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Hausdorff dimension and a generalized form of simultaneous Diophantine approximation

by

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1. Introduction. Suppose that m and n are positive integers, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_m) \in \mathbb{R}_+^m$ is a vector of strictly positive numbers, and $Q \subset \mathbb{Z}^n$ is an infinite set of integer vectors. Let X denote a general point in $\mathbb{R}^{mn}$, which we will write in the form $X = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$, with $\mathbf{x}_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, and define the set

$$W_Q(m,n;\boldsymbol{\tau}) = \{X \in \mathbb{R}^{mn} : \|\mathbf{x}_i \cdot \mathbf{q}\| < |\mathbf{q}|^{-\tau_i},\ 1 \le i \le m,\ \text{for infinitely many } \mathbf{q} \in Q\}$$

(where, for any $z \in \mathbb{R}$, $\|z\|$ denotes the distance from z to the nearest integer). In the special case $\boldsymbol{\tau} = \boldsymbol{\tau}(\tau) = (\tau, \ldots, \tau)$, for $\tau > 0$, and $Q = \mathbb{Z}^n$, the set $W_{\mathbb{Z}^n}(m,n;\boldsymbol{\tau}(\tau))$ has been studied by many authors; in particular, its Hausdorff dimension has been obtained. Jarník [8] and Besicovitch [1] showed that if $\tau > 1$, then $\dim W_{\mathbb{Z}}(1,1;\boldsymbol{\tau}(\tau)) = 2/(1+\tau)$ (dim denotes Hausdorff dimension). Later Jarník [9] and Eggleston [7] showed that if $\tau > 1/m$, then $\dim W_{\mathbb{Z}}(m,1;\boldsymbol{\tau}(\tau)) = (m+1)/(1+\tau)$. Furthermore, Eggleston obtained the dimension of $W_Q(m,1;\boldsymbol{\tau}(\tau))$ for certain infinite sets $Q \subset \mathbb{Z}$ and Bovey and Dodson [3] obtained the dimension of $W_Q(m,n;\boldsymbol{\tau}(\tau))$ for certain $Q \in \mathbb{Z}^n$. These results were extended to arbitrary infinite sets $Q \subset \mathbb{Z}$ by Borosh and Fraenkel [2] and to arbitrary $Q \subset \mathbb{Z}^n$ by Rynne [10].

To state their results we need the following definition. Suppose that $Q \subset \mathbb{Z}^n$ is an arbitrary infinite set and let

$$\nu(Q) = \inf\Big\{\nu \in \mathbb{R} : \sum_{\mathbf{q} \in Q} |\mathbf{q}|^{-\nu} < \infty\Big\}.$$

Clearly, $0 \le \nu(Q) \le n$. It is shown in [10] that if $\tau \ge \nu(Q)/m$, then

$$\dim W_Q(m,n;\boldsymbol{\tau}(\tau)) = m(n-1) + \frac{m+\nu(Q)}{1+\tau}.$$

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This result was extended in [11] to the set $W_Q(m,1;\boldsymbol{\tau})$ for general $\boldsymbol{\tau}$. Such an extension also exists for $m=2$ and $n=1$ for the simultaneous approximation of real numbers by algebraic numbers of bounded degree [6]. In the present paper we will obtain the Hausdorff dimension of $W_Q(m,n;\boldsymbol{\tau})$ for general n.

Without loss of generality we will suppose throughout that $\tau_1 \geq \ldots \geq \tau_m$. Let $\sigma(\boldsymbol{\tau}) = \sum_{i=1}^m \tau_i$, and define the number

$$D_Q(m,n;\boldsymbol{\tau}) = m(n-1) + \min_{1\le k\le m}\left\{\frac{m+\nu(Q)+\sum_{i=k}^m(\tau_k-\tau_i)}{1+\tau_k}\right\}.$$

Theorem 1.1. *If* $\sigma(\boldsymbol{\tau}) \geq \nu(Q)$, *then*

$$\dim W_Q(m,n;\boldsymbol{\tau}) = D_Q(m,n;\boldsymbol{\tau}).$$

If $\sigma(\boldsymbol{\tau}) \leq \nu(Q)$, *then* $\dim W_Q(m,n;\boldsymbol{\tau}) = mn$.

Remark 1.2. It will be shown at the end of the proof of Theorem 1.1 that if $\sigma(\boldsymbol{\tau}) = \nu(Q)$ then $D_Q(m,n;\boldsymbol{\tau}) = mn$ so the results in the two cases in the theorem are consistent.

The above problem can be generalized in the manner considered in [4]. Let $\boldsymbol{\psi} = (\psi_1,\ldots,\psi_m)$ be a collection of non-negative functions on $\mathbb{Z}^n$ (the functions ψ_i need only be defined on Q, but for simplicity we ignore this). Now define the set

$$W_Q(m,n;\boldsymbol{\psi}) = \{X\in\mathbb{R}^{mn} : \|\mathbf{x}_i\cdot\mathbf{q}\| < \psi_i(\mathbf{q}),\ 1\le i\le m,\ \text{for infinitely many } \mathbf{q}\in Q\}.$$

Under a further assumption on the limiting behaviour of the functions ψ_i we can obtain the dimension of $W_Q(m,n;\boldsymbol{\psi})$. Suppose that the limits

$$\lambda(\psi_i) = \lim_{|\mathbf{q}|\to\infty}\frac{-\log\psi_i(\mathbf{q})}{\log|\mathbf{q}|},\qquad i=1,\ldots,m,$$

exist and are positive, and put $\boldsymbol{\tau}(\boldsymbol{\psi}) := (\lambda(\psi_1),\ldots,\lambda(\psi_m))$. Then from Theorem 1.1 we obtain the following result.

Corollary 1.3. *If* $\sigma(\boldsymbol{\tau}(\boldsymbol{\psi})) \geq \nu(Q)$, *then*

$$\dim W_Q(m,n;\boldsymbol{\psi}) = D_Q(m,n;\boldsymbol{\tau}(\boldsymbol{\psi})).$$

If $\sigma(\boldsymbol{\tau}(\boldsymbol{\psi})) \leq \nu(Q)$, *then* $\dim W_Q(m,n;\boldsymbol{\psi}) = mn$.

Proof. From the hypotheses on the functions ψ_i we have, for any $\varepsilon>0$ and each $i=1,\ldots,m$,

$$|\mathbf{q}|^{-\lambda(\psi_i)-\varepsilon} \le \psi_i(\mathbf{q}) \le |\mathbf{q}|^{-\lambda(\psi_i)+\varepsilon},$$

for all sufficiently large $|\mathbf{q}|\in Q$. Thus, letting $\boldsymbol{\varepsilon} = (\varepsilon,\ldots,\varepsilon)$, it follows that

$$W_Q(m,n;\boldsymbol{\tau}(\boldsymbol{\psi})+\boldsymbol{\varepsilon}) \subset W_Q(m,n;\boldsymbol{\psi}) \subset W_Q(m,n;\boldsymbol{\tau}(\boldsymbol{\psi})-\boldsymbol{\varepsilon}).$$

Now, letting $\varepsilon \to 0$, the result follows from these inclusions and the continuity with respect to $\boldsymbol{\tau}$ of the dimension result in Theorem 1.1 (see Remark 1.2). ∎

2. Proof of Theorem 1.1. To fix our notation we first recall the (standard) definition of the Hausdorff dimension of an arbitrary set $E \subset \mathbb{R}^r$, for any positive integer r. Let $\mathcal{I}$ be a countable collection of bounded sets $I \subset \mathbb{R}^r$. For any $\varrho > 0$, the ϱ-volume of the collection $\mathcal{I}$ is defined to be

$$V_\varrho(\mathcal{I}) = \sum_{I\in\mathcal{I}} d(I)^\varrho,$$

where $d(I) = \sup\{|\mathbf{x} - \mathbf{y}|_2 : \mathbf{x}, \mathbf{y} \in I\}$ is the diameter of I and $|\cdot|_2$ denotes the usual Euclidean norm in $\mathbb{R}^r$. For every $\eta > 0$ define

$$m_\varrho(\eta, E) = \inf V_\varrho(\mathcal{I}),$$

where the infimum is taken over all countable collections, $\mathcal{I}$, of sets I with diameter $d(I) \le \eta$, that cover E. Now define the *ϱ-dimensional Hausdorff outer measure* of E to be

$$m_\varrho(E) = \sup_{\eta>0} m_\varrho(\eta, E).$$

The *Hausdorff dimension* of E is defined to be

$$\dim E = \inf\{\varrho : m_\varrho(E) = 0\}.$$

We also require some further notation. For any finite set A we let $|A|$ denote the cardinality of A. The notation $a \ll b$ (respectively $a \gg b$) will denote an inequality of the form $a \le cb$ (respectively $a \ge cb$), where $c > 0$ is a constant which depends at most on m, n, $\nu(Q)$, $\boldsymbol{\tau}$ and δ (which will be introduced below); similarly, $c_1, c_2, \ldots$ will denote positive constants which depend at most on m, n, $\nu(Q)$, $\boldsymbol{\tau}$ and δ. If $a \ll b \ll a$ then we write $a \approx b$. A set of the form $B = \{\mathbf{x} \in \mathbb{R}^r : |\mathbf{x} - \mathbf{b}|_2 \le d/2\}$, for any $r \ge 1$, is said to be a *ball* of diameter d and centre $\mathbf{b}$. If $\alpha > 0$ is a real number then αB will denote the ball with centre $\mathbf{b}$ and diameter αd. Let U_n denote the unit cube

$$U_n = \{\mathbf{x} \in \mathbb{R}^n : 0 \le x_i \le 1,\ i = 1, \ldots, n\},$$

and let U $(= U_{mn})$ be the Cartesian product $U = \bigtimes_{i=1}^m U_n \subset \mathbb{R}^{mn}$.

We can now begin the proof of the theorem. Since $W_Q(m, n; \boldsymbol{\tau})$ is invariant under translations by integer vectors it suffices to consider the set $W_Q(m, n; \boldsymbol{\tau}) \cap U$. The proof is in two parts—we obtain, separately, an upper bound and a lower bound for $\dim W_Q(m, n; \boldsymbol{\tau}) \cap U$. The proof of the upper bound $\dim W_Q(m, n; \boldsymbol{\tau}) \cap U \le D_Q(m, n; \boldsymbol{\tau})$, for $\sigma(\boldsymbol{\tau}) \ge \nu(Q)$, is relatively straightforward and follows from combining the corresponding arguments in [10] and in [11] (the bound $\dim W_Q(m, n; \boldsymbol{\tau}) \le mn$ is trivial). For brevity we will omit the details.

To prove the reverse inequality for $\dim W_Q(m,n;\boldsymbol{\tau}) \cap U$ we first require some lemmas. Suppose, for now, that $\nu = \nu(Q) > 0$ and $\sigma(\boldsymbol{\tau}) > \nu$, and let $\delta > 0$ be an arbitrarily small number satisfying

$$0 < \delta < \min\{\nu, \sigma(\boldsymbol{\tau}) - \nu, 1\} \tag{1}$$

(the cases where the above assumptions do not hold will be dealt with at the end of the proof). Some other restrictions will be imposed on δ below, but essentially δ is a fixed "sufficiently small" number. Since the case $n = 1$ was dealt with in [11] we will also suppose that $n \geq 2$.

We also suppose that the series $\sum_{\mathbf{q}\in Q} |\mathbf{q}|^{-\nu}$ is divergent. If this assumption does not hold we replace ν with $\nu - \varepsilon$, $\varepsilon > 0$, throughout the following argument to obtain

$$\dim W_Q(m,n;\boldsymbol{\tau}) \geq m(n-1) + \min_{1\leq k\leq m} \left\{ \frac{m + \nu - \varepsilon + \sum_{i=k}^m (\tau_k - \tau_i)}{1 + \tau_k} \right\},$$

which yields the result since $\varepsilon > 0$ is arbitrary.

Lemma 2.1 (Lemma 2.1 of [10]). *For any integer $k_0 > 0$ there exists an integer $k > k_0$ such that*

$$\sum_{\substack{\mathbf{q}\in Q \\ 2^k \leq |\mathbf{q}| < 2^{k+1}}} 1 \geq 2^{k\nu}/k^2. \tag{2}$$

From now on, N will always denote an integer of the form 2^k, where k is such that (2) holds. By Lemma 2.1 there are infinitely many such integers. Thus, writing

$$Q(N) = \{\mathbf{q} \in Q : N \leq |\mathbf{q}| < 2N\},$$

we have

$$|Q(N)| \geq N^{\nu - \delta/2},$$

for all sufficiently large N (of the above form). Now, for any vector $\mathbf{q} \in Q(N)$, let $[\mathbf{q}] \subset Q$ denote the set of all those vectors $\mathbf{q}' \in Q(N)$ which are linearly dependent on $\mathbf{q}$. Clearly the relation of linear dependence is an equivalence relation on the set $Q(N)$ and we let $[Q(N)]$ denote the corresponding set of equivalence classes $[\mathbf{q}]$.

Lemma 2.2 (Lemma 2.2 of [10]). *There exists a number α, with $\delta \leq \alpha \leq \nu$, and a subset $\widetilde{Q} \subset Q$ such that, for infinitely many N,*

$$|[\widetilde{Q}(N)]| \approx N^{\alpha - \delta}, \tag{3}$$

$$|[\mathbf{q}]| \approx N^{\nu - \alpha}, \tag{4}$$

for all equivalence classes $[\mathbf{q}] \in [\widetilde{Q}(N)]$. Thus

$$|\widetilde{Q}(N)| \approx N^{\nu - \delta}. \tag{5}$$

It should be noted that the number α here was denoted by γ in [10]. We now suppose that $\nu - \alpha > 0$. The case where this does not hold will be discussed at the end of the proof.

LEMMA 2.3 (Lemma 1 of [11]). *The following result holds for almost all collections in the set* $\{\boldsymbol{\tau} \in \mathbb{R}_+^m : \sigma(\boldsymbol{\tau}) \ge \nu\}$ *(here, "almost all" is with respect to Lebesgue measure in* $\mathbb{R}^m$*). There exists an integer* $K = K(\boldsymbol{\tau})$, $1 \le K \le m$, *and a number* $\delta_0 = \delta_0(\boldsymbol{\tau}) > 0$ *such that for any* $\delta \in (0, \delta_0)$ *there exists a collection of numbers* $\widetilde{\boldsymbol{\tau}} = \widetilde{\boldsymbol{\tau}}(\delta) = (\widetilde{\tau}_1(\delta), \ldots, \widetilde{\tau}_m(\delta)) \in \mathbb{R}_+^m$, *with the following properties*:

(τ1) $\tau_i - \delta/m = \widetilde{\tau}_i \ge \tau_{i+1} + \delta/m$ *for each* $i = K+1, \ldots, m$;
(τ2) $\tau_K - 2\delta/m \ge \widetilde{\tau}_1 = \ldots = \widetilde{\tau}_K \ge \tau_{K+1} + \delta/m$;
(τ3) $\sum_{i=1}^m \widetilde{\tau}_i = \nu$.

In particular, $\widetilde{\tau}_1 \ge \ldots \ge \widetilde{\tau}_m$.

REMARK 2.4. If $K = m$ then condition (τ1) and the second inequality in condition (τ2) are to be ignored. We adopt the convention that any arguments relating to situations which cannot occur for a particular choice of numbers are to be ignored in that particular case.

Let G denote the set of collections $\boldsymbol{\tau}$ for which the conclusions of Lemma 2.3 hold. By the continuity argument following the proof of Lemma 1 in [11], we need only prove the required lower bound for $\dim W_Q(m, n; \boldsymbol{\tau})$ for all $\boldsymbol{\tau} \in G$. Thus from now on we consider a fixed $\boldsymbol{\tau} \in G$ and write σ for $\sigma(\boldsymbol{\tau})$.

We now require some further notation. For any $\mathbf{q} \in \mathbb{Z}^n$, $t \in \mathbb{Z}$, let $H(\mathbf{q}, t) \subset \mathbb{R}^n$ denote the $(n-1)$-dimensional hyperplane $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}\cdot\mathbf{q} + t = 0\}$. If $\mathbf{t} \in \mathbb{Z}^m$, let $H(\mathbf{q}, \mathbf{t}) = \times_{i=1}^m H(\mathbf{q}, t_i) \subset \mathbb{R}^{mn}$. The next lemma is an adaptation of Lemma 4 in [2], Lemma 2.3 of [10] and Lemma 2 of [11].

LEMMA 2.5. *For any number* L *with* $0 < L < 1$, *there exist arbitrarily large integers* N *such that, for every ball* $C \subset U$ *with diameter* L, *and every equivalence class* $[\mathbf{q}] \in [\widetilde{Q}(N)]$, *there is a set* $S = S(C, [\mathbf{q}])$, *consisting of pairs* $(\mathbf{q}, \mathbf{t})$, $\mathbf{q} \in [\mathbf{q}]$ *and* $\mathbf{t} \in \mathbb{Z}^m$, *with the properties*:

(i) *for all* $(\mathbf{q}, \mathbf{t}) \in S$, $H(\mathbf{q}, \mathbf{t}) \cap \frac{1}{2}C \ne \emptyset$,

(ii) *for all distinct pairs* $(\mathbf{q}^1, \mathbf{t}^1), (\mathbf{q}^2, \mathbf{t}^2) \in S$, *there is an integer* i *for which*

$$|H(\mathbf{q}^1, t_i^1) - H(\mathbf{q}^2, t_i^2)|_2 \ge c_1 N^{-1-\widetilde{\tau}_i + \alpha/m - \delta/m}; \tag{6}$$

(iii) *the number of pairs* $(\mathbf{q}, \mathbf{t})$ *in* S *satisfies*

$$|S| \gg L^m \chi([\mathbf{q}]) \gg L^m N^{m+\nu-\alpha-\delta/2}, \tag{7}$$

where $\chi([\mathbf{q}]) = \sum_{\mathbf{q}\in[\mathbf{q}]} \phi(|\mathbf{q}|)^m$ *and* ϕ *is the Euler function*;

(iv) *for any set* $I \subset C$ *with* $d(I) > N^{-1+\delta}$, *let* S_I *denote the set of pairs* $(\mathbf{q},\mathbf{t}) \in S$ *for which* $H(\mathbf{q},\mathbf{t}) \cap I \neq \emptyset$. *Then*

$$|S_I| \ll d(I)^m \chi([\mathbf{q}]).$$

Proof. The proof of Lemma 2.3 in [10] is based on the results in Lemma 4 of [2]. The present lemma can be proved in a similar manner, but based on the results in Lemma 2 of [11] (which in turn was based on the proof of Lemma 4 in [2]). We will omit the details. ∎

We now suppose that L and $C \subset U$, with $d(C) = L$, are fixed, and choose N so that Lemma 2.5 holds. We now wish to construct a collection of balls in C lying "close" to the planes $H(\mathbf{q},\mathbf{t})$, $(\mathbf{q},\mathbf{t}) \in S([\mathbf{q}])$, where $S([\mathbf{q}])$ is the set constructed in Lemma 2.5 (to simplify the notation slightly we have suppressed the dependence of S on C). To ensure that the balls from different such collections do not intersect we need the following rather complicated construction.

For any equivalence class $[\mathbf{q}] \in [\widetilde{Q}(N)]$ let

$$E([\mathbf{q}]) = \bigcup_{(\mathbf{q},\mathbf{t})\in S([\mathbf{q}])} \big(H(\mathbf{q},\mathbf{t}) \cap \tfrac{3}{4}C\big).$$

Since the planes $H(\mathbf{q},\mathbf{t})$, with $(\mathbf{q},\mathbf{t}) \in S([\mathbf{q}])$, pass through the ball $\frac{1}{2}C$, the $m(n-1)$-dimensional Lebesgue measure (which we denote by $\mu_{m(n-1)}$) of the set $H(\mathbf{q},\mathbf{t}) \cap \frac{3}{4}C$ satisfies $\mu_{m(n-1)}\big(H(\mathbf{q},\mathbf{t}) \cap \frac{3}{4}C\big) \gg L^{m(n-1)}$, and hence by (7),

$$\mu_{m(n-1)}(E([\mathbf{q}])) \gg L^{mn}\chi([\mathbf{q}]) \gg L^{mn}N^{m+\nu-\alpha-\delta}. \tag{8}$$

Now, for any $\mathbf{p} \in \widetilde{Q}(N)$, $\mathbf{p} \notin [\mathbf{q}]$ and any pair $(\mathbf{q},\mathbf{t}) \in S([\mathbf{q}])$, let

$$F(\mathbf{p};\mathbf{q},\mathbf{t}) = \big\{X \in H(\mathbf{q},\mathbf{t}) \cap \tfrac{3}{4}C : \|\mathbf{x}_i \cdot \mathbf{p}\| < 8nN^{-\widetilde{\tau}_i-\delta/m},\ i = 1,\ldots,m\big\}.$$

Let

$$F([\mathbf{q}]) = \bigcup_{\substack{\mathbf{p}\in\widetilde{Q}(N)\\ \mathbf{p}\notin[\mathbf{q}]}} \ \bigcup_{(\mathbf{q},\mathbf{t})\in S([\mathbf{q}])} F(\mathbf{p};\mathbf{q},\mathbf{t}).$$

Lemma 2.6 (Lemma 2.4 of [10]). *For any* $[\mathbf{q}] \in \widetilde{Q}(N)$,

$$\frac{\mu_{m(n-1)}(F([\mathbf{q}]))}{\mu_{m(n-1)}(E([\mathbf{q}]))} \ll L^{-mn}N^{-\delta}.$$

Proof. For any $\mathbf{p} \neq \mathbf{0}$ and any $\eta \geq 0$, let

$$A_{\mathbf{p}}(\eta) = \{\mathbf{x} \in U_n : \|\mathbf{x} \cdot \mathbf{p}\| \leq \eta\}.$$

It is shown in [5] or [12] that if $\mathbf{p}$ and $\mathbf{p}'$ are linearly independent integer vectors then, for any η, $\eta' > 0$,

$$\mu_n(A_{\mathbf{p}}(\eta) \cap A_{\mathbf{p}'}(\eta')) = 4\eta\eta'. \tag{9}$$

Now, by definition,

$$F([\mathbf{q}]) \subset \bigcup_{\substack{\mathbf{p}\in\widetilde{Q}(N)\\ \mathbf{p}\notin[\mathbf{q}]}} \bigcup_{\mathbf{q}\in[\mathbf{q}]} \bigtimes_{i=1}^{m} (A_{\mathbf{p}}(N^{-\widetilde{\tau}_i-\delta/m}) \cap A_{\mathbf{q}}(0)),$$

so

$$\mu_{m(n-1)}(F([\mathbf{q}])) \le \sum_{\substack{\mathbf{p}\in\widetilde{Q}(N)\\ \mathbf{p}\notin[\mathbf{q}]}} \sum_{\mathbf{q}\in[\mathbf{q}]} \prod_{i=1}^{m} \mu_{n-1}(A_{\mathbf{p}}(N^{-\widetilde{\tau}_i-\delta/m}) \cap A_{\mathbf{q}}(0)).$$

For each $\eta > 0$, the set $A_{\mathbf{p}}(N^{-\widetilde{\tau}_i-\delta/m}) \cap A_{\mathbf{q}}(\eta)$ is an n-dimensional "thickening" of the set $A_{\mathbf{p}}(N^{-\widetilde{\tau}_i-\delta/m}) \cap A_{\mathbf{q}}(0)$ (which consists of portions of $(n-1)$-dimensional planes) with "thickness" $2\eta|\mathbf{q}|_2^{-1}$. Thus

$$\begin{aligned}\mu_{n-1}(A_{\mathbf{p}}(N^{-\widetilde{\tau}_i-\delta/m}) \cap A_{\mathbf{q}}(0)) &= \lim_{\eta\to 0} \mu_n(A_{\mathbf{p}}(N^{-\widetilde{\tau}_i-\delta/m}) \cap A_{\mathbf{q}}(\eta))/2\eta|\mathbf{q}|_2^{-1} \\ &\ll N^{1-\widetilde{\tau}_i-\delta/m},\end{aligned}$$

by (9). Hence by (τ3), (4) and (5),

$$\begin{aligned}\mu_{m(n-1)}(F([\mathbf{q}])) &\ll \sum_{\substack{\mathbf{p}\in\widetilde{Q}(N)\\ \mathbf{p}\notin[\mathbf{q}]}} \sum_{\mathbf{q}\in[\mathbf{q}]} \prod_{i=1}^{m} N^{1-\widetilde{\tau}_i-\delta/m} \\ &\ll N^{\nu-\delta} N^{\nu-\alpha} N^{m-\nu-\delta} = N^{m+\nu-\alpha-2\delta},\end{aligned}$$

so the result follows from (8). ■

Now, it follows from Lemma 2.6 that for N sufficiently large we can choose a collection $\mathcal{B}^0([\mathbf{q}])$ of pairwise disjoint balls $B \subset \frac{3}{4}C$, in $\mathbb{R}^{mn}$, with diameter $n^{-1}(2N)^{-(1+\tau_1)}$, whose centres Z lie on $E([\mathbf{q}]) \setminus F([\mathbf{q}])$, and satisfy

(10) $$|Z - Z'|_2 \ge 4N^{-(1+\tau_1)} \quad \text{if } Z \ne Z',$$

and such that

(11) $$|\mathcal{B}^0([\mathbf{q}])| \gg \frac{\mu_{m(n-1)}(E([\mathbf{q}]))}{(N^{-(1+\tau_1)})^{m(n-1)}} \gg L^{mn}\chi([\mathbf{q}])N^{m(n-1)(1+\tau_1)}$$

(by (8)). Since each $B \in \mathcal{B}^0([\mathbf{q}])$ has diameter $n^{-1}(2N)^{-(1+\tau_1)}$, and lies on some plane $H(\mathbf{q}, \mathbf{t})$, with $\mathbf{q} \in [\mathbf{q}]$, it follows that if $X = (\mathbf{x}_1, \ldots, \mathbf{x}_m) \in B$ then for each $i = 1, \ldots, m$,

$$\|\mathbf{x}_i \cdot \mathbf{q}\| \le n^{-1}(2N)^{-(1+\tau_1)}|\mathbf{q}|_2 < (2N)^{-\tau_1} \le |\mathbf{q}|^{-\tau_i}$$

(using $|\mathbf{q}|_2 < 2nN$ for all $\mathbf{q} \in \widetilde{Q}(N)$), so B has the property:

(12) if $X \in B$ then there exists $\mathbf{q} \in [\mathbf{q}]$ such that $\|\mathbf{x}_i \cdot \mathbf{q}\| < |\mathbf{q}|^{-\tau_i}$, $i = 1, \ldots, m$.

Now choose an arbitrary ball $B^0 \in \mathcal{B}^0([\mathbf{q}])$, with centre $Z^0 = (\mathbf{z}_1^0, \ldots, \mathbf{z}_m^0)$. For each vector $\mathbf{r} \in \mathbb{Z}^m$, with

$$r_1 = 0, \quad |r_i| < (8n)^{-1} 2^{-\tau_1} N^{\tau_1 - \tau_i}, \quad i = 2, \ldots, m, \tag{13}$$

let $B^{\mathbf{r}}(B^0)$ be the ball with diameter $n^{-1}(2N)^{-(1+\tau_1)}$ and centre $Z^{\mathbf{r}} = (\mathbf{z}_1^{\mathbf{r}}, \ldots, \mathbf{z}_m^{\mathbf{r}})$, where

$$\mathbf{z}_i^{\mathbf{r}} = \mathbf{z}_i^0 + r_i 4N^{-(1+\tau_1)} \mathbf{q}/|\mathbf{q}|_2, \quad i = 1, \ldots, m,$$

(note that the unit vector $\mathbf{q}/|\mathbf{q}|_2$ is orthogonal to the plane $H(\mathbf{q}, t)$ in $\mathbb{R}^n$, for any $t \in \mathbb{R}$). We let $\mathcal{B}(B^0)$ denote the collection $\mathcal{B}(B^0) = \bigcup_{\mathbf{r}} B^{\mathbf{r}}(B^0)$ (where the union is over all vectors $\mathbf{r}$ satisfying (13)). If N is sufficiently large, then each ball $B \in \mathcal{B}(B^0)$ satisfies $B \subset C$ and property (12) (by a similar calculation to the above, using (13)). Furthermore, (if $c_1 N^{\alpha/m} \ge 4$) from (6) and the above construction, if the balls B^1, B^2 in $\mathcal{B}^0([\mathbf{q}])$ lie on different planes $H(\mathbf{q}, t)$ then the centres Z, Z' of any two balls $B \in \mathcal{B}(B^1)$, $B' \in \mathcal{B}(B^2)$, satisfy

$$|\mathbf{z}_i - \mathbf{z}_i'|_2 \ge N^{-1-\widetilde{\tau}_i - \delta/m}, \quad \text{for some } i, \tag{14}$$

(again using $|\mathbf{q}|_2 < 2nN$ for all $\mathbf{q} \in \widetilde{Q}(N)$, and also $\tau_1 - \delta/m \ge \widetilde{\tau}_i + \delta/m$ for all i).

Repeating this process for all $B^0 \in \mathcal{B}^0([\mathbf{q}])$ we obtain the collection

$$\mathcal{B}([\mathbf{q}]) = \bigcup_{B^0 \in \mathcal{B}^0([\mathbf{q}])} \mathcal{B}(B^0).$$

Each $B \in \mathcal{B}([\mathbf{q}])$ has the property (12), and it follows from (14) that all the balls in $\mathcal{B}([\mathbf{q}])$ are disjoint, and so, from (11) and the number of vectors $\mathbf{r}$ satisfying (13), we have

$$\begin{aligned} |\mathcal{B}([\mathbf{q}])| &\gg L^{mn} \chi([\mathbf{q}]) N^{m(n-1)(1+\tau_1)} \prod_{i=1}^m N^{\tau_1 - \tau_i} \\ &\gg L^{mn} \chi([\mathbf{q}]) N^{m(n-1)(1+\tau_1) + \gamma}, \end{aligned} \tag{15}$$

where $\gamma = \sum_{i=1}^m (\tau_1 - \tau_i) = m\tau_1 - \sigma$.

Repeating the above constructions for each $[\mathbf{q}] \in [\widetilde{Q}(N)]$ we obtain the collection

$$\mathcal{B} = \bigcup_{[\mathbf{q}] \in [\widetilde{Q}(N)]} \mathcal{B}([\mathbf{q}]).$$

If $[\mathbf{q}] \ne [\mathbf{q}']$ and $B \in \mathcal{B}([\mathbf{q}])$, $B' \in \mathcal{B}([\mathbf{q}'])$ then it follows from the definition of the sets $F(\mathbf{p}; \mathbf{q}, \mathbf{t})$ and the above construction that the centres of these balls, Z and Z' respectively, satisfy (14). Hence, in particular, all the balls in the collection $\mathcal{B}$ are disjoint.

Using these constructions we can now prove the following lemma, which is similar to Lemmas 2.5 and 2.6 of [10], or Lemma 3 of [11]. For the reader's convenience we summarize here certain relationships between the various numbers we have introduced above:

$$\nu = \sum_{i=1}^{m} \widetilde{\tau}_i, \quad \sigma = \sum_{i=1}^{m} \tau_i, \quad \gamma = \sum_{i=1}^{m} (\tau_1 - \tau_i) = m\tau_1 - \sigma.$$

LEMMA 2.7. *For any number L with $0 < L < 1$, there exist arbitrarily large integers N such that for any ball $C \subset U$ with diameter L there is a collection $\mathcal{B}$ of disjoint balls $B \subset C$, such that*:

(i) *each $B \in \mathcal{B}$ has diameter $n^{-1}(2N)^{-(1+\tau_1)}$ and the centres of any two balls in $\mathcal{B}$ are at least a distance $4N^{-(1+\tau_1)}$ apart*;

(ii) *for each $B \in \mathcal{B}$, (12) holds for some $[\mathbf{q}] \in [\widetilde{Q}(N)]$*;

(iii) $|\mathcal{B}| \ge c_2 L^{mn} X(N) N^{m(n-1)(1+\tau_1)+\gamma}$, *where*

$$X(N) = \sum_{[\mathbf{q}] \in [\widetilde{Q}(N)]} \chi([\mathbf{q}]) \gg N^{m+\nu-3\delta/2};$$

(iv) *if I is a set in $\mathbb{R}^{mn}$ with $d(I) \ge n^{-1} N^{-(1+\tau_1)}$, which intersects h of the balls B in $\mathcal{B}$, then*:

(a) *suppose that $N^{-(1+\tau_k)} < d(I) \le N^{-(1+\tau_{k+1})}$, for some k with $1 \le k \le m-1$*:

- *if $k < K$, then*

(16)
$$h \le c_3 d(I)^{mn-k} N^{(mn-k)(1+\tau_1)+\sum_{i=1}^{k}(\tau_1-\tau_i)};$$

- *if $k = K$, then*

(17)
$$h \le c_3 d(I)^{mn-k} N^{(mn-k)(1+\tau_1)+\sum_{i=1}^{k}(\tau_1-\tau_i)} + c_3 d(I)^{mn} N^{m(n-1)(1+\tau_1)+m+\nu+\gamma+\delta};$$

- *if $k > K$, then*

(18)
$$h \le c_3 d(I)^{mn} N^{m(n-1)(1+\tau_1)+m+\nu+\gamma+\delta};$$

(b) *if $N^{-(1+\tau_m)} < d(I) \le N^{-1+\delta}$, then*

(19)
$$h \le c_3 d(I)^{mn} N^{m(n-1)(1+\tau_1)+m+\nu+\gamma};$$

(c) *if $N^{-1+\delta} < d(I)$, then*

(20)
$$h \le c_3 d(I)^{mn} X(N) N^{m(n-1)(1+\tau_1)+\gamma}.$$

Proof. It is clear that the collection of balls $\mathcal{B}$ constructed above has the properties (i) and (ii) for N sufficiently large (the estimate on the distance between the centres of the balls in $\mathcal{B}$ follows from (10) and (14)). The estimate for $|\mathcal{B}|$ in (iii) follows from (15) and the definition of $\mathcal{B}$, while the estimate for $X(N)$ follows from (3) and (7). We now prove (iv).

For any $[\mathbf{q}] \in [\widetilde{Q}(N)]$ and any pair $(\mathbf{q}, \mathbf{t}) \in S([\mathbf{q}])$, let $\mathcal{B}(\mathbf{q}, \mathbf{t})$ be the set of all balls $B \in \mathcal{B}([\mathbf{q}])$ which belong to any collection $\mathcal{B}(B^0)$ for which the centre of B^0 lies on the plane $H(\mathbf{q},\mathbf{t})$ (i.e., $\mathcal{B}(\mathbf{q},\mathbf{t})$ is the set of all balls $B\in\mathcal{B}(\mathbf{q},\mathbf{t})$ which lie "close" to the plane $H(\mathbf{q},\mathbf{t})$). It follows from the above constructions that if $(\mathbf{q},\mathbf{t})\neq(\mathbf{q}',\mathbf{t}')$ and $B\in\mathcal{B}(\mathbf{q},\mathbf{t})$, $B'\in\mathcal{B}(\mathbf{q}',\mathbf{t}')$ then their centres Z, Z' satisfy (14).

Now suppose that $d(I)$ satisfies the inequalities in case (a) for some k, $1\le k\le m-1$. We begin by estimating the number $h(\mathbf{q},\mathbf{t})$ of balls $B\in\mathcal{B}(\mathbf{q},\mathbf{t})$ which can intersect I. Since the balls $B\in\mathcal{B}(\mathbf{q},\mathbf{t})$ have diameters $n^{-1}(2N)^{-(1+\tau_1)}$, their centres are a distance at least $N^{-(1+\tau_1)}$ apart, and they all lie "close" to the $m(n-1)$-dimensional plane $H(\mathbf{q},\mathbf{t})$, it follows from the geometry of the situation and the construction of the collection $\mathcal{B}(\mathbf{q},\mathbf{t})$ that the number $h(\mathbf{q},\mathbf{t})$ of balls $B\in\mathcal{B}(\mathbf{q},\mathbf{t})$ which can intersect I satisfies

$$\text{(21)}\qquad h(\mathbf{q},\mathbf{t})\ll\left(\frac{d(I)}{N^{-(1+\tau_1)}}\right)^{n(m-k)}\prod_{i=1}^{k}N^{\tau_1-\tau_i}\left(\frac{d(I)}{N^{-(1+\tau_1)}}\right)^{n-1}$$
$$\le d(I)^{mn-k}N^{(mn-k)(1+\tau_1)+\sum_{i=1}^{k}(\tau_1-\tau_i)}.$$

Now, if $k<K$ then by (τ2), (14) and the above construction, if N is sufficiently large, I can intersect balls from at most one collection $\mathcal{B}(\mathbf{q},\mathbf{t})$ with $(\mathbf{q},\mathbf{t})\in\bigcup_{[\mathbf{q}]\in[\widetilde{Q}(N)]}S([\mathbf{q}])$. Thus (16) follows from (21). Next, if $k>K$ then by (τ1), (τ2), (14) and the above construction, if N is sufficiently large the number of collections $\mathcal{B}(\mathbf{q},\mathbf{t})$ which contain balls intersecting I is

$$\text{(22)}\qquad \ll\prod_{i=1}^{k}\frac{d(I)}{N^{-1-\widetilde{\tau}_i-\delta/m}}=d(I)^kN^{k+\sum_{i=1}^{k}\widetilde{\tau}_i+k\delta/m}.$$

Therefore, in this case it follows from (21) and (22) that the total number of balls intersecting I is $\ll d(I)^{mn}N^{\zeta}$, where

$$\begin{aligned}
\zeta&=k+\sum_{i=1}^{k}\widetilde{\tau}_i+k\delta/m+(mn-k)(1+\tau_1)+\sum_{i=1}^{k}(\tau_1-\tau_i)\\
&=m(n-1)(1+\tau_1)+m+\sum_{i=1}^{m}\widetilde{\tau}_i-\sum_{i=k+1}^{m}\widetilde{\tau}_i+\sum_{i=1}^{m}(\tau_1-\tau_i)\\
&\qquad+\sum_{i=k+1}^{m}\tau_i+k\delta/m\\
&=m(n-1)(1+\tau_1)+m+\nu+\gamma+\sum_{i=k+1}^{m}(\tau_i-\widetilde{\tau}_i)+(k\delta)/m\\
&\le m(n-1)(1+\tau_1)+m+\gamma+\nu+\delta
\end{aligned}$$

(using (τ2) and (τ3)). This proves (18). Finally (in case (a)), suppose that $k=K$. Then, using the above arguments, if $d(I)<N^{-1-\widetilde{\tau}_K-\delta/m}$ we obtain

the estimate (16), while if $d(I) \ge N^{-1-\widetilde{\tau}_K-\delta/m}$ we obtain the estimate (18). Adding these estimates yields (17), which completes the proof of case (a).

Next, consider case (b). For a fixed equivalence class $[\mathbf{q}] \in [\widetilde{Q}(N)]$, it follows from (6) that the number of collections $\mathcal{B}(\mathbf{q},\mathbf{t})$ with $(\mathbf{q},\mathbf{t}) \in S([\mathbf{q}])$, which have at least one ball intersecting the set I, is

$$\ll \prod_{i=1}^{m} \frac{d(I)}{N^{-1-\widetilde{\tau}_i+\alpha/m-\delta/m}} = d(I)^m N^{m+\nu-\alpha+\delta},$$

and the number of balls B in each such collection $\mathcal{B}(\mathbf{q},\mathbf{t})$ is

$$\ll \left(\frac{d(I)}{N^{-(1+\tau_1)}}\right)^{m(n-1)} \prod_{i=1}^{m} N^{\tau_1-\tau_i} = d(I)^{m(n-1)} N^{m(n-1)(1+\tau_1)+\gamma}. \tag{23}$$

Hence the number of balls corresponding to a single equivalence class which intersect I is

$$\ll d(I)^{mn} N^{m+\nu-\alpha+\delta+m(n-1)(1+\tau_1)+\gamma}.$$

The number of possible equivalence classes is $\ll N^{\alpha-\delta}$ which, together with the above estimate, gives (19).

Finally, in case (c) it follows from (iv) of Lemma 2.5 that the number of collections $\mathcal{B}(\mathbf{q},\mathbf{t})$ with $(\mathbf{q},\mathbf{t}) \in S([\mathbf{q}])$, which have at least one ball intersecting the set I is $\ll d(I)^m \chi([\mathbf{q}])$. Using the estimate (23) for the number of balls in each such collection and summing over the set of equivalence classes $[\mathbf{q}] \in [\widetilde{Q}(N)]$ yields (20). This completes the proof of Lemma 2.7. ■

Now, it will be shown that if $\delta > 0$ is sufficiently small then we have $\dim W_Q(m,n;\boldsymbol{\tau}) \ge \varrho := D_Q(m,n;\boldsymbol{\tau}) - 4\delta$. On letting $\delta \to 0$ this yields the required lower bound for $\dim W_Q(m,n;\boldsymbol{\tau})$, which will complete the proof, subject to the additional conditions imposed above.

Choose $N_0 > 0$ sufficiently large that

$$4c_3 N_0^{-(\sigma-\nu)-\delta(1+\tau_1)} \le c_2 \tag{24}$$

(this is possible since $\sigma > \nu$). Let $\mathcal{F}$ be any countable family of sets I in $\mathbb{R}^n$ of positive diameter $d(I) \le \frac{1}{2} n^{-1}(2N_0)^{-(1+\tau_1)}$ with

$$V_\varrho(\mathcal{F}) = \sum_{I\in\mathcal{F}} d(I)^\varrho < 1. \tag{25}$$

We will show that the family $\mathcal{F}$ cannot cover the set $W_Q(m,n;\boldsymbol{\tau}) \cap U$ and hence, by definition, $m_\varrho(W_Q(m,n;\boldsymbol{\tau})) > 0$, which proves $\dim W_Q(m,n;\boldsymbol{\tau}) \ge \varrho$. To do this we construct a sequence of sets $U \supset J_0 \supset J_1 \supset \ldots$, where $J_j \subset \mathbb{R}^{mn}$ is the union of $M_j > 0$ pairwise disjoint balls and integers $N_0 < N_1 < \ldots$, such that for $j \ge 1$, the following conditions are satisfied:

(i)$_j$ J_j intersects no $I \in \mathcal{F}$ with $d(I) > \frac{1}{2}n^{-1}(2N_j)^{-(1+\tau_1)}$;

(ii)$_j$ each ball of J_j has diameter $n^{-1}(2N_j)^{-(1+\tau_1)}$ and their centres are at least a distance $4N_j^{-(1+\tau_1)}$ apart;

(iii)$_j$ if $X \in J_j$, there is a $\mathbf{q} \in \widetilde{Q}(N_j)$ such that $\|\mathbf{x}_i \cdot \mathbf{q}\| < |\mathbf{q}|^{-\tau_i}$, for $i = 1, \ldots, m$;

(iv)$_j$ $M_j \ge 4c_3c_2^{-1}2^{mn(1+\tau_1)}N_j^{-(\sigma-\nu)+mn(1+\tau_1)-\delta(1+\tau_1)}$ (we suppose that δ is sufficiently small that the exponent of N_j here is positive).

Supposing that such sequences exist, let

$$J_\infty = \bigcap_{j=0}^{\infty} J_j.$$

Since the sequence $J_j, j = 0, 1, \ldots,$ is a decreasing sequence of non-empty closed bounded sets in $\mathbb{R}^{mn}$, J_∞ is non-empty. By (i)$_j$, J_∞ does not intersect any set $I \in \mathcal{F}$, while by (iii)$_j$, $J_\infty \subset W_Q(m, n; \boldsymbol{\tau})$. Thus, $\mathcal{F}$ does not cover $W_Q(m, n; \boldsymbol{\tau})$.

The construction is by induction. Let J_0 be the ball of diameter 1 and centre $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$, and let N_0 be as above. Now suppose that $J_0, J_1, \ldots, J_{j-1}$, $N_0, N_1, \ldots, N_{j-1}$ have already been constructed satisfying the above conditions, for some $j \ge 1$. We will construct J_j and N_j. Let D be a ball of J_{j-1} and let $C = \frac{1}{4}D$. Applying Lemma 2.7 to C we choose $N_j = N$ such that $N_j^{-1+\delta} < n^{-1}(2N_{j-1})^{-(1+\tau_1)}$, and we obtain the corresponding collection of balls $\mathcal{B} = \mathcal{B}(D)$. Let

$$\mathcal{G}_j = \bigcup_{D \in J_{j-1}} \mathcal{B}(D),$$

and let

$$\begin{aligned}
\mathcal{F}_j^{1,k} &= \{I \in \mathcal{F} : N_j^{-(1+\tau_k)} < d(I) \le N_j^{-(1+\tau_{k+1})}\}, \qquad k = 1, \ldots, m-1,\\
\mathcal{F}_j^{2} &= \{I \in \mathcal{F} : N_j^{-(1+\tau_m)} < d(I) \le N_j^{-1+\delta}\},\\
\mathcal{F}_j^{3} &= \{I \in \mathcal{F} : N_j^{-1+\delta} < d(I) \le N_{j-1}^{-(1+\tau_1)}\}.
\end{aligned}$$

Taking $\mathcal{H}_j$ to be the set of balls in $\mathcal{G}_j$ which intersect a set $I \in \bigcup_k \mathcal{F}_j^{1,k} \cup \mathcal{F}_j^2 \cup \mathcal{F}_j^3$, we define J_j to be the union of the balls in the collection $\mathcal{G}_j \setminus \mathcal{H}_j$. Thus, we have $J_j \subset J_{j-1}$ and (i)$_j$ holds (because $d(I) \le \frac{1}{2}n^{-1}(2N_0)^{-(1+\tau_1)}$, $I \in \mathcal{F}$, if $j = 1$, and because of (i)$_{j-1}$ if $j > 1$). Also, (ii)$_j$ and (iii)$_j$ follow from (i) and (ii) of Lemma 2.7. It remains to consider (iv)$_j$.

If $I \in \bigcup_k \mathcal{F}_j^{1,k} \cup \mathcal{F}_j^2 \cup \mathcal{F}_j^3$, then I cannot intersect balls in $\mathcal{B}(D)$ for two distinct balls $D \in J_{j-1}$ (because of (ii)$_{j-1}$, if $j > 1$). Therefore, by part (iv) of Lemma 2.7,

$$c_3^{-1}|\mathcal{H}_j| \le \sum_{k=1}^{K} \sum_{I\in\mathcal{F}_j^{1,k}} d(I)^{mn-k} N_j^{(mn-k)(1+\tau_1)+\sum_{i=1}^{k}(\tau_1-\tau_i)} + \sum_{k=K}^{m-1} \sum_{I\in\mathcal{F}_j^{1,k}} d(I)^{mn} N_j^{m(n-1)(1+\tau_1)+m+\nu+\gamma+\delta} + \sum_{I\in\mathcal{F}_j^{2}} d(I)^{mn} N_j^{m(n-1)(1+\tau_1)+m+\nu+\gamma} + \sum_{I\in\mathcal{F}_j^{3}} d(I)^{mn} X(N_j) N_j^{m(n-1)(1+\tau_1)+\gamma}. \tag{26}$$

We now estimate the various sums in (26). First we consider the integers k such that $1 \le k \le K$, and suppose that $mn-k-\varrho \le 0$. Then, by the definition of $\mathcal{F}_j^{1,k}$, we have

$$d(I)^{mn-k} = d(I)^{\varrho} d(I)^{mn-k-\varrho} \le d(I)^{\varrho} N_j^{-(mn-k-\varrho)(1+\tau_k)},$$

and so, using (25), we obtain

$$\sum_{I\in\mathcal{F}_j^{1,k}} d(I)^{mn-k} N_j^{(mn-k)(1+\tau_1)+\sum_{i=1}^{k}(\tau_1-\tau_i)} \ll N_j^{\zeta},$$

where

$$\zeta = -(mn-k-\varrho)(1+\tau_k) + (mn-k)(1+\tau_1) + \sum_{i=1}^{k}(\tau_1-\tau_i).$$

Now, by the definition of $D_Q(m,n;\boldsymbol{\tau})$,

$$\begin{aligned} -(mn-k-\varrho)(1+\tau_k) \le &\; -(m-k)(1+\tau_k) + m + \nu \\ &+ \sum_{i=k}^{m}(\tau_k-\tau_i) - 4\delta(1+\tau_k) \\ =&\; k + \nu - \sum_{i=k+1}^{m} \tau_i - 4\delta(1+\tau_k), \end{aligned}$$

so

$$\zeta \le m(n-1)(1+\tau_1) + m + \nu + \gamma - 4\delta(1+\tau_k).$$

If $mn-k-\varrho > 0$ similar calculations yield

$$\zeta = -(mn-k-\varrho)(1+\tau_{k+1}) + (mn-k)(1+\tau_1) + \sum_{i=1}^{k}(\tau_1-\tau_i),$$

and

$$
\begin{aligned}
&-(mn-k-\varrho)(1+\tau_{k+1})\\
&\quad\le -(m-k)(1+\tau_{k+1})+m+\nu+\sum_{i=k+1}^{m}(\tau_{k+1}-\tau_i)-4\delta(1+\tau_{k+1})\\
&\quad= k+\nu-\sum_{i=k+1}^{m}\tau_i-4\delta(1+\tau_{k+1}),
\end{aligned}
$$

so

$$\zeta \le m(n-1)(1+\tau_1)+m+\nu+\gamma-4\delta(1+\tau_{k+1}).$$

Next we consider k such that $K\le k\le m-1$. In this case we use

$$mn-\varrho \ge \frac{m\tau_1-\nu-\sum_{i=1}^{m}(\tau_1-\tau_i)}{1+\tau_1}+4\delta = \frac{\sigma-\nu}{1+\tau_1}+4\delta > 4\delta > 0 \tag{27}$$

(since $\sigma>\nu$), to obtain the estimate

$$\sum_{I\in\mathcal{F}_j^{1,k}} d(I)^{mn}N_j^{m(n-1)(1+\tau_1)+m+\nu+\gamma+\delta} \ll N_j^{\zeta},$$

where

$$
\begin{aligned}
\zeta &= -(mn-\varrho)(1+\tau_{k+1})+m(n-1)(1+\tau_1)+m+\nu+\gamma+\delta\\
&< m(n-1)(1+\tau_1)+m+\nu+\gamma-3\delta,
\end{aligned}
$$

for δ sufficiently small.

For the summation over $\mathcal{F}_j^2$ in (26) we again use (27) to obtain a similar estimate with

$$
\begin{aligned}
\zeta &= -(mn-\varrho)(1-\delta)+m(n-1)(1+\tau_1)+m+\nu+\gamma+\delta\\
&< m(n-1)(1+\tau_1)+m+\nu+\gamma-3\delta,
\end{aligned}
$$

for δ sufficiently small.

Finally, for the summation over $\mathcal{F}_j^3$ in (26) we obtain (using (27))

$$
\begin{aligned}
&\sum_{I\in\mathcal{F}_j^3} d(I)^{mn}X(N_j)N_j^{m(n-1)(1+\tau_1)+\gamma}\\
&\qquad\qquad\le N_{j-1}^{-(mn-\varrho)(1+\tau_1)}X(N_j)N_j^{m(n-1)(1+\tau_1)+\gamma}\\
&\qquad\qquad\le N_{j-1}^{-(\sigma-\nu)-\delta(1+\tau_1)}X(N_j)N_j^{m(n-1)(1+\tau_1)+\gamma}.
\end{aligned}
$$

Combining the above estimates, we obtain

$$|\mathcal{H}_j| \le 2c_3 N_{j-1}^{-(\sigma-\nu)-\delta(1+\tau_1)}X(N_j)N_j^{m(n-1)(1+\tau_1)+\gamma}, \tag{28}$$

for sufficiently large N_j (using the estimate $X(N)\gg N^{m+\nu-3\delta/2}$ in Lemma 2.7).

Now suppose that $j = 1$. By (iii) of Lemma 2.7 (with $d(C) = 1$), together with (24) and (28),

$$|\mathcal{G}_1| \ge c_2 X(N_1) N_1^{m(n-1)(1+\tau_1)+\gamma} \ge 2|\mathcal{H}_1|.$$

Hence,

$$M_1 \ge |\mathcal{G}_1| - |\mathcal{H}_1| \ge c_2 2^{-1} X(N_1) N_1^{m(n-1)(1+\tau_1)+\gamma},$$

so $(\mathrm{iv})_1$ holds for sufficiently large N_1.

Next suppose that $j > 1$. Then, by (iii) of Lemma 2.7, $(\mathrm{ii})_{j-1}$, $(\mathrm{iv})_{j-1}$ and (28),

$$\begin{aligned} |\mathcal{G}_j| &\ge M_{j-1} c_2 (2N_{j-1})^{-mn(1+\tau_1)} X(N_j) N_j^{m(n-1)(1+\tau_1)+\gamma} \\ &\ge 4c_3 N_{j-1}^{-(\sigma-\nu)-\delta(1+\tau_1)} X(N_j) N_j^{m(n-1)(1+\tau_1)+\gamma} \ge 2|\mathcal{H}_j|. \end{aligned} \tag{29}$$

Thus, $M_j \ge |\mathcal{G}_j| - |\mathcal{H}_j| \ge \frac{1}{2}|\mathcal{G}_j|$, and it follows from (29) that $(\mathrm{iv})_j$ holds for sufficiently large N_j if δ is sufficiently small. This completes the proof of the theorem under the various particular assumptions made in the course of the argument, viz., $\nu > \alpha > 0$ and $\sigma > \nu$.

We now remove these assumptions. Firstly, we note that the cases when $\nu = \alpha > 0$ and when $\nu = 0$ (with $\sigma > \nu$), can be dealt with by a similar method to that described in the final paragraph of [10]. Next, when $\sigma = \nu$ (for any $\nu \ge 0$) the estimate $\dim W_Q(m,n;\boldsymbol{\tau}) \ge D_Q(m,n;\boldsymbol{\tau})$ follows from the result just proved by using the continuity argument following Lemma 1 in [11] (elements $\boldsymbol{\tau} \in G$ have $\sigma > \nu$, but any $\boldsymbol{\tau}$ for which $\sigma = \nu$ lies on the boundary of G).

Now suppose that $\sigma \le \nu$. Then, for each k with $1 \le k \le m$,

$$\begin{aligned} \frac{m + \nu + \sum_{i=k}^m (\tau_k - \tau_i)}{1+\tau_k} &\ge \frac{m + \sum_{i=1}^m \tau_i + \sum_{i=k}^m (\tau_k - \tau_i)}{1+\tau_k} \\ &= \frac{m + \sum_{i=1}^{k-1} \tau_i + \sum_{i=k}^m \tau_k}{1+\tau_k} \\ &\ge \frac{m + \sum_{i=1}^m \tau_k}{1+\tau_k} = m, \end{aligned}$$

and hence, by the definition, $D_Q(m,n;\boldsymbol{\tau}) \ge mn$. Furthermore, if $\sigma = \nu$ then for $k = 1$,

$$\frac{m + \nu + \sum_{i=1}^m (\tau_1 - \tau_i)}{1+\tau_1} = \frac{m + \sum_{i=1}^m \tau_1}{1+\tau_1} = m,$$

so, together with the previous estimates, this shows that in this case $\dim W_Q(m,n;\boldsymbol{\tau}) = D_Q(m,n;\boldsymbol{\tau}) = mn$.

Now suppose that $\sigma < \nu$. Then, by increasing the components of the vector $\boldsymbol{\tau}$ appropriately, we can construct a vector $\overline{\boldsymbol{\tau}}$ such that $\sigma(\overline{\boldsymbol{\tau}}) = \nu$, and hence, since $W_Q(m,n;\overline{\boldsymbol{\tau}}) \subset W_Q(m,n;\boldsymbol{\tau})$, the above result for the case $\sigma = \nu$

gives

$$\dim W_Q(m,n;\boldsymbol{\tau}) \geq \dim W_Q(m,n;\overline{\boldsymbol{\tau}}) = mn,$$

which finally completes the proof of the theorem.

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