq 0$. In view of parts (i) and (ii) of Theorem 1 there are signs $\epsilon_\pm$ such that

$$\epsilon(X_i) = \begin{cases} \epsilon_-, & \text{for } i = -\ell, \ldots, -1. \\ \epsilon_0, & \text{for } i = 0. \\ \epsilon_+, & \text{for } i = 1, \ldots, r. \end{cases}$$

Next let σ be a generator of D. It follows from [D, Corollary 1.9] that $X_0(\sigma) = -\varepsilon_0\gamma_0|N_0 : E|\varphi_0(1)$. So $X_i(\sigma) = (-1)^i X_0(\sigma)$, as $X_i + X_{i+1}$ is a projective character of G, for $i = -\ell, \ldots, r-1$ (see [F, VII,2.19(ii)]).

Recall from Section (5) that there are $|N_0 : E|$ blocks of C_0 which induce to B; these are the blocks b_0^τ, where τ ranges over N_0/E. We note also that $X'_{0,1}({}^\tau\sigma) = \varphi_0(1)$. Now [B, Theorem(4B)] is an immediate consequence of [B, Theorem(4A)]. In our context, this states that

$$\sum_{i=-\ell}^{r} \epsilon(X_i)X_i(\sigma) = |N_0 : E|\epsilon(X'_{0,1})X'_{0,1}(\sigma).$$

In view of the previous paragraph this simplifies to

$$\sum_{i=1}^{\ell}(-1)^i\epsilon_- + \epsilon_0 + \sum_{i=1}^{r}(-1)^i\epsilon_+ = -\varepsilon_0\gamma_0\nu_0. \tag{7}$$

We consider a number of cases.

Suppose first that $\epsilon_0 \neq 0$. Then $\epsilon_- = \epsilon_0 = \epsilon_+$, by part (iii) of Theorem 1. So (7) becomes

$$-\varepsilon_0\gamma_0\nu_0\epsilon_0 = \begin{cases} (-1)^\ell, & \text{if } s \text{ is odd.} \\ 0, & \text{if } s \text{ is even.} \end{cases} \tag{8}$$

In particular b_0 is not real if s is even. As e is odd when s is even, this already follows from Corollary 12.

Suppose then that $\epsilon_0 = 0$. Now (7) evaluates as

$$-\varepsilon_0\gamma_0\nu_0 = \begin{cases} \epsilon_-, & \text{if } \ell \text{ is odd and } r \text{ is even.} \\ \epsilon_- + \epsilon_+, & \text{if } \ell \text{ and } r \text{ are both odd.} \\ \epsilon_+, & \text{if } \ell \text{ is even and } r \text{ is odd.} \\ 0, & \text{if } \ell \text{ and } r \text{ are both even.} \end{cases} \tag{9}$$

proof of part (iv) of Theorem 1. The hypothesis is that $\epsilon_0 = 0$, at least one of ϵ_-, ϵ_+ is not zero and $\ell \equiv r \equiv 1 \pmod 2$. Now B has e non-exceptional characters, of which $\ell + r$ are real-valued. So $e \equiv \ell + r$ is even. Then b_0 is not real, according to Corollary 12. This in turn implies that $\nu_0 = 0$. So $\epsilon_- + \epsilon + = 0$, according to (9). We conclude that $\epsilon_-\epsilon_+ = -1$, which gives the conclusion of (iv). □

6. Passing from B to its canonical character

Let $i = 0, \ldots, a-1$. Then N_i contains the normalizer N_0 of D in G. So by Brauer's first main theorem there is a unique p-block B_i of N_i such that $B_i^G = B$. As $(B_i^o)^G = B^o = B$, the uniqueness forces $B_i^o = B_i$. Now B_i has defect group D and inertial index $e = |EC_i : C_i|$. So $\ell(B_{a-1}) = e$ and $k(B_{a-1}) = e + \frac{p^a-1}{e}$. We first consider the block B_{a-1} of the largest subgroup N_{a-1}. Following [D, Section 7], write

$$\mathrm{IBr}(B_{a-1}) = \{\tilde{\chi}_1, \ldots, \tilde{\chi}_e\}, \quad \mathrm{Irr}(B_{a-1}) = \{\tilde{X}_1, \ldots, \tilde{X}_e\} \bigcup \{\tilde{X}_\lambda \mid \lambda \in \Lambda\},$$

and set $\tilde{X}_0 = \sum \tilde{X}_\lambda$.

Proposition 14. *The exceptional characters in B and B_{a-1} have the same F-S indicators.*

Proof. Suppose first that $|\Lambda| \geq 2$. According [D, (7.2)] there is a sign d such that

$$(\tilde{X}_\lambda - \tilde{X}_\mu)^G = d(X_\lambda - X_\mu), \quad \text{for all } \lambda, \mu \in \Lambda.$$

It follows that $\langle \tilde{X}_\lambda, X_\lambda \rangle$ or $\langle \tilde{X}_\mu, X_\lambda \rangle$ is odd. So in view of part (i) of Theorem 1, the conclusion holds in this case.

From now on we suppose that $|\Lambda| = 1$. Then E has a single orbit on $\mathrm{Irr}(D)^\times$, which forces $|D| = p$ and $e = p-1$. As $\tilde{X}_0$ is the unique exceptional character in B_{a-1}, it is real valued. Then it follows from part (iii) of Theorem 1 that all real irreducible characters in B_{a-1} have the same F-S indicators.

Now by [D, (7.3), (7.8), first two paragraphs of p40], there is a sign ε_0' such that

$$(\tilde{X}_0 - \sum_{i=1}^{p-1} \tilde{X}_i)^G = \varepsilon_0' \sum_{i=0}^{p-1} \varepsilon_i X_i.$$

Here $\varepsilon_0, \ldots, \varepsilon_{p-1}$ are as introduced earlier and X_0 can be chosen to be real, as p is odd. Taking inner-products of characters, and reading modulo 2, we see that $\langle \tilde{X}_i^G, X_0 \rangle$ is odd, for some real $\tilde{X}_i$. So $\epsilon(\tilde{X}_i) = \epsilon(X_0)$. Then by the previous paragraph $\epsilon(\tilde{X}_0) = \epsilon(X_0)$. □

Proposition 15. *All exceptional characters in $B_0, \ldots, B_{a-1}$ and B have the same F-S indicators.*

Proof. We prove this by induction on $|D|$. The base case $|D| = p$ holds, by Proposition 14. Suppose that $|D| > p$. We assume that the conclusion holds for all p-blocks with a cyclic defect group of order strictly less than $|D|$.

We use the bar notation for subgroups and objects associated with the quotient group N_{a-1}/D_{a-1}. Let $i = 0, \ldots, a-1$. Then $\overline{N}_i$ is the normalizer of $\overline{D}_i$ in $\overline{N}_{a-1}$. As C_i centralizes D_{a-1}, Theorem 5.8.11 of [NT] shows that b_i dominates a unique block $\overline{b_i}$ of $\overline{C}_i$. Moreover $\overline{b}_i$ has cyclic defect group $\overline{D}$. Now b_i has the unique irreducible Brauer character φ_i, and we can and do identify φ_i with the unique irreducible Brauer character in $\overline{b}_i$. Then the inertia group of $\overline{b}_i$ in $\overline{N}_i$ is the inertia group of φ_i in $\overline{N}_i$, which is $\overline{EC_i}$.

According to [D, Section 4], there is a unique p-block of N_i, denoted here by $\overline{B}_i$, which lies over $\overline{b}_i$. Moreover $\overline{B_i}$ has cyclic defect group $\overline{D}$. As inflation and induction of characters commute, this block is dominated by B_i. Now B_i and $\overline{B_i}$ have the same inertial index as $|EC_i : C_i| = |\overline{EC_i} : \overline{C}_i|$. So by inflation $\mathrm{IBr}(\overline{B_i}) = \mathrm{IBr}(B_i)$. In particular $\overline{B_i}$ is the unique block of $\overline{N_i}$ that is dominated by B_i. Also by inflation $\mathrm{Irr}(\overline{B_i}) \subseteq \mathrm{Irr}(B_i)$.

As $|\overline{D}| < |D|$, all exceptional characters in $\overline{B}_0, \ldots, \overline{B}_{a-1}$ have the same F-S indicators, by our inductive hypothesis. But the inclusion $\mathrm{Irr}(\overline{B_i}) \subseteq \mathrm{Irr}(B_i)$ identifies the exceptional characters in $\overline{B}_i$ with exceptional characters in B_i. It now follows from part (i) of Theorem 1 that all exceptional characters in $B_0, \ldots, B_{a-1}$ have the same F-S indicators. $\square$

Recall that b_0 has a unique irreducible Brauer character φ_0. This is the canonical character of B, in the sense of [NT, 5.8.3]. For the next theorem, we simplify the notation of (4) for the irreducible characters in b_0 by writing χ_λ in place of $X'_{0,\lambda}$, for all $\lambda \in \mathrm{Irr}(D)$. Then according to W. Reynolds [NT, 5.8.14], for $c \in C_0$ we have

$$\chi_\lambda(c) = \begin{cases} \lambda(c_p)\varphi_0(c'_p), & \text{if } c_p \in D. \\ 0, & \text{if } c_p \notin D. \end{cases} \tag{10}$$

Then $\mathrm{Irr}(b_0) = \{\chi_\lambda \mid \lambda \in \mathrm{Irr}(D)\}$. Notice that χ_1 is the unique irreducible character in b_0 whose kernel contains D.

Theorem 16. *Suppose that B has a real exceptional character. Then N_0/C_0 has a unique subgroup T/C_0 of order 2, and all exceptional characters in B have F-S indicator equal to the Gow indicator $\epsilon_{T/C_0}(\chi_1)$.*

Proof. Recall that B has a real exceptional character if b_0 is real and e is even, or if b_0 is not real and e is odd. In both these cases $|N_0 : C_0|$ is even. As N_0/C_0 is also cyclic, it has a unique subgroup T/C_0 of order 2.

In view of Proposition 15, we may assume that $G = N_0$. So $B = B_0$, D and C_0 are normal subgroups of G and E is the stabilizer of b_0 in G. Then Λ is a set of representatives for the orbits of N_0 on $\mathrm{Irr}(D)^\times$. Set E^* as the stabilizer of $\{b_0, b_0^o\}$ in G. Clifford correspondence defines a bijection between the irreducible characters of E^* which lie over b_0 and the irreducible characters in B. This bijection preserves reality, and hence F-S indicators. So from now on we assume that $G = E^*$.

As χ_1 is invariant in E and E/C_0 is cyclic, χ_1 has e extensions to E, which we denote by $\eta_1, \ldots, \eta_e$. Then $X_i := \eta_i^G$, for $i = 1, \ldots, e$, give the e non-exceptional characters in B. Moreover $X_\lambda := \chi_\lambda^G$, for all $\lambda \in \Lambda$, give the exceptional characters in B.

Following Corollary 12, there are three cases we must consider:

Case 1: b_0 is real, e is even and B has real non-exceptional characters. Then according to part (iii) of Theorem 1 all real irreducible characters in B have the same F-S indicators. We choose notation so that X_1 is real. As $X_1{\downarrow_T}$ is a real extension of χ_1 to T, it follows that $\epsilon(X_1) = \epsilon(X_1{\downarrow_T}) = \epsilon_{T/C_0}(\chi_1)$. This concludes Case 1.

Case 2: b_0 is real, e is even but B has no real non-exceptional characters. As χ_1 does not extend to a real character of E, it does not extend to a real character of T, according to Lemma 4. So $\epsilon_{T/C_0}(\chi_1) = -\epsilon(\chi_1)$, by the definition of the Gow indicator.

Now consider the notation used in the proof of part (i) of Theorem 1. Here $C_i = C_0$ and $\varphi_i = \varphi_0$ and $X'_{i,1} = \chi_1$, for $i = 0, \ldots, a-1$. If $\lambda \in \Lambda$ then $(X_\lambda){\downarrow_{C_0}} = \sum_{\tau \in G/C_0} \chi_{\lambda^\tau}$. So $d^{(x)}_{X_\lambda, \varphi_i} = \sum_{\tau \in G/C_0} \lambda({}^\tau x)$, for all $x \in D^\times$. This means that $\varepsilon_0 \gamma_i = 1$, for $i = 0, \ldots, a-1$. Now in (6), the term σ is 0, as none of $X_1, \ldots, X_e$ are real. So the first equation in (6) simplifies here to $-\epsilon(X_\lambda) = \epsilon(\chi_1)$, for all $\lambda \in \Lambda_1$. So $\epsilon(X_\lambda) = \epsilon_{T/C_0}(\chi_1)$, for all $\lambda \in \Lambda$, by the previous paragraph and Proposition 15.

Case 3: The final case is that b_0 is not real and e is odd. As B has an odd number e of non-exceptional characters, at least one of them must be real valued. So we assume that X_1 is real. Then, just as in Case 1, all real irreducible characters in B have the same F-S indicators.

As $|E : C_0|$ is odd and $|G : E| = 2$, we have $G/C_0 = E/C_0 \times T/C_0$. Now T/C_0 conjugates $\mathrm{Irr}(b_0)$ into $\mathrm{Irr}(b_0^o)$. So χ_1 is T-conjugate to $\overline{\chi}_1$. In particular $\chi_1{\uparrow^T}$ is irreducible and real valued. Now $X_1 = (\eta_1){\uparrow^G}$ and $(\eta_1){\downarrow_{C_0}} = \chi_1$. So $(X_1){\downarrow_T} = (\chi_1){\uparrow^T}$, by Mackey's theorem.

Now from above $\epsilon(X_\lambda) = \epsilon(X_1)$, for all $\lambda \in \Lambda$. Also $\epsilon(X_1) = \epsilon((X_1){\downarrow_T})$, as both are real valued. Finally $\epsilon((X_1){\downarrow_T}) = \epsilon_{T/C_0}(\chi_1)$, by the definition. This completes Case 3. $\square$

Finally, we prove the application to ordinary characters as stated in the Introduction:

Proof of Theorem 2. Let x be a weakly real p-element of G of maximal order and set $Q := \langle x \rangle$ and $N := \mathrm{N}_G(Q)$. Let λ be a faithful linear character of Q. Then $N_\lambda = \mathrm{C}_N(x)$ and $N_\lambda^* = \mathrm{C}_N^*(x)$. So N_λ^* does not split over N_λ. By Lemma 3 there exists $\chi \in \mathrm{Irr}(N \mid \lambda)$ such that $\epsilon(\chi) = -1$.

Let $\tilde{B}$ be the p-block of N which contains χ and let D be a defect group of $\tilde{B}$. Then $Q \subseteq D$ and $\mathrm{N}_G(D) \subseteq N$. In particular $B := \tilde{B}^G$ is defined and B has defect group D. So $Q = D_i$, $N = N_i$ and $\tilde{B} = B_i$ for some $i \geq 0$, in cyclic defect group notation.

Notice that λ is non-trivial. So $D \not\subseteq \ker(\chi)$. This means that χ is an exceptional character in B_i. So all exceptional characters in B_i, and hence also in B, are symplectic. The number of exceptional characters in B is $\frac{|D|-1}{e}$, where e is the inertial index of B. The number of weakly real p-conjugacy classes of G is equal to the number of N-orbits on $Q^\times$, which equals $\frac{|D_i|-1}{|N_i:C_i|}$. As $|D_i| \leq |D|$ and $e \leq |N_i : C_i|$, we conclude that the number of symplectic irreducible characters of G is not less than the number of weakly real p-conjugacy classes of G. □

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