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# The quadratic type of the 2-principal indecomposable modules of the double covers of alternating groups

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The principal indecomposable modules of the double cover $2.\mathcal{A}_n$ of the alternating group over a field of characteristic 2 are enumerated using the partitions of $n$ into distinct parts. We determine which of these modules afford a non-degenerate $2.\mathcal{A}_n$-invariant quadratic form. Our criterion depends on the alternating sum and the number of odd parts of the corresponding partition.



## 1. Introduction

Recall that an element of a finite group $G$ is said to be 2-regular if it has odd order and real if it is conjugate to its inverse. Moreover a real element is strongly real if it is

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inverted by an involution and otherwise it is said to be weakly real. If $k$ is a field, then a $kG$-module is said to have quadratic type if it affords a non-degenerate $G$-invariant $k$-valued quadratic form. The following is a recent result of R. Gow and the author [3]:

**Proposition 1.** *Suppose that $k$ is an algebraically closed field of characteristic 2. Then for any finite group $G$, the number of isomorphism classes of quadratic type principal indecomposable $kG$-modules is equal to the number of strongly real 2-regular conjugacy classes of $G$.*

Our focus here is on the double cover $2.\mathcal{A}_n$ of the alternating group $\mathcal{A}_n$. All real 2-regular elements of $\mathcal{A}_n$ are strongly real. So every self-dual principal indecomposable $k\mathcal{A}_n$-module has quadratic type. On the other hand, $2.\mathcal{A}_n$ may have real 2-regular elements which are not strongly real. In this note we determine which principal indecomposable $k(2.\mathcal{A}_n)$-modules have quadratic type.

Let $\mathcal{S}_n$ be the symmetric group of degree $n$ and let $\mathcal{D}(n)$ be the set of partitions of $n$ which have distinct parts. In [6, 11.5] G. James constructed an irreducible $k\mathcal{S}_n$-module $D^\mu$ for each partition $\mu \in \mathcal{D}(n)$. Moreover, he showed that the $D^\mu$ are pairwise non-isomorphic, and every irreducible $k\mathcal{S}_n$-modules is isomorphic to some $D^\mu$.

As $\mathcal{A}_n$ has index 2 in $\mathcal{S}_n$, Clifford theory shows that the restriction $D^\mu{\downarrow_{\mathcal{A}_n}}$ is either irreducible or splits into a direct sum of two non-isomorphic irreducible $k\mathcal{A}_n$-modules. Moreover, every irreducible $k\mathcal{A}_n$-module is a direct summand of some $D^\mu{\downarrow_{\mathcal{A}_n}}$.

D. Benson determined [1] which $D^\mu{\downarrow_{\mathcal{A}_n}}$ are reducible and we recently determined [8] when the irreducible direct summands of $D^\mu{\downarrow_{\mathcal{A}_n}}$ are self-dual (see below for details). Throughout this paper we use $D^\mu_A$ to denote an irreducible direct summand of $D^\mu{\downarrow_{\mathcal{A}_n}}$.

As the centre of $2.\mathcal{A}_n$ acts trivially on any irreducible module, $D^\mu_A$ can be considered as an irreducible $k(2.\mathcal{A}_n)$-module, and all irreducible $k(2.\mathcal{A}_n)$-modules arise in this way.

The alternating sum of a partition $\mu$ is $|\mu|_a := \sum(-1)^{j+1}\mu_j$. We use $\ell_o(\mu)$ to denote the number of odd parts in $\mu$. So $|\mu|_a \equiv \ell_o(\mu) \pmod 2$ and $|\mu|_a \geq \ell_o(\mu)$, if $\mu$ has distinct parts. Our result is:

**Theorem 2.** *Let $\mu$ be a partition of $n$ into distinct parts and let $P^\mu$ be the projective cover of the simple $k(2.\mathcal{A}_n)$-module $D^\mu_A$. Then $P^\mu$ has quadratic type if and only if*

$$\frac{n-|\mu|_a}{2} \leq 4m \leq \frac{n-\ell_o(\mu)}{2}, \quad \textit{for some integer } m.$$

Note that $P^\mu$ is a principal indecomposable $k(2.\mathcal{A}_n)$-module, but is not a $k\mathcal{A}_n$-module. Throughout the paper all our modules are left modules.

## 2. Notation

### 2.1. Principal indecomposable modules

This section consists of statements of well known facts. See [10, Sections 1.1, 1.10, 3.1, 3.6] for details and proofs.

The group algebra of a finite group $G$ over a field $k$ is a $k$-algebra $kG$ together with a distinguished $k$-basis whose elements are identified with the elements of $G$. So each element of $kG$ is unique expressible as $\sum_{g \in G} \lambda_g g$, where $\lambda_g \in k$ for all $g \in G$. The algebra multiplication in $kG$ is the $k$-linear extension of the group multiplication in $G$.

Multiplication on the left makes $kG$ into a module over itself, the so-called regular $kG$-module. The indecomposable direct summands of $kG$ are called the principal indecomposable $kG$-modules. Each such module has the form $kGe$, where $e$ is a primitive idempotent in $kG$.

Let $P$ be a principal indecomposable $kG$-module. The sum of all simple submodules of $P$ is a simple $kG$-module $S$. Moreover, $P/J(P) \cong S$, where $J(P)$ is the sum of all proper submodules of $P$. So $P$ is the projective cover of $S$. Moreover $P \leftrightarrow S$ establishes a one-to-one correspondence between the isomorphism classes of principal indecomposable $kG$-modules and the isomorphism classes of irreducible $kG$-modules.

Let $(K, R, k)$ be a $p$-modular system for $G$, where $p$ is prime. So $R$ is discrete valuation ring of characteristic 0, with unique maximal ideal $J$ containing $p$, and $R$ is complete with respect to the topology induced by the valuation. Also $K$ is the field of fractions of $R$, $k = R/J$ is the residue field of $R$ and $k$ has characteristic $p$. We assume that $K$ and $k$ are splitting fields for all subgroups of $G$.

In this context every principal indecomposable $kG$-module $P$ has a unique lift to a principal indecomposable $RG$-module $\hat{P}$ (this means that $\hat{P}$ is a finitely generated free $RG$-module, which is projective as $RG$-module, and the $kG$-module $\hat{P}/J\hat{P}$ is isomorphic to $P$).

A conjugacy class of $G$ is said to be $p$-regular if its elements have order coprime to $p$. The number of isomorphism classes of irreducible $kG$-modules equals the number of $p$-regular conjugacy classes of $G$. So the number of isomorphism classes of principal indecomposable $kG$-modules equals the number of $p$-regular conjugacy classes of $G$.

### 2.2. Symplectic and quadratic forms

A good reference for this section is [5, VII, 8]. A $kG$-module $M$ is said to be self-dual if it is isomorphic to its dual $M^* = \mathrm{Hom}_k(M, k)$. This occurs if and only if $M$ affords a non-degenerate $G$-invariant $k$-valued bilinear form. A self-dual $M$ has quadratic, orthogonal or symplectic type if it affords a non-degenerate $G$-invariant quadratic form, symmetric bilinear form or symplectic bilinear form, respectively.

If $p \neq 2$, R. Gow showed that an indecomposable $kG$-module is self-dual if and only if it has orthogonal or symplectic type, and these types are mutually exclusive. See [5, VII, 8.11]. W. Willems, and independently J. Thompson [12], showed that the type of a principal indecomposable module coincides with the type of its socle.

If $p = 2$, P. Fong noted that each non-trivial self-dual irreducible $kG$-module has symplectic type. This form is unique up to scalars, by Schur's Lemma. See [5, VII, 8.13]. However now it is possible that the projective cover has neither orthogonal nor symplectic type.

The correspondence $P \leftrightarrow S$ between principal indecomposable $kG$-modules and simple $kG$-modules respects duality. So $P$ is self-dual if and only if $S$ is self-dual. As the number of isomorphism classes of self-dual irreducible $kG$-modules equals the number of real $p$-regular conjugacy classes of $G$, it follows that the number of isomorphism classes of self-dual principal indecomposable $kG$-modules equals the number of real $p$-regular conjugacy classes of $G$.

Recall that $g \mapsto g^{-1}$, for $g \in G$, extends to a $k$-algebra anti-automorphism $x \mapsto x^o$ on $kG$ called the contragredient map.

**Proposition 3.** *Let $(K, R, k)$ be a 2-modular system for $G$ and let $\hat{P}$ be a principal indecomposable $RG$-module. Set $P = \hat{P}/J\hat{P}$ and $S = P/\operatorname{rad}(P)$, let $\Phi$ be the character of $\hat{P}$ and let $\varphi$ be the Brauer character of $S$. Then the following are equivalent:*

(i) *$\hat{P}$ has quadratic type.*
(ii) *$P$ has quadratic type.*
(iii) *$P$ has symplectic type.*
(iv) *There is an involution $t$ in $G$ and a primitive idempotent $e$ in $kG$ such that $P \cong kGe$ and $t^{-1}et = e^o$.*
(v) *If $B$ is a symplectic form on $S$, then $B(ts, s) \neq 0$, for some involution $t$ in $G$ and some $s$ in $S$.*
(vi) *$\varphi(g) \notin 2R$, for some strongly real 2-regular elements $g$ of $G$.*
(vii) *$\frac{\Phi(g)}{|C_G(g)|} \in 2R$, for all weakly real 2-regular elements $g$ of $G$.*

The equivalence of (i), (ii), (iii) and (iv) was proved in [4] and that of (ii), (vi) and (vii) in [3]. We only need the equivalence of (ii) and (v) to prove Theorem 2. This was first demonstrated in [7].

## 3. The double covers of alternating groups

### 3.1. Strongly real classes

The alternating group $\mathcal{A}_n$ is the subgroup of even permutations in the symmetric group $\mathcal{S}_n$. So $\mathcal{A}_5, \mathcal{A}_6, \dots$ is an infinite family of finite simple groups. For $n \geq 4$, $\mathcal{A}_n$ has a unique double cover $2.\mathcal{A}_n$. Then $2.\mathcal{A}_n$ is a subgroup of each double cover $2.\mathcal{S}_n$ of $\mathcal{S}_n$. Moreover $2.\mathcal{A}_n$ is a Schur covering group of $\mathcal{A}_n$, if $n = 5$ or $n \geq 8$. In this section we describe the conjugacy classes and characters of these groups. See [11] for an elegant exposition of this theory.

Given distinct $i_1, i_2, \dots, i_m \in \{1, \dots, n\}$, we use $(i_1, i_2, \dots, i_m)$ to denote an $m$-cycle in $\mathcal{S}_n$. So $(i_1, i_2, \dots, i_m)$ maps $i_j$ to $i_{j+1}$, for $j = 1, \dots, m-1$, sends $i_m$ to $i_1$ and fixes all $i \neq i_1, \dots, i_m$. Now each permutation $\sigma \in \mathcal{S}_n$ has a unique factorization as a product of disjoint cycles. If we arrange the lengths of these cycles in a non-increasing sequence, we get a partition of $n$, which is called the cycle type of $\sigma$. The set of permutations with a fixed cycle type $\lambda$ is a conjugacy class of $\mathcal{S}_n$, here denoted $C_\lambda$. In particular the 2-regular conjugacy classes of $\mathcal{S}_n$ are indexed by the set $\mathcal{O}(n)$ of partitions of $n$ whose parts are odd.

A transposition in $\mathcal{S}_n$ is a 2-cycle $(i, j)$ where $i, j$ are distinct elements of $\{1, \dots, n\}$. So $(i, j)$ has cycle type $(2, 1^{n-2})$. It is clear that there is one conjugacy class of involutions for each partition $(2^m, 1^{n-2m})$ of $n$, with $1 \leq m \leq n/2$. We call a product of $m$-disjoint transpositions an $m$-involution in $\mathcal{S}_n$. It follows that $\mathcal{S}_n$ has $\lfloor \frac{n}{2} \rfloor$ conjugacy classes of involutions; the $m$-involutions, for $1 \leq m \leq n/2$.

Suppose that $\pi = (i_1, i_{1+m})(i_2 i_{2+m}) \dots (i_m, i_{2m})$ is an $m$-involution in $\mathcal{S}_n$. Then we say that $(i_1, i_{1+m}), (i_2 i_{2+m}), \dots, (i_m, i_{2m})$ are the transpositions in $\pi$ and write $(i_j, i_{j+m}) \in \pi$, for $j = 1, \dots, m$. Notice that each $\{i_j, i_{j+m}\}$ is a non-singleton orbit of $\pi$ on $\{1, \dots, n\}$.

Let $\lambda$ be a partition of $n$. We use $\ell(\lambda)$ to denote the number of parts in $\lambda$, and we say that $\lambda$ is even if $n \equiv \ell(\lambda) \mod 2$. Then $C_\lambda \subseteq \mathcal{A}_n$ if and only if $\lambda$ is even, and if $\lambda$ is even, then $C_\lambda$ is a union of two conjugacy classes of $\mathcal{A}_n$ if $\lambda$ has distinct odd parts and otherwise $C_\lambda$ is a single conjugacy class of $\mathcal{A}_n$. In either case we use $C_{\lambda,A}$ to denote an $\mathcal{A}_n$-conjugacy class contained in $C_\lambda$. If $\lambda$ has distinct odd parts then $C_{\lambda,A}$ is a real conjugacy class of $\mathcal{A}_n$ if and only if $n \equiv \ell(\lambda) \mod 4$.

Next let $z \in 2.\mathcal{A}_n$ be the involution which generates the centre of $2.\mathcal{A}_n$. As $\langle z \rangle$ is a central 2-subgroup of $2.\mathcal{A}_n$, there is a one-to-one correspondence between the 2-regular conjugacy classes of $2.\mathcal{A}_n$ and the 2-regular conjugacy classes of $\mathcal{A}_n \cong (2.\mathcal{A}_n)/\langle z \rangle$; if $\lambda$ is an odd partition of $n$ the preimage of $C_{\lambda,A}$ in $2.\mathcal{A}_n$ consists of a single class $\hat{C}_{\lambda,A}$ of odd order elements and another class $z\hat{C}_{\lambda,A}$ of elements whose 2-parts equal $z$.

Notice that an $m$-involution belongs to $\mathcal{A}_n$ if and only if $m$ is even. Moreover, the $2m$-involutions form a single conjugacy class of $\mathcal{A}_n$. So $\mathcal{A}_n$ has $\lfloor \frac{n}{4} \rfloor$ conjugacy classes of involutions; the $2m$-involutions, for $1 \leq m \leq n/4$. Now each $2m$-involution in $\mathcal{A}_n$ is the image of two involutions in $2.\mathcal{A}_n$, if $m$ is even, or is the image of two elements of order 4 in $2.\mathcal{A}_n$, if $m$ is odd.

Set $m_o(\lambda)$ as the number of parts which occur with odd multiplicity in $\lambda$.

**Lemma 4.** *If $\lambda$ is a partition of $n$ with all parts odd then $\hat{C}_{\lambda,A}$ is a strongly real conjugacy class of $2.\mathcal{A}_n$ if and only if there is an integer $m$ such that $\frac{n-\ell(\lambda)}{2} \leq 4m \leq \frac{n-m_o(\lambda)}{2}$.*

**Proof.** Let $\sigma \in \mathcal{A}_n$ have cycle type $\lambda$ and let $\pi$ be an $m$-involution in $\mathcal{S}_n$ which inverts $\sigma$. Set $\ell := \ell(\lambda)$, and let $X_1, \ldots, X_\ell$ be the orbits of $\sigma$ on $\{1, \ldots, n\}$. Then $\pi$ permutes the sets $X_1, \ldots, X_\ell$.

If $\pi X_j = X_j$, for some $j$, then $\pi$ fixes a unique element of $X_j$, and hence acts as an $\frac{|X_j|-1}{2}$-involution on $X_j$. If instead $\pi X_j \neq X_j$, then $\pi$ is a bijection $X_j \to \pi X_j$. So $\pi$ acts as an $|X_j|$-involution on $X_j \cup \pi X_j$. We may order the $X_j$ and choose $k \geq 0$ such that $\pi X_j = X_{j+k}$, for $j = 1, \ldots, k$, and $\pi X_j = X_j$, for $j = 2k+1, 2k+2, \ldots, \ell$. Then from above

$$m = \sum_{j=1}^{k} \frac{|X_j| + |X_{j+k}|}{2} + \sum_{j=2k+1}^{\ell} \frac{|X_j| - 1}{2} = \frac{n + 2k - \ell}{2}.$$

Now the maximum value of $2k$ is $2k = \ell - m_o(\lambda)$, when $\pi$ pairs the maximum number of orbit of $\sigma$ which have equal size. This implies that $m \leq \frac{n-m_o(\lambda)}{2}$. The minimum value of $2k$ is 0. This occurs when $\pi$ fixes each orbit of $\sigma$. It follows from this that $m \geq \frac{n-\ell(\lambda)}{2}$.

Conversely, it is clear that for each $m > 0$ with $\frac{n-\ell}{2} \leq m \leq \frac{n-m_o(\lambda)}{2}$, there is an $m$-involution $\pi \in \mathcal{S}_n$ which inverts $\sigma$; $\pi$ pairs $\ell+2m-n$ orbits of $\sigma$ and fixes the remaining $n-2m$ orbits of $\sigma$. The conclusion of the Lemma now follows from our description of the involutions in $2.\mathcal{A}_n$. □

### *3.2. Irreducible modules*

By an $n$-tabloid we mean an indexed collection $R = (R_1, \ldots R_\ell)$ of non-empty subsets of $\{1, \ldots, n\}$ which are pairwise disjoint and whose union is $\{1, \ldots, n\}$ (also known as an ordered partition of $\{1, \ldots, n\}$). We shall refer to $R_1, \ldots, R_\ell$ as the rows of $R$. Set $\lambda_i := |R_i|$. Then we may choose indexing so that $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition of $n$, which we call the type of $R$. Now $\mathcal{S}_n$ acts on all $\lambda$-tabloids; the corresponding permutation module (over $\mathbb{Z}$) is denoted $M^\lambda$.

Next recall that the Young diagram of $\lambda$ is a collection of boxes in the plane, oriented in the Anglo-American tradition: the first row consists of $\lambda_1$ boxes. Then for $i = 2, \ldots, \ell$ in turn, the $i$-th row consists of $\lambda_i$ boxes placed directly below the $(i-1)$-th row, with the leftmost box in row $i$ directly below the leftmost box in row $i-1$.

By a $\lambda$-tableau we shall mean a bijection $t : [\lambda] \to \{1, \ldots, n\}$, or a filling of the boxes in the Young diagram with the symbols $1, \ldots, n$. So for $1 \leq r \leq \ell$ and $1 \leq c \leq \lambda_r$, we use $t(r, c)$ to denote the image of the position $(r, c) \in [\lambda]$ in $\{1, \ldots, n\}$. Conversely, given $i \in \{1, \ldots, n\}$, there is a unique $r = r_t(i)$ and $c = c_t(i)$ such that $t(r, c) = i$. We say that $i$ is in row $r$ and column $c$ of $t$.

Clearly there are $n!$ tableaux of type $\lambda$ and $\mathcal{S}_n$ acts regularly on the set of $\lambda$-tableau. For $\sigma \in \mathcal{S}_n$, we define $\sigma t : [\lambda] \to \{1, \ldots, n\}$ as the composition $(\sigma t)(r, c) = \sigma(t(r, c))$, for all $(r, c) \in [\lambda]$. In other words, the permutation module of $\mathcal{S}_n$ acting on tableau is (non-canonically) isomorphic to the regular module $\mathbb{Z}\mathcal{S}_n$; once we fix $t$, we may identify $\sigma \in \mathcal{S}_n$ with the tableau $\sigma t$.

Associated with $t$, we have two important subgroups of $\mathcal{S}_n$. The column stabilizer of $t$ is $C_t := \{\sigma \in \mathcal{S}_n \mid c_t(i) = c_t(\sigma i)$, for $i = 1, \ldots, n\}$ and the row stabilizer of $t$ is $R_t := \{\sigma \in \mathcal{S}_n \mid r_t(i) = r_t(\sigma i)$, for $i = 1, \ldots, n$.

We use $\{t\}$ to denote the tabloid formed by the rows of $t$. So $\{t\}_r := \{t(r, c) \mid 1 \leq c \leq \lambda_r\}$, for $r = 1, \ldots, \ell$. Also $\{s\} = \{t\}$ if and only if $s = \sigma t$, for some $\sigma \in R_t$. Notice that the actions of $\mathcal{S}_n$ on tableau and tabloids are compatible, in the sense that $\sigma\{t\} = \{\sigma t\}$. In other words, the map $t \mapsto \{t\}$ induces a surjective $\mathcal{S}_n$-homomorphism $\mathbb{Z}\mathcal{S}_n \to M^\lambda$. The kernel of this homomorphism is the $\mathbb{Z}$-span of $\{\sigma t \mid \sigma \in R_t\}$.

The polytabloid $e_t$ associated with $t$ is the following element of $M^\lambda$:

$$e_t := \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma)\{\sigma t\}.$$

We use $\operatorname{supp}(t) := \{\{\sigma t\} \mid \sigma \in C_t\}$ to denote the set of tabloids which occur in the definition of $e_t$. Note that $e_{\pi t} = \operatorname{sgn}(\pi) e_t$, for all $\pi \in C_t$. In particular, if $r_t(i) = r_t(j)$, then $e_{(i,j)t} = -e_t$. Also if $\pi \in \mathcal{S}_n$, then $C_{\pi t} = \pi C_t \pi^{-1}$ and $R_{\pi t} = \pi R_t \pi^{-1}$. So $e_{\pi t} = \pi e_t$ and $\operatorname{supp}(\pi t) = \pi \operatorname{supp}(t)$.

The $\mathbb{Z}$-span of all $\lambda$-polytabloids is a $\mathcal{S}_n$-submodule of $M^\lambda$ called the Specht module. It is denoted by $S^\lambda$. So $S^\lambda$ is a finitely generated free $\mathbb{Z}$-module ($\mathbb{Z}$-lattice).

### *3.3. Involutions and bilinear forms*

Let $\mu$ be a partition of $n$ which has distinct parts and let $\langle\ ,\ \rangle$ be the symmetric bilinear form on $M^\mu$ with respect to which the $\mu$-tabloids form an orthonormal basis. Now let $k$ be a field of characteristic 2. Then according to James, $D^\mu := S^\mu / S^\mu \cap (S^\mu)^\perp$ is a non-zero irreducible $k\mathcal{S}_n$-module. Here $(S^\mu)^\perp := \{m \in M^\mu \mid \langle m, s\rangle \in 2\mathbb{Z}, \forall s \in S^\mu\}$.

Suppose that $\mu$ has parts $\mu_1 > \cdots > \mu_{2s-1} > \mu_{2s} \geq 0$. Benson's classification of the irreducible $k\mathcal{A}_n$-modules [1], and our classification of the self-dual irreducible $k\mathcal{A}_n$-modules [8], are given by:

**Lemma 5.** *$D^\mu \downarrow_{\mathcal{A}_n}$ is reducible if and only if for each $j > 0$*

$$\textit{(i)}\ \ \mu_{2j-1} - \mu_{2j} = 1 \textit{ or } 2 \qquad \textit{and} \qquad \textit{(ii)}\ \ \mu_{2j-1} + \mu_{2j} \not\equiv 2 \pmod 4.$$

*If $D^\mu \downarrow_{\mathcal{A}_n}$ is reducible, its irreducible direct summands are self-dual if and only if $\sum_{j>0} \mu_{2j}$ is even.*

Let $\overline{n}$ denote the residue of an integer $n$ mod 2. Then

**Lemma 6.** *Let $\phi : S^\mu \to D^\mu$ be the $\mathbb{Z}\mathcal{S}_n$-projection. Then $B(\phi x, \phi y) := \overline{\langle x, y\rangle}$, for $x, y \in S^\mu$, defines a non-zero symplectic bilinear form on $D^\mu$, if $\mu \neq (n)$.*

**Remark 7.** Notice that if $x, y \in D^\mu$ and $\pi$ is an involution in $\mathcal{S}_n$ then

$$B(\pi(x+y), x+y) = B(\pi x, x) + B(\pi y, y).$$

So we can focus on a single polytabloid in $S^\lambda$.

**Lemma 8.** *If $t$ is a $\mu$-tableau and $\pi$ is an involution in $\mathcal{S}_n$, then*

$$\langle \pi e_t, e_t\rangle \equiv |\{T \in \operatorname{supp}(\pi t) \cap \operatorname{supp}(t) \mid \pi T = T\}| \mod 2.$$

*In particular, if $\langle \pi e_t, e_t\rangle$ is odd, then $\pi \in R_{\sigma t}$, for some $\sigma \in C_t$.*

**Proof.** We have

$$\begin{aligned}\langle \pi e_t, e_t\rangle &= \sum_{\sigma_1,\sigma_2\in C_t} \operatorname{sgn}(\pi\sigma_1\pi^{-1})\operatorname{sgn}(\sigma_2)\langle(\pi\sigma_1\{t\}, \sigma_2\{t\}\rangle \\ &\equiv |\{(\sigma_1,\sigma_2) \in C_t \times C_t \mid \pi\sigma_1\{t\} = \sigma_2\{t\}\}| \pmod 2 \\ &= |\operatorname{supp}(\pi t) \cap \operatorname{supp}(t)|.\end{aligned}$$

Now notice that $T \mapsto \pi T$ is an involution on $\operatorname{supp}(\pi t)\cap\operatorname{supp}(t)$. So $|\operatorname{supp}(\pi t)\cap\operatorname{supp}(t)| \equiv |\{T \in \operatorname{supp}(\pi t) \cap \operatorname{supp}(t) \mid \pi T = T\}| \mod 2$.

Suppose that $\langle \pi e_t, e_t\rangle$ is odd. Then by the above, there exists $\sigma \in C_t$ such that $\pi\{\sigma t\} = \{\sigma t\}$. This means that $\pi \in R_{\sigma t}$. □

**Lemma 9.** *Let $t$ be a $\mu$-tableau and let $m$ be a positive integer such that $\langle \pi e_t, e_t\rangle$ is odd, for some $m$-involution $\pi \in \mathcal{S}_n$. Then $m \leq \frac{n-\ell_o(\mu)}{2}$ and $\pi$ fixes at most one entry in each column of $t$.*

**Proof.** By the previous Lemma, we may assume that $\pi \in R_t$. Now $R_t \cong \mathcal{S}_\mu$. For $i > 0$, there is $j$-involution in $\mathcal{S}_i$ for $j = 1, \dots \lfloor \frac{i}{2} \rfloor$. So there is an $m$-involution in $R_t$ if and only if

$$m \leq \sum \left\lfloor \frac{\mu_i}{2} \right\rfloor = \sum_{\mu_i \text{ even}} \frac{\mu_i}{2} + \sum_{\mu_i \text{ odd}} \frac{\mu_i - 1}{2} = \frac{n - \ell_o(\mu)}{2}$$

Let $i, j$ belong to a single column of $t$. We claim that $i, j$ belong to different columns of $\pi t$. For suppose otherwise. Then $(i,j) \in C_t \cap C_{\pi t}$. So the map $T \mapsto (i,j)T$ is an involution on $\operatorname{supp}(\pi t) \cap \operatorname{supp}(t)$ which has no fixed-points. In particular $|\operatorname{supp}(\pi t) \cap \operatorname{supp}(t)|$ is even, contrary to hypothesis. This proves the last assertion. □

We can now prove a key technical result:

**Lemma 10.** *Let $t$ be a $\mu$-tableau and let $m$ be a positive integer such that $\langle \pi e_t, e_t\rangle$ is odd, for some $m$-involution $\pi \in \mathcal{S}_n$. Then $m \geq \frac{n-|\mu|_a}{2}$.*

**Proof.** Let $T \in \operatorname{supp}(\pi t) \cap \operatorname{supp}(t)$ such that $\pi T = T$. Write $\pi_j$ for the restriction of $\pi$ to the rows $T_{2j-1}$ and $T_{2j}$ of $T$, for each $j > 0$. Then there is $m_j \geq 0$ such that $\pi_j$ is an $m_j$-involution, for each $j > 0$. So $m = \sum m_j$ and $\pi = \pi_1 \pi_2 \dots \pi_{\lfloor \frac{\ell(\mu)+1}{2} \rfloor}$.

We assume for the sake of contradiction that $m < \frac{n-|\mu|_a}{2}$. Now $\frac{n-|\mu|_a}{2} = \sum_{j>0} \mu_{2j}$. So $m_j < \mu_{2j}$ for some $j > 0$, and we choose $j$ to be the smallest such positive integer.

There is a unique $\sigma \in C_t$ such that $T = \{\sigma t\}$. Set $s = \sigma t$. So $\pi \in R_s$. We define the graph $\operatorname{Gr}_\pi(s)$ of $\pi$ on $s$ as follows. The vertices of $\operatorname{Gr}_\pi(s)$ are labels $1, \dots, \mu_{2j-1}$ of the columns which meet row $\mu_{2j-1}$ of $s$. There is an edge $c_1 \longleftrightarrow c_2$ if and only if one of the two transpositions $(s(2j-1, c_1), s(2j-1, c_2))$ or $(s(2j, c_1), s(2j, c_2))$ belongs to $\pi_j$. As there are at most two entries in each column of $s$ which are moved by $\pi_j$, it follows that each connected component of $\operatorname{Gr}_\pi(s)$ is either a line segment or a simple closed curve.

We claim that $\operatorname{Gr}_\pi(s)$ has a component with a vertex set contained in $\{1, \dots, \mu_{2j}\}$. For otherwise every component $\Gamma$ of $\operatorname{Gr}_\pi(s)$ is a line segment and $|\operatorname{Edge}(\Gamma)| \geq |\operatorname{Vx}(\Gamma) \cap \{1, \dots, \mu_{2j}\}|$. Summing over all $\Gamma$ we get the contradiction

$$\mu_{2j} = \sum_{\Gamma} |\operatorname{Vx}(\Gamma) \cap \{1, \dots, \mu_{2j}\}| \leq \sum_{\Gamma} |\operatorname{Edge}(\Gamma)| = m_j.$$

Now let $X$ be the union of the component of $\operatorname{Gr}_\pi(s)$ which are contained in $\{1, \dots, \mu_{2j}\}$ and let $\Gamma$ be the component of $\operatorname{Gr}_\pi(s)$ which contains $\min(X)$. In particular $\operatorname{Vx}(\Gamma) \subseteq \{1, \dots, \mu_{2j}\}$.

Consider the involution $\sigma_\Gamma := \prod_{c \in \operatorname{Vx}(\Gamma)} (t(2j-1, c), t(2j, c))$. This transposes the entries between rows $2j-1$ and $2j$ in each column in $\operatorname{Vx}(\Gamma)$. Now it is clear that $\pi$ is in the row stabilizer of $\sigma_\Gamma s$. So $\{\sigma_\Gamma s\} \in \operatorname{supp}(\pi t) \cap \operatorname{supp}(t)$. Moreover, $\operatorname{Gr}_\pi(s) = \operatorname{Gr}_\pi(\sigma_\Gamma s)$ and $s = \sigma_\Gamma(\sigma_\Gamma s)$. It follows that the pair $T \neq \sigma_\Gamma T$ of tabloids makes zero contribution to $\langle \pi e_t, e_t \rangle$ modulo 2. But $T$ is an arbitrary $\pi$-fixed tabloid in $\operatorname{supp}(\pi t) \cap \operatorname{supp}(t)$. So $\langle \pi e_t, e_t \rangle$ is even, according to Lemma 8. This contradiction completes the proof. □

### *3.4. Proof of Theorem 2*

Suppose first that $P^\mu$ has quadratic type. Then by (ii)$\Longleftrightarrow$(v) in Proposition 3, $B(\hat{\pi} x, x\rangle \neq 0$, for some $x \in D^\mu_A$ and involution $\hat{\pi} \in 2.\mathcal{A}_n$. Let $\pi$ be the image of $\hat{\pi}$ in $\mathcal{A}_n$. Then Remark 7 implies that there is a $\mu$-tableau $t$ such that $\langle \pi e_t, e_t \rangle$ is odd. Now $\pi$ is a $4m$-involution, for some $m > 0$, and Lemmas 9 and 10 imply that $\frac{n-|\mu|_a}{2} \leq 4m \leq \frac{n-\ell_o(\mu)}{2}$. This proves the 'only if' part of the Theorem.

According to Lemma 4, the strongly real 2-regular conjugacy classes of $2.\mathcal{A}_n$ are enumerated by $\lambda \in \mathcal{O}(n)$ such that there is a positive integer $m$ with $\frac{n-\ell(\lambda)}{2} \leq 4m \leq \frac{n-m_o(\lambda)}{2}$ (if $\lambda$ has distinct parts, $\frac{n-\ell(\lambda)}{2} = \frac{n-m_o(\lambda)}{2}$ and there are two 2-regular classes of $2.\mathcal{A}_n$ labelled by $\lambda$, in all other cases there is a single 2-regular class of $2.\mathcal{A}_n$ labelled by $\lambda$).

By Theorem 2.1 in [2] (or the main result in [9]) there is a bijection $\phi : \mathcal{O}(n) \to \mathcal{D}(n)$ such that $\ell(\lambda) = |\phi(\lambda)|_a$ and $m_o(\lambda) = \ell_o(\phi(\lambda))$, for all $\lambda \in \mathcal{O}(n)$. Then from the previous paragraph the number of strongly real 2-regular conjugacy classes of $2.\mathcal{A}_n$ coincides with the number of irreducible $k(2.\mathcal{A}_n)$-modules enumerated by $\mu \in \mathcal{D}(n)$ such that $\frac{n-|\mu|_a}{2} \leq 4m \leq \frac{n-\ell_o(\mu)}{2}$ for some integer $m$. However, from earlier in the proof, these are the only $P^\mu$ which can be of quadratic type. We conclude from Proposition 1 that each of these $P^\mu$ is of quadratic type, and furthermore that no other $P^\mu$ is of quadratic type. □

*3.5. Example with* $2.\mathcal{A}_{13}$

The 18 distinct partitions of 13 give rise to 21 principal indecomposable $k(2.\mathcal{A}_{13})$-modules. The types are:

| $\mu$ | $\frac{n-\|\mu\|_a}{2}$ | $\frac{n-\ell_o(\mu)}{2}$ | type |
|---|---|---|---|
| $(7,6)$ | 6 | 6 | 2 non-quadratic |
| $(8,5)$ | 5 | 6 | non-quadratic |
| $(6,5,2)$ | 5 | 6 | non-quadratic |
| $(6,4,2,1)$ | 5 | 6 | non-quadratic |
| $(5,4,3,1)$ | 5 | 5 | 2 not self-dual |
| $(7,5,1)$ | 5 | 5 | 2 not self-dual |
| $(9,4)$ | 4 | 6 | quadratic |
| $(7,4,2)$ | 4 | 6 | quadratic |
| $(6,4,3)$ | 4 | 6 | quadratic |
| $(8,4,1)$ | 4 | 6 | quadratic |
| $(7,3,2,1)$ | 4 | 5 | quadratic |
| $(10,3)$ | 3 | 6 | quadratic |
| $(8,3,2)$ | 3 | 6 | quadratic |
| $(9,3,1)$ | 3 | 5 | quadratic |
| $(11,2)$ | 2 | 6 | quadratic |
| $(10,2,1)$ | 2 | 6 | quadratic |
| $(12,1)$ | 1 | 6 | quadratic |
| $(13)$ | 0 | 6 | quadratic |

Using (i) and (ii) in Lemma 5, we see that $D^\mu{\downarrow}_{\mathcal{A}_{13}}$ is a sum of two non-isomorphic irreducible $k(2.\mathcal{A}_{13})$-modules for $\mu = (7,6), (5,4,3,1)$ or $(7,5,1)$. For all other $\mu$, $D^\mu{\downarrow}_{\mathcal{A}_{13}}$ is irreducible. So there are $21 = 18 + 3$ projective indecomposable $k(2.\mathcal{A}_{13})$-modules.

By the last statement in Lemma 5, the two irreducible $k(2.\mathcal{A}_{13})$-modules $D_A^{(5,4,3,1)}$ are duals of each other, as are the two irreducible $k(2.\mathcal{A}_{13})$-modules $D_A^{(7,5,1)}$. By the same result both irreducible $k(2.\mathcal{A}_{13})$-modules $D_A^{(7,6)}$ are self-dual. However $6 \equiv 2 \pmod 4$. So neither principal indecomposable $k(2.\mathcal{A}_{13})$-module $P^{(7,6)}$ is of quadratic type.

Next if $\mu = (8,5), (6,5,2)$ or $(6,4,2,1)$ we have $\frac{n-|\mu|_a}{2} = 5$ and $\frac{n-\ell_o(\mu)}{2} = 6$. So the principal indecomposable $k(2.\mathcal{A}_{13})$-module $P^\mu$ is not of quadratic type for any of these $\mu$'s. For each of the remaining partitions $\mu$, the principal indecomposable $k(2.\mathcal{A}_{13})$-module $P^\mu$ is of quadratic type, according to Theorem 2.

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