J. DIFFERENTIAL GEOMETRY
45 (1997) 638-649

EXOTIC SPHERES WITH POSITIVE RICCI CURVATURE

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Abstract

We show that a certain class of manifolds admit metrics of positive Ricci curvature. This class includes many exotic spheres, including all homotopy spheres which represent elements of bP_{2n}.

§0.

In this paper we investigate the Ricci curvature of a certain class of manifolds which includes many exotic spheres. In particular we will be concerned with constructing metrics of positive Ricci curvature. Our main result is as follows:

Theorem 2.1. *Homotopy spheres which bound parallelisable manifolds admit metrics of positive Ricci curvature.*

The diffeomorphism classes of homotopy spheres bounding parallelisable manifolds of dimension m form an abelian group under the connected sum operation. This group is denoted bP_m. It was shown by Kervaire and Milnor in [5] that $bP_{odd} = 0$, bP_{4k+2} is either 0 or $\mathbb{Z}_2$ (depending on k), and bP_{4k} is cyclic. In [4] Hernandez showed that a certain class of Brieskorn manifolds carry Ricci positive metrics. This class includes homotopy spheres representing the non-trivial element of those groups bP_{4k+2} which are isomorphic to $\mathbb{Z}_2$ (a case previously covered by Cheeger in [3]]), as well as many elements in bP_{4k}. Until now, however, it was an open question whether in fact all such homotopy spheres admit Ricci positive metrics. Theorem 2.1 will actually

Received February 28, 1996.

follow as a corollary of a more general theorem, which states that certain manifolds arising from a construction known as 'plumbing' admit Ricci positive metrics.

This paper is comprised of two sections: §1 concerns the plumbing of disk bundles and §2 contains the main results and their proofs.

The author would like to thank Stephan Stolz for suggesting this problem and for many valuable conversations.

§1.

The technique of plumbing is closely related to that of surgery. With it we can explicitly construct manifolds with a prescribed even intersection form. A general reference for the background material is [2, Chapter 5]. Below we give a brief description of those aspects which we will need.

The building blocks which we use for plumbing are the disk bundles associated to smooth, oriented, metric vector bundles, *i.e.*, oriented vector bundles with a smoothly varying inner product on the fibres. Assume the fibres, base and total spaces are oriented compatibly. We will restrict our attention to bundles whose base spaces are spheres. Though we need not restrict ourselves in this way, it will prove sufficient for our purposes. Note that (metric) oriented vector bundles over a sphere of dimension k are classified by the group $\pi_k BSO_k$.

Suppose we have two such disk bundles

$$\alpha : E_\alpha \to S^r,$$
$$\beta : E_\beta \to S^t,$$

where the fibres of α and β are D^t and D^r respectively. We can 'plumb' these bundles together as follows. Choose two small disks $d_r \subset S^r$ and $d_t \subset S^t$. Since these are contractable, there exist unique trivialisations of the bundles restricted to these regions. In other words we have diffeomorphisms:

$$\alpha\,|_{d_r} \cong D^r \times D^t,$$
$$\beta\,|_{d_t} \cong D^t \times D^r.$$

We can now use these diffeomorphisms to make a 'cross-identification', *i.e.*, fibre disk of α with base disk of β and vice versa. To do this we must choose diffeomorphisms

$$\theta_1 : D^r_\alpha \longrightarrow D^r_\beta, \qquad \theta_2 : D^t_\alpha \longrightarrow D^t_\beta.$$

The object thus produced is said to be the result of plumbing α and β. It can be made differentiable by simply straightening out the angles.

The diffeomorphisms θ_1 and θ_2 can be chosen to either preserve or reverse orientation. We shall say we plumb with sign $+1$ if both θ_1 and θ_2 are orientation preserving, and sign -1 if both are orientation reversing. Note that the result of plumbing two disk bundles is oriented compatibly with the given orientations, irrespective of sign, if at least one of r and t is even.

We can represent the plumbing by a schematic diagram in the following way. For each bundle we draw a dot, which should be labelled with the appropriate element of the group $\pi_k BSO_k$. Each time we plumb two of the bundles together, and join the appropriate dots with a line. If both r and t are odd, then we should label this line with the sign of the plumbing. In this way we construct a graph. This graph reflects some of the homotopy properties of the manifold. Precisely: the graph has the same homotopy type as the 1-skeleton, assuming r and t are > 1. Thus, for example, if the graph is simply connected, the same will be true of the manifold.

If we restrict ourselves further to using only stably-trivial bundles with

$$\text{base dimension} = \text{fibre dimension} = k \in 2\mathbb{Z},$$

we can associate to our graph a symmetric matrix M over $\mathbb{Z}$ with even entries on the diagonal. We do this as follows: begin with n bundles over the k-sphere. Arrange these in some order. Suppose that the i^{th} bundle is represented by $\lambda_i \tau_{S^k} \in \pi_k BSO_k$ where $\tau_{S^k} \in \pi_k BSO_k$ represents the tangent bundle of S^k. Suppose further that the plumbings between any two bundles have the same sign. Let $M_{ii} = 2\lambda_i$. For $i \neq j$ let

$$M_{ij} = \begin{cases} p & \text{if bundles } i \text{ and } j \text{ are plumbed together } p \text{ times} \\ & \text{with sign } +1, \\ -p & \text{if bundles } i \text{ and } j \text{ are plumbed together } p \text{ times} \\ & \text{with sign } -1, \\ 0 & \text{otherwise .} \end{cases}$$

Clearly $M_{ij} = M_{ji}$.

The matrix M defines a quadratic form on the free n-dimensional $\mathbb{Z}$-module V after an ordered basis has been chosen. Intersection homology theory tells us that this quadratic form is the same as the intersection form of the manifold obtained by plumbing the original graph (See [2, Theorem V.2.1]).

The transition from graph to quadratic form can be reversed, and in so doing the assertion at the start of §1, that plumbing gives a way of constructing manifolds of prescribed even intersection form, can be established.

Given an even quadratic form q on V, choose a basis for V and write down the corresponding $(n \times n)$ symmetric matrix M with even diagonal entries. Draw n dots, order them, and for each $i \leq n$ label vertex i with

$$\frac{M_{ii}}{2}\tau_{S^k} \quad (\in \pi_k BSO_k).$$

For each off-diagonal matrix entry M_{ij}, draw $|M_{ij}|$ lines joining vertices i and j. Label each according to the sign of M_{ij}. By plumbing the resulting graph we create a manifold whose intersection matrix is M when written down relative to a homology basis given by the zero sections of the bundles used.

It is easy to see that each component of this plumbed manifold has the homotopy type of a wedge of k-spheres and 1-spheres. Moreover, provided $k > 2$ each component has a free fundamental group, which is isomorphic to the fundamental group of the boundary. The former statement can be seen by examining the corresponding graph. Performing surgeries on embedded circles in the boundary we can render both component and boundary simply connected.

Consider now the connected sum of the (simply connected) boundaries. Call this object X.

Proposition 1.1. *X is a homotopy sphere $\Longleftrightarrow$ det $M = \pm 1$.*

For the proof see [2, Lemma V.2.7].

For our applications, we need to avoid both surgeries on $1-$spheres and performing connected sum operations. We therefore restrict our attention to plumbings involving simply connected graphs and base spheres of dimension ≥ 3.

Lemma 1.2. *The result of plumbing any two stably-trivial disk bundles over stably parallelisable manifolds is a parallelisable manifold.*

Proof. Consider the composition:

$$\widetilde{KO}(E \triangle F) \cong \widetilde{KO}(M \vee N) \cong \widetilde{KO}(M) \oplus \widetilde{KO}(N),$$

where E^{n+m} is a disk bundle over M^m, F^{n+m} is a disk bundle over N^n, and the symbol $\triangle$ denotes plumbing.

The first isomorphism is due in part to the fact that

$$E \triangle F \simeq M \vee N,$$

and both isomorphisms rely on the fact that $\widetilde{KO}$ is a (reduced) cohomology theory.

We need to show that the tangent bundle $T(E \triangle F)$ is stably-trivial. Since $E \triangle F$ is a manifold with boundary, it will follow from this (for example by [5, Lemma 3.4]) that $E \triangle F$ is parallelisable.

Clearly, the composition maps $T(E \triangle F)$ onto the direct sum of $\widetilde{KO}$-theory elements representing the tangent bundles of M and N. If these latter bundles are stably trivial, then this means the image of $T(E \triangle F)$ is zero. Since the composition is an isomorphism, we deduce $T(E \triangle F)$ represents the zero element $\widetilde{KO}(E \triangle F)$ and therefore $T(E \triangle F)$ is stably-trivial.

Note. More generally we see that plumbing stably-trivial disk bundles according to any simply-connected graph will yield a parallelisable manifold.

Before proceeding further we introduce two algebraic results.

Lemma 1.3. *Let q_1 and q_2 be even quadratic forms on $\mathbb{Z}$-modules V_1 respectively V_2, q_1 unimodular, whose associated graphs with respect to some $\mathbb{Z}$-bases of V_1, V_2 are simply connected. Then there exists a basis for $V_1 \bigoplus V_2$ such that the graph for $q_1 \bigoplus q_2$ is simply connected.*

Proof. Suppose the basis for V_1 is $e_1, ..., e_n$, and the basis of V_2 is $f_1, ..., f_m$. Consider the basis $e_1, ..., e_n, \theta, f_2, ..., f_m$ for $V_1 \bigoplus V_2$, where

$$\theta = f_1 + \sum_{i=1}^{n} \mu_i e_i$$

for some integers μ_i.

In order for the corresponding graph to be simply connected it suffices to demand that

$$\langle e_i, \theta \rangle = \begin{cases} 1 \text{ if } i = n, \\ 0 \text{ otherwise.} \end{cases} \qquad (*)$$

We have

$$\langle e_i, \theta \rangle = \left\langle e_i, \sum_{i=1}^{n} \mu_i e_i \right\rangle,$$

and since q_1 is unimodular it follows that we can find values of μ_i so that $(*)$ is satisfied.

Proposition 1.4. *For each positive integer n it is possible to find an even, unimodular quadratic form over $\mathbb{Z}$ having signature $8n$ and whose associated graph is simply-connected.*

Proof. We proceed by induction. For the case $n = 1$ we consider the simply-connected graph E_8. Assuming all vertices are labelled with τ_{S^k} and all lines with $+1$, the associated matrix is clearly:

$$\begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{pmatrix}$$

This matrix defines an even, unimodular quadratic form of signature 8 on some free $\mathbb{Z}-$module V_1.

Now assume the result is true for $n = r-1$. We therefore have a unimodular even quadratic form q_{r-1} on a $\mathbb{Z}$-module V_{r-1}, with signature $8(r-1)$ and an associated graph that is simply connected. Consider the form $q_{r-1} \bigoplus E_8$ on $V_{r-1} \bigoplus V_1$. This is an even unimodular form with signature $8r$. By Lemma 1.3 we can rechoose the basis for this latter module so that the associated graph is simply connected. Setting $V_r = V_{r-1} \bigoplus V_1$ and $q_r = q_{r-1} \bigoplus E_8$ completes the induction step.

We now come to the main result of this section.

Proposition 1.5. *A representative of any non-trivial element in bP_{2k} arises as the boundary of a manifold constructed by plumbing stably-trivial $k-$disk bundles over $k-$spheres according to a simply connected graph.*

Proof. This result is well-known in the case of Kervaire spheres, (the exotic spheres arising in dimensions congruent to 1 modulo 4 - see for example [6, p.162]). We therefore only need give our attention to the groups bP_{4k}. About these we know the following facts [5, §7]:

(1) Let σ_k be the quantity

$$\sigma_k = 2^{2k+1}(2^{2k-1} - 1).\text{numerator}\frac{4B_k}{k},$$

where B_k is the k^{th} Bernoulli number. There is an isomorphism

$$bP_{4k} \longrightarrow \mathbb{Z}_{\frac{\sigma_k}{8}},$$

which is realised by the mapping

$$[\Sigma^{4k-1}] \longmapsto \frac{\sigma \quad (\text{mod } \sigma_k)}{8},$$

where σ is the signature of a parallelisable manifold which Σ bounds.

(2) A given integer σ occurs as $\sigma(M)$ for some s-parallelisable M bounded by a homotopy sphere $\iff$ $\sigma \equiv 0$ mod 8.

It is clear from this that if we can produce a homotopy sphere of dimension $4k-1$ bounding a parallelisable manifold of signature $8n$ for all integers $k > 1$, $n \geq 1$, then this collection will contain a set of representaives of each of the diffeomorphism classes.

Given n, k as above, consider the matrix of signature $8n$ guaranteed by Proposition 1.4. Plumb $2k-$disk bundles over $2k-$spheres according to this matrix. The resulting manifold has signature $8n$ and is parallelisable by Lemma 1.2. Moreover, Proposition 1.1 shows the $(4k-1)-$dimensional boundary is a homotopy sphere, as required.

§2.

Our main results are as follows:

Theorem 2.1. *Homotopy spheres which bound parallelisable manifolds admit metrics of positive Ricci curvature.*

Theorem 2.2. *The boundary of any manifold obtained by plumbing $n-$disk bundles over $n-$spheres ($n \geq 3$) according to a simply connected graph admits a metric of positive Ricci curvature.*

Theorem 2.3. *The boundary of a manifold obtained by plumbing together two disk bundles over spheres admits a Ricci positive metric provided the fibre disks and base spheres have dimension ≥ 3.*

Notice in this last theorem that we are not insisting the base and fibres have the same dimension. This is interesting for two reasons. Firstly, we can deduce that the boundary of the manifold obtained by plumbing the 5–dimensional vector bundle over S^4 generating $KO(S^4)$ with the non-trivial 4–dimensional vector bundle over S^5, admits a metric of positive Ricci curvature. This manifold is a homotopy sphere, but does not bound a parallelisable manifold. Thus the converse to Theorem 2.1 is not true.

Secondly, it can be shown that the same homotopy sphere has non-trivial 'improved Witten genus' (see [7] for definitions and details). This shows that a conjecture of Stolz and Höhn claiming the vanishing of the Witten genus for a Ricci positive spin manifold with $\frac{1}{2}p_1 M = 0$ cannot be refined by replacing the Witten genus by the 'improved Witten genus'.

We will prove the above theorems in reverse order. First though, we need to discuss the boundaries of plumbed manifolds in some detail.

Suppose W_1^{n+m+1} is an $(m+1)$–disk bundle over an n–sphere, with boundary M_1. Let W_2^{n+m+1} be an n–disk bundle over an $(m+1)$–sphere, with boundary M_2. Consider the boundary of a plumbing between W_1 and W_2. We can describe this in terms of the sphere bundles M_1 and M_2 as follows.

Choose embedded disks d_{M_1} and d_{M_2} of maximal dimension in the base spheres of M_1 and M_2 respectively. There exists a unique trivialisation ϕ_i of M_i over d_{M_i}, $i = 1, 2$. Restricted to the boundaries, these give trivialisations $\dot{\phi}_1$ and $\dot{\phi}_2$ thus:

$$\dot{\phi}_1 : M_1|_{\partial d_{M_1}} \longrightarrow S^{n-1} \times S^m,$$

$$\dot{\phi}_2 : M_2|_{\partial d_{M_2}} \longrightarrow S^m \times S^{n-1}.$$

Now remove the interior of d_{M_1} and d_{M_2}, and the portion of the respective sphere bundles lying above. We are then left with two sphere bundles over disks, each equipped with a boundary trivialisation. Denote these bundles α_1 and α_2. To complete the 'boundary plumbing' simply identify the two boundaries using the trivialisations in the canonical way. Symbolically this can be represented

$$\partial(W_1 \triangle W_2) = (\alpha_1 \amalg \alpha_2)/\sim.$$

Notice that we could also describe the manifold $(\alpha_1 \amalg \alpha_2)/\sim$ as being the result of performing an appropriate surgery on M_1. Equally, by reversing the roles of M_1 and M_2 we could regard the construction as a surgery on M_2.

Let us assign a metric to M_1 using the Vilms method (see [1, Theorem 9.59]). For this we need to specify a metric on the base, a metric on an abstract fibre which is invariant under the action of the structural group, and a connection on the associated principal bundle.

We choose round metrics of radii N and ρ on the base and fibre spheres respectively. We choose a principal connection in the following way. The trivialisation ϕ_1 is a diffeomorphism onto a product, namely $d_{M_1} \times S^m$. Regarding this as a trivial bundle over d_{M_1}, we note that there is a canonical principal connection on the associated principal bundle $d_{M_1} \times SO(m+1)$. Pulling this back via ϕ_1, we obtain a flat principal connection for that portion of the $SO(m+1)$ bundle associated to M_1 which lies above d_{M_1}. Now extend this connection over the whole principal bundle. (In general, of course, it will not be possible to do this, so the connection is globally flat.)

With these pieces of data, the Vilms construction gives us a submension metric on M_1 with totally geodesic fibres, which is isometric to a product over d_{M_1}. Without loss of generality, we will assume from now on that d_{M_1} is a geodesic disk of radius R.

It is well known (see for example [1, Proposition 9.70]) that if the number ρ is chosen small enough, the resulting metric on M_1 will have positive Ricci curvature. We will suppose this is the case. Of course, this metric will restrict to give a positive Ricci curvature metric on α_1.

Lemma 2.4. *Assume $n \geq m+1 \geq 3$. There exists κ depending on n, m, W_1 and the ratio $\frac{R}{N}$, such that if $\frac{\rho}{N} < \kappa$ then we can choose a Riemannian metric on α_2 which gels smoothly with the metric on α_1 to give a Ricci positive metric on $(\alpha_1 \amalg \alpha_2)/\sim$.*

Proof. This becomes a trivial consequence of [8, Theorem 2.1] after observing that W_1 determines the pair $(\alpha_1, \dot{\phi}_1)$ in an essentially unique way.

Lemma 2.5. *Given $\kappa' > 0$, there exists $\kappa > 0$ such that if $\frac{\rho}{N} < \kappa$ we can arrange for the metric on α_2 described in Lemma 2.4 to satisfy the following additional conditions:*

(1) Above a small disc of radius R' in the interior of the base of α_2, we can choose the metric so we have an isometry with $D^n_{R'}(N') \times S^m(\rho')$ for some ρ' and N'. Here $D^n_{R'}(N')$ is a geodesic disc of radius R' in a round $n-$sphere of radius N', and $S^m(\rho')$ is a round $m-$sphere of radius ρ'.

(2) $\frac{\rho'}{N'} < \kappa'$.

(3) $\frac{R'}{N'} = \Delta$ *where* Δ *is a fixed constant depending on* n *and* m *only.*

Proof. This is just Proposition 2.2 of [8].

Proof of Theorem 2.3. Consider a plumbing between the bundles W_1^{n+m+1} and W_2^{n+m+1}. Recall that the base of W_1 is an $n-$sphere, and

$$\partial(W_1 \triangle W_2) = (\alpha_1 \amalg \alpha_2)/\sim .$$

Lemma 2.4 asserts that this manifold admits a positive Ricci curvature metric provided $n \geq m+1 \geq 3$ and $\frac{\rho}{N} < \kappa$, some $\kappa < 0$. Assume for the moment that $n \geq m+1$, (we have that n and $m+1$ are ≥ 3 by hypothesis). The statement of Lemma 2.4 implies that ρ and κ are independent. Therefore we can certainly arrange to have $\frac{\rho}{N} < \kappa$ by simply choosing ρ smaller if necessary.

Suppose now that $n < m+1$, so Lemma 2.4 will not work. There is nothing to prevent us reversing the roles of W_1 and W_2 (and therefore α_1 and α_2) in this lemma and its preceeding discussion. We would then have that $\partial(W_1 \triangle W_2)$ can be equipped with a Ricci positive metric provided $m+1 \geq n \geq 3$ and $\frac{\rho}{N} < \kappa_1$, some $\kappa_1 > 0$. The dimensional condition is now satisfied by assumption, and we can arrange for the second condition to hold in the same manner as before.

Remark. In [9], the author generalises this theorem to cover plumbings with straight line graphs of any (finite) length.

Proof of Theorem 2.2. The fact that we can restrict our attention here to $n-$disk bundles over $n-$spheres ($n \geq 3$) means the dimensional requirement of Lemma 2.4 is always satisfied.

Suppose we define a Ricci positive Vilms metric on the boundary of a disk bundle W_1 and use Lemma 2.4 to obtain a Ricci positive metric on the boundary of some plumbing $W_1 \triangle W_2$ as described earlier. We consider three extensions to this construction.

(i) Suppose we wish to plumb a further bundle to W_1. Call this bundle W_3. By rechoosing the principal connection if necessary, arrange for the metric on ∂W_1 in a neighbourhood of the proposed plumbing to be isometric with a product in the usual way. To invoke Lemma 2.4 we will need some condition $\frac{\rho}{N} < \kappa_2$ to be satisfied. But this can be arranged by choosing a smaller value for ρ. It is clear that then Lemma 2.4 can be successfully applied to both plumbings to guarantee a Ricci positive metric on

$\partial(W_3 \triangle W_1 \triangle W_2)$. Of course, this generalises to any number of further plumbings on W_1.

(ii) Suppose we wish to plumb a second bunde to W_2. Again, we will refer to this bundle as W_3. To see that $\partial(W_1 \triangle W_2 \triangle W_3)$ admits a Ricci positive metric we need Lemma 2.5. Assertion (1) of this lemma says we can arrange our Ricci positive metric on $\partial(W_1 \triangle W_2)$ to take a form on some neighbourhood inside ∂W_2 which permits further applications of Lemma 2.4.

However, if we aim to apply Lemma 2.4, some condition of the form $\frac{\rho'}{N'} < \kappa'$ will need to be satisfied. Now by Lemma 2.5 (3), κ' only depends on W_3, as all other parameters are fixed.

Given a value for κ', we can find a corresponding value for κ which is possibly smaller than that used for the first plumbing. If so, we may need to rechoose ρ to ensure that the condition $\frac{\rho}{N} < \kappa$ is still satisfied.

We then perform the plumbing inside the region sitting over the disk of radius R' guaranteed by Lemma 2.5 (1). Assertion (2) of this lemma then states that our condition $\frac{\rho'}{N'} < \kappa'$ is satisfied, so by Lemma 2.4 we can conclude the existence of a Ricci positive metric on

$$\partial(W_1 \triangle W_2 \triangle W_3).$$

Should we wish to plumb a further bundle, W_4, to W_3, the same approach will work. We obtain a value κ'' for the third plumbing (depending on W_4), which gives a value for κ' and in turn a value for κ, and this imposes an upper bound on ρ.

(iii) Suppose once more we begin with $W_1 \triangle W_2$, but this time want to plumb both W_3 *and* W_4 to W_2. To apply Lemma 2.4 to each of these plumbings we will need to satisfy conditions involving constants κ'_3 and κ'_4. Set $\kappa' = min(\kappa'_3, \kappa'_4)$. We ensure $\frac{\rho'}{N'} < \kappa'$ as described in (ii), and perform both plumbings where the metric on ∂W_2 is 'nice' in the sense of Lemma 2.5 (1). We can then invoke Lemma 2.4 to guarantee the existence of a Ricci positive metric on the boundary of the resulting manifold.

Having presented these three special cases, it should now be clear how to inductively build a Ricci positive metric on the boundary of any plumbing with a simply-connected graph.

Proof of Theorem 2.1. In Proposition 1.5 we see that any homotopy sphere bounding a parallelisable manifold can be expressed as the boundary of a manifold plumbed according to a simply connected graph. The result follows by Theorem 2.2.

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