

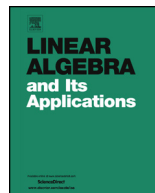


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Diagonal Riccati stability and applications



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ABSTRACT

We consider the question of diagonal Riccati stability for a pair of real matrices A, B . A necessary and sufficient condition for diagonal Riccati stability is derived and applications of this to two distinct cases are presented. We also describe some motivations for this question arising in the theory of generalised Lotka–Volterra systems.

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1. Introduction and preliminaries

We consider the following problem. Given $A, B \in \mathbb{R}^{n \times n}$, determine conditions for the existence of *diagonal* positive definite matrices P, Q in $\mathbb{R}^{n \times n}$ such that

$$A^T P + PA + Q + PBQ^{-1}B^T P \prec 0, \quad (1)$$

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where $M \prec 0$ denotes that $M = M^T$ is negative definite. Throughout the paper, $M \succ 0$ ($M \succeq 0$) denotes that M is positive definite (positive semi-definite); $M \preceq 0$ denotes that M is negative semi-definite.

When diagonal positive definite solutions P, Q of (1) exist, we say that the pair A, B is *diagonally Riccati stable*.

Our interest in the question stems from the stability of time-delay systems. Specifically, when such a pair P, Q exists, the linear time-delay system associated with A, B admits a Lyapunov–Krasovskii functional of a particularly simple form [8]. The more general question of when positive definite solutions of (1), not necessarily diagonal, exist was highlighted in [17] and some preliminary results on this question were also described in this reference.

We now introduce some notation and terminology, and recall some basic results that will be needed later in the paper.

A matrix A in $\mathbb{R}^{n \times n}$ is *Metzler* if its off-diagonal elements are nonnegative: formally, $a_{ij} \geq 0$ for $i \neq j$. As is standard, a *nonnegative* matrix A is one satisfying $a_{ij} \geq 0$ for $1 \leq i, j \leq n$. For vectors v, w in \mathbb{R}^n , $v \geq w$ is understood componentwise and means that $v_i \geq w_i$ for $1 \leq i \leq n$. Similarly, $v > w$ means $v \geq w$, $v \neq w$ and $v \gg w$ means $v_i > w_i$ for $1 \leq i \leq n$. Similar notation is used for matrices of compatible dimensions. In particular, $A \geq 0$, ($A \leq 0$) denotes that A is entrywise nonnegative (nonpositive).

For a positive integer n , we denote by $Sym(n, \mathbb{R})$ the space of $n \times n$ real symmetric matrices. For $A \in \mathbb{R}^{n \times n}$, A^T denotes the transpose of A . A matrix $A \in \mathbb{R}^{n \times n}$ is *Hurwitz* if all of its eigenvalues have negative real parts. It is classical that A is Hurwitz if and only if there exists some $P \succ 0$ with $A^T P + P A \prec 0$.

The following result recalls some well-known facts concerning Metzler matrices [11].

Proposition 1.1. *Let $A \in \mathbb{R}^{n \times n}$ be Metzler. The following are equivalent:*

- (i) A is Hurwitz;
- (ii) there exists some vector $v \gg 0$ in \mathbb{R}^n with $Av \ll 0$;
- (iii) there exists some positive definite diagonal matrix $D \in \mathbb{R}^{n \times n}$ with $A^T D + D A \prec 0$;
- (iv) for every non-zero $v \in \mathbb{R}^n$, there is some index i with $v_i(Av)_i < 0$;
- (v) $A^{-1} \leq 0$.

The next result concerning the Riccati equation is based on the Schur complement and follows from Theorem 7.7.6 in [10].

Lemma 1.1. *Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n}$ be given. Then the matrices $P \succ 0$, $Q \succ 0$ in $\mathbb{R}^{n \times n}$ satisfy (1) if and only if:*

$$S := \begin{pmatrix} A^T P + P A + Q & P B \\ B^T P & -Q \end{pmatrix} \prec 0. \tag{2}$$

Theorem 1.1. (See [4].) Let C_1, C_2 be non-empty convex subsets of a Euclidean Space E with inner product $\langle \cdot, \cdot \rangle$. Further, assume that C_2 is a cone and $C_1 \cap C_2$ is empty. Then there exists a non-zero vector $v \in E$ such that

$$\begin{aligned} \langle v, x \rangle &\geq 0 \text{ for all } x \in C_1, \\ \langle v, x \rangle &\leq 0 \text{ for all } x \in C_2. \end{aligned}$$

In this paper, we apply the above result to convex sets in the Euclidean space $Sym(m, \mathbb{R})$ of $m \times m$ real symmetric matrices equipped with the inner product

$$\langle A, B \rangle = \text{trace}(AB).$$

2. Diagonal solutions to the Riccati equation

In a recent paper [13], it was shown that if A is Metzler, B is nonnegative and $A + B$ is Hurwitz, then there exist $P \succ 0$ diagonal and $Q \succ 0$ satisfying (1) (Q need not be diagonal). Later, we shall show that it is in fact possible to choose Q to be diagonal in this case.

We first present necessary and sufficient conditions for the pair A, B to be diagonally Riccati stable. This result echoes the condition for the existence of diagonal solutions to the Lyapunov equation given in [3]: see also [16,5,12] for more recent work on diagonal solutions of the Lyapunov equation.

Theorem 2.1. Let $A, B \in \mathbb{R}^{n \times n}$ be given. The following are equivalent.

- (i) There exist $P \succ 0, Q \succ 0$ diagonal satisfying (1).
- (ii) For every non-zero positive semi-definite

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{pmatrix} \tag{3}$$

in $Sym(2n, \mathbb{R})$ with $\text{diag}(H_{11}) \geq \text{diag}(H_{22})$, the matrix

$$AH_{11} + BH_{12}^T$$

has a negative diagonal entry.

Proof. (i) \Rightarrow (ii): If there exist diagonal matrices $P \succ 0, Q \succ 0$ satisfying (1), it is immediate from Lemma 1.1 that the matrix

$$R := \begin{pmatrix} A^T P + PA + Q & PB \\ B^T P & -Q \end{pmatrix} \prec 0. \tag{4}$$

Let a non-zero $H \in \text{Sym}(2n, \mathbb{R})$ as in (3) be given satisfying $\text{diag}(H_{11}) \geq \text{diag}(H_{22})$. As R is negative definite and H is non-zero, it follows that

$$\text{trace}(HR) < 0. \tag{5}$$

Expanding the left hand side of (5) we see that

$$\text{trace}(H_{11}A^T P + H_{11}PA + H_{11}Q + H_{12}B^T P + H_{12}^T PB - H_{22}Q) < 0 \tag{6}$$

which implies that

$$\text{trace}(2PAH_{11} + 2PBH_{12}^T + Q(H_{11} - H_{22})) < 0.$$

As $\text{diag}(H_{11}) \geq \text{diag}(H_{22})$ and $Q \succ 0$ is diagonal, it follows that $\text{trace}(Q(H_{11} - H_{22})) \geq 0$ and thus we can conclude that

$$\text{trace}(P(AH_{11} + BH_{12}^T)) < 0.$$

As $P \succ 0$ is diagonal, this implies that the matrix

$$AH_{11} + BH_{12}^T$$

has at least one negative diagonal entry.

(ii) \Rightarrow (i): We shall prove the contrapositive. Suppose that there exist no positive definite diagonal matrices P, Q satisfying (1). Consider the sets

$$\mathcal{C}_1 := \left\{ \left(\begin{array}{cc} A^T P + PA + Q & PB \\ B^T P & -Q \end{array} \right) \mid Q, P \succ 0 \text{ diagonal} \in \mathbb{R}^{n \times n} \right\}, \tag{7}$$

and

$$\mathcal{C}_2 := \{H \in \text{Sym}(2n, \mathbb{R}) \mid H \prec 0\}.$$

Then it is easy to see that \mathcal{C}_1 and \mathcal{C}_2 are convex cones and by assumption we have that

$$\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset.$$

It follows from Theorem 1.1 that there exists a non-zero matrix H in $\text{Sym}(2n, \mathbb{R})$ such that

$$\begin{aligned} \text{trace}(HX) &\leq 0 \text{ for all } X \in \mathcal{C}_2 \\ \text{trace}(HX) &\geq 0 \text{ for all } X \in \mathcal{C}_1. \end{aligned}$$

The first of the above two inequalities implies that $H \succeq 0$. The second implies that

$$\text{trace}(H_{11}A^T P + H_{11}PA + H_{12}B^T P + H_{12}^T PB + (H_{11} - H_{22})Q) \geq 0 \tag{8}$$

for all choices of diagonal matrices $P \succ 0, Q \succ 0$. Rearranging, we see that for all $P \succ 0, Q \succ 0$ diagonal

$$\text{trace}(2P(AH_{11} + BH_{12}^T)) \geq \text{trace}(Q(H_{22} - H_{11})). \quad (9)$$

If there exists a diagonal $Q \succ 0$ such that $\text{trace}(Q(H_{22} - H_{11})) > 0$, then for a given diagonal $P \succ 0$, we could obtain a contradiction to (9) by considering tQ for $t > 0$ sufficiently large. This implies that $\text{trace}(Q(H_{22} - H_{11})) \leq 0$ for all diagonal $Q \succ 0$. It follows from this that $\text{diag}(H_{22}) \leq \text{diag}(H_{11})$. Moreover, for every diagonal $P \succ 0$, we must have

$$\text{trace}(P(AH_{11} + BH_{12}^T)) \geq 0.$$

Now suppose that $AH_{11} + BH_{12}^T$ has a negative diagonal entry: say in the i th position. Then we could choose $P \succ 0$ diagonal such that $\text{trace}(P(AH_{11} + BH_{12}^T)) < 0$ by setting the i th entry of P to be large enough and keeping the other entries of P small and positive. It follows that under the assumption that the pair A, B is not diagonally Riccati stable, that all diagonal entries of $AH_{11} + BH_{12}^T$ are nonnegative. Hence, if $AH_{11} + BH_{12}^T$ has a negative diagonal entry for every H satisfying the conditions of the theorem, there must exist diagonal $P \succ 0, Q \succ 0$ satisfying (1). This completes the proof. \square

A simple application of Schur complements shows that if the pair A, B is diagonally Riccati stable and P, Q satisfy (1), then

$$\begin{aligned} A^T P + P A &< 0 \\ (A + B)^T P + P(A + B) &< 0. \end{aligned}$$

In the paper [13], it was asked whether or not the converse of this statement held. It is possible to use Theorem 2.1 to answer this and show that it is not the case.

Example 2.1. Consider

$$A = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix}.$$

It is easy to verify that

$$D = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies $A^T D + D A < 0, (A + B)^T D + D(A + B) < 0$. However, if we consider H of the form (3) with

$$H_{11} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, H_{12} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, H_{22} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

then $H \succeq 0$ and $\text{diag}(H_{22}) \leq \text{diag}(H_{11})$. However, it can be verified by direct computation that $AH_{11} + BH_{12}^T$ has no negative diagonal entries. Hence, by [Theorem 2.1](#), there do not exist diagonal $P \succ 0, Q \succ 0$ satisfying (1).

3. Applications

We now present a number of applications of the general result [Theorem 2.1](#); first considering the case when A is Metzler, and B is nonnegative. This situation was previously considered in [13] and our next result strengthens the result given in [Theorem 3.1](#) of [13] by showing that the matrix $Q \succ 0$ can be taken to be diagonal.

Theorem 3.1. *Let $A \in \mathbb{R}^{n \times n}$ be Metzler and $B \in \mathbb{R}^{n \times n}$ be nonnegative. There exist diagonal matrices $P \succ 0, Q \succ 0$ satisfying (1) if and only if $A + B$ is Hurwitz.*

Proof. We know that if such matrices P, Q exist, then there exists a diagonal $D \succ 0$ such that $(A + B)^T D + D(A + B) \prec 0$ and, in particular, that $A + B$ is Hurwitz.

Conversely, suppose that $A + B$ is Hurwitz. Let a non-zero

$$H = \begin{pmatrix} U & V \\ V^T & W \end{pmatrix} \tag{10}$$

in $\text{Sym}(2n, \mathbb{R})$ be given with $\text{diag}(W) \leq \text{diag}(U)$. We wish to show that the matrix $AU + BV^T$ has a negative diagonal entry. To this end, note that as $H \succeq 0$, it follows that $U \succeq 0$ and hence that

$$|u_{ji}| \leq \sqrt{u_{ii}}\sqrt{u_{jj}}$$

for $1 \leq i, j \leq n$ and of course $u_{ii} \geq 0$ for $1 \leq i \leq n$. Furthermore, as $H \succeq 0$, we also have that

$$|v_{ij}| \leq \sqrt{u_{ii}}\sqrt{w_{jj}} \leq \sqrt{u_{ii}}\sqrt{u_{jj}}.$$

Writing g for the vector in \mathbb{R}^n whose i th component is given by $\sqrt{u_{ii}}$, $g \neq 0$ as otherwise $H = 0$. As A is Metzler and B is nonnegative, we have

$$\begin{aligned} (AU + BV^T)_{ii} &= \sum_{j=1}^n a_{ij}u_{ji} + \sum_{j=1}^n b_{ij}v_{ij} \\ &\leq \sum_{j=1}^n a_{ij}g_jg_i + \sum_{j=1}^n b_{ij}g_jg_i \\ &= g_i[(A + B)g]_i. \end{aligned}$$

As $A + B$ is Metzler and Hurwitz and $g \neq 0$, it follows from Proposition 1.1 that there exists some index i such that $g_i[(A + B)g]_i < 0$ and hence that $(AU + BV^T)_{ii}$. It now follows from Theorem 2.1 that there exist diagonal matrices $P \succ 0, Q \succ 0$ satisfying (1) as claimed. \square

Our next result gives conditions under which lower triangular pairs A, B are diagonally Riccati stable.

Theorem 3.2. *Let $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n}$ be lower triangular matrices. Then there exist diagonal matrices $P \succ 0, Q \succ 0$ satisfying (1) if and only if A is Hurwitz and $|b_{ii}| < |a_{ii}|$ for $1 \leq i \leq n$.*

Proof. If there exist diagonal matrices $P \succ 0, Q \succ 0$ satisfying (1) then $A^T D + DA \prec 0$ and hence A is Hurwitz. To show that $|b_{ii}| < |a_{ii}|$ for all i , we argue by contradiction. Assume that there is some i with $|b_{ii}| \geq |a_{ii}|$. First assume that $b_{ii} \geq 0$ (the case $b_{ii} \leq 0$ can be handled identically). Consider the matrix $H \in \text{Sym}(2n, \mathbb{R})$ of the form (10) where

$$U = V = W = e_i e_i^T.$$

So U and V are both the diagonal matrix with all entries zero apart from the i th diagonal entry. It is then easy to see that $H \succeq 0$ and that $\text{diag}(W) \leq \text{diag}(U)$. The only non-zero diagonal entry in $AU + BV^T$ is

$$\begin{aligned} (AU + BV^T)_{ii} &= a_{ii}u_{ii} + b_{ii}v_{ii} \\ &= a_{ii} + b_{ii} \\ &\geq 0 \end{aligned}$$

as $b_{ii} \geq 0$ and $|b_{ii}| \geq |a_{ii}|$. It follows from Theorem 2.1 that there cannot exist diagonal matrices $P \succ 0, Q \succ 0$ satisfying (1) which is a contradiction. For the case where $b_{ii} \leq 0$, choose H with $U = W = e_i e_i^T, V = -e_i e_i^T$.

For the converse, assume A is Hurwitz and that $|b_{ii}| < |a_{ii}|$ for $1 \leq i \leq n$. Consider $H \succeq 0$ in $\text{Sym}(2n, \mathbb{R})$ given by (10) with $\text{diag}(W) \leq \text{diag}(U)$. As H is non-zero, let i be the lowest index such that $u_{ii} > 0$. As U is positive semi-definite, it follows that $u_{ji} = 0$ for all $j < i$. Moreover, from our assumptions on H , we must have

$$\begin{aligned} |v_{ij}| &\leq \sqrt{u_{ii}}\sqrt{w_{jj}} \\ &\leq \sqrt{u_{ii}}\sqrt{u_{jj}} \\ &= 0 \end{aligned}$$

for all $j < i$. Moreover, we also have $|v_{ii}| \leq |u_{ii}|$. Using these observations and the fact that A, B are lower triangular, we see that

$$\begin{aligned}
 (AU + BV^T)_{ii} &= \sum_{j=1}^n a_{ij}u_{ji} + \sum_{j=1}^n b_{ij}v_{ij} \\
 &= \sum_{j \leq i} a_{ij}u_{ji} + \sum_{j \leq i} b_{ij}v_{ij} \\
 &= a_{ii}u_{ii} + b_{ii}v_{ii}.
 \end{aligned}$$

The result now follows from the simple observation that $a_{ii}u_{ii} < 0$ and $|b_{ii}v_{ii}| < |a_{ii}u_{ii}|$. \square

An identical argument where we choose the index i to be equal to the largest index for which $u_{ii} > 0$ will show that a corresponding result also hold for upper triangular matrices.

Comment. Note that [Example 2.1](#) shows that even when both A and B are lower triangular and A and $A + B$ are both Hurwitz, it is possible that the pair A, B is not diagonally Riccati stable.

4. Metzler and nonnegative matrices – a closer look

In the previous section, we showed that if A is Metzler, B nonnegative and $A + B$ Hurwitz, there exist diagonal matrices $P \succ 0, Q \succ 0$ satisfying [\(1\)](#). We have also noted that for any such pair P, Q , the matrix P must be a simultaneous diagonal solution of the Lyapunov inequalities associated with $A, A + B$. We next show that for any diagonal $P \succ 0$ satisfying $A^T P + P A \prec 0, (A + B)^T P + P(A + B) \prec 0$, there must exist a *diagonal* $Q \succ 0$ such that P, Q satisfy [\(1\)](#). Note that in this case, as B is nonnegative, from $(A + B)^T P + P(A + B) \prec 0$ it follows that $A^T P + P A \prec 0$.

This provides more insight into the pairs of diagonal matrices $P \succ 0, Q \succ 0$ satisfying [\(1\)](#) and, moreover, the method of proof gives us a constructive approach to finding P and Q .

Theorem 4.1. *Let $A \in \mathbb{R}^{n \times n}$ be Metzler and $B \in \mathbb{R}^{n \times n}$ be nonnegative and assume that $A + B$ is Hurwitz. Let $P \succ 0$ be diagonal and such that*

$$(A + B)^T P + P(A + B) \prec 0. \tag{11}$$

Then there exists $Q \succ 0$, diagonal such that P, Q satisfy [\(1\)](#).

Proof. As $P \succ 0$ is diagonal it is straightforward to verify that $(A + B)^T P + P(A + B)$ is again Metzler and, moreover as it is negative definite, it is Hurwitz. From [Proposition 1.1](#), there exist vectors $v \gg 0, w \gg 0$ such that

$$((A + B)^T P + P(A + B))v = -w. \tag{12}$$

Now choose a diagonal $Q \succ 0$ such that

$$Qv = B^T P v + \frac{1}{2}w. \tag{13}$$

A simple calculation now shows that

$$\begin{pmatrix} A^T P + PA + Q & PB \\ B^T P & -Q \end{pmatrix} \begin{pmatrix} v \\ v \end{pmatrix} = \begin{pmatrix} -w/2 \\ -w/2 \end{pmatrix}. \tag{14}$$

Noting that

$$M = \begin{pmatrix} A^T P + PA + Q & PB \\ B^T P & -Q \end{pmatrix}$$

is Metzler, it follows from (14) that M is Hurwitz. However as M is symmetric this implies that $M \prec 0$ and hence that P, Q satisfy (1). This completes the proof. \square

5. Generalised Lotka–Volterra systems

In this section, we highlight how the results described above can be applied to a generalised class of Lotka–Volterra models occurring in population dynamics. Specifically, we consider the following generalised Lotka–Volterra model of population dynamics in an n species community:

$$\dot{x}_i(t) = g_i(x_i(t)) \left(c_i + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} f_j(x_j(t - \tau)) \right), \tag{15}$$

for $1 \leq i \leq n$.

For systems without time-delay, similar models were considered in [14], while more recently a switched system version without time-delay was studied in [2,1]. Systems of the form (15) with time delay and with some special forms of functions f_i were studied, for example, in [7,6].

Here $x_i(t)$ is the population density of the i -th species; the functions $g_i : [0, +\infty) \rightarrow [0, +\infty)$ and $f_i : [0, +\infty) \rightarrow [0, +\infty)$ possess special properties described below and allow for a generalisation of the classical Lotka–Volterra model; c_i, a_{ij}, b_{ij} are constant coefficients. The coefficients c_i characterise the intrinsic growth rate of the i -th population; the self-interaction terms $p_{ii}g_i(x_i)f_i(x_i)$ with $p_{ii} < 0$ reflect the limited resources available in the environment; the terms $p_{ij}g_i(x_i)f_j(x_j)$ for $j \neq i$ describe the influence of population j on population i .

Let \mathbb{R}_+^n be the nonnegative cone of \mathbb{R}^n :

$$\mathbb{R}_+^n := \{x \in \mathbb{R}^n \mid x \geq 0\}.$$

Also, $\text{int } \mathbb{R}_+^n$ is the interior of \mathbb{R}_+^n .

We make use of some standard results and concepts from the Lyapunov stability theory of Functional Differential Equations [8]. In particular, for a given real number $\tau > 0$, $C([-\tau, 0], \text{int } \mathbb{R}_+^n)$ denotes the space of continuous functions $\phi : [-\tau, 0] \rightarrow \text{int } \mathbb{R}_+^n$ with the uniform norm $\|\phi\|_\tau = \max_{\theta \in [-\tau, 0]} \|\phi(\theta)\|$, and $\|\cdot\|$ denotes the Euclidean norm of a vector. We assume that initial functions for (15) belong to the space $C([-\tau, 0], \text{int } \mathbb{R}_+^n)$. For a solution $x(t)$ of (15), we denote by x_t the element in $C([-\tau, 0], \text{int } \mathbb{R}_+^n)$ given by $x_t(u) = x(t+u)$ for $u \in [-\tau, 0]$. For ϕ in $C([-\tau, 0], \text{int } \mathbb{R}_+^n)$, ϕ_j denotes its j th component.

We consider functions f_i and g_i satisfying the following properties, which are consistent with the standard assumptions made in [14,2] and elsewhere. Moreover, these assumptions are clearly satisfied by the classical Lotka–Volterra model [9].

- (i) g_i and f_i are continuous on $[0, +\infty)$.
- (ii) For each initial condition in $\text{int } \mathbb{R}_+^n$, there exists a unique, globally defined solution to (15).
- (iii) $g_i(0) = f_i(0) = 0$, $g_i(x_i) > 0$, $f_i(x_i) > 0$ for $x_i > 0$.
- (iv) f_i is a strictly increasing function on $[0, +\infty)$, and $f_i(x_i) \rightarrow +\infty$ as $x_i \rightarrow +\infty$.
- (v) $\int_1^{+\infty} f_i(\zeta)/g_i(\zeta) d\zeta = +\infty$;
- (vi) $\int_0^1 1/g_i(\zeta) d\zeta = +\infty$.

It is possible to state technical Lipschitz-type conditions on the functions f_i and g_i that are sufficient for the global existence and uniqueness of solutions. However, such conditions are typically conservative, and in view of the biological application motivating the work, it seems more natural to simply assume the existence of solutions as stated in condition (ii) above.

The assumptions on the system imply that $\text{int } \mathbb{R}_+^n$ is an invariant set for (15). For biological reasons, we will consider (15) with respect to the state space $\text{int } \mathbb{R}_+^n$. The following result demonstrates the relevance of diagonal Riccati stability to Lotka–Volterra systems and provides a practical motivation for the question considered in this paper.

Throughout the remainder of this section, A and B denote the matrices $A = \{a_{ij}\}_{i,j=1}^n$, $B = \{b_{ij}\}_{i,j=1}^n$. We recall here that a class \mathcal{K} function α [15] is a continuous function from $[0, \infty)$ to $[0, \infty)$ which is strictly increasing and satisfies $\alpha(0) = 0$.

Theorem 5.1. *Consider the system described by (15). Assume that:*

- 1. *there exists an interior equilibrium point \bar{x} ;*
- 2. *there exist diagonal matrices $P \succ 0$, $Q \succ 0$ satisfying (1).*

Then the equilibrium point \bar{x} is globally asymptotically stable in $\text{int } \mathbb{R}_+^n$ for any value of the delay τ .

Proof. Denote $f(x) = (f_1(x_1), \dots, f_n(x_n))^T$, $c = (c_1, \dots, c_n)^T$.

By assumption, \bar{x} is an equilibrium point in $\text{int } \mathbb{R}_+^n$ so it satisfies:

$$c + (A + B)f(x) = 0.$$

Using this, the system (15) can be rewritten as follows

$$\dot{x}_i(t) = g_i(x_i(t)) \left(\sum_{j=1}^n a_{ij}(f_j(x_j(t)) - f_j(\bar{x}_j)) + \sum_{j=1}^n b_{ij}(f_j(x_j(t - \tau)) - f_j(\bar{x}_j)) \right), \quad i = 1, \dots, n. \tag{16}$$

We show that for the system (16), there exists a Lyapunov–Krasovskii functional of the form

$$V(\phi) = \sum_{i=1}^n p_i \int_{\bar{x}_i}^{\phi(0)} \frac{f_i(\zeta) - f_i(\bar{x}_i)}{g_i(\zeta)} d\zeta + \sum_{j=1}^n \mu_j \int_{-\tau}^0 (f_j(\phi_j(u)) - f_j(\bar{x}_j))^2 du, \tag{17}$$

where p_i and μ_j are positive coefficients.

In fact, we know by assumption that there exist diagonal matrices $P \succ 0, Q \succ 0$ such that

$$M = \begin{pmatrix} A^T P + P A + Q & P B \\ B^T P & -Q \end{pmatrix} \prec -2\beta I$$

for some $\beta > 0$. It follows readily that for $x, y \in \mathbb{R}^n$,

$$x^T P A x + x^T P B y + 1/2(x^T Q x - y^T Q y) < -\beta(x^T x + y^T y). \tag{18}$$

Choosing p_1, \dots, p_n and μ_1, \dots, μ_n so that $P = \text{diag}(p_1, \dots, p_n), Q = \text{diag}(2\mu_1, \dots, 2\mu_n)$, we claim that V given by (17) defines a Lyapunov–Krasovskii functional for the system (16).

First of all, it follows from the assumed properties of the functions f_i and g_i that

$$\chi_1(\|\varphi(0) - \bar{x}\|) \leq V(\varphi) \leq \chi_2(\|\varphi - \bar{x}\|_\tau),$$

where $\chi_1(\zeta)$ and $\chi_2(\zeta)$ are class \mathcal{K} (in the sense of Hahn) functions [15], and $V(\varphi) \rightarrow +\infty$ as $\varphi(0)$ tends to the boundary of $\text{int } \mathbb{R}_+^n$.

Differentiating the functional (17) with respect to the system (16), we obtain

$$\begin{aligned} \dot{V}(x_t) &= \sum_{i=1}^n p_i (f_i(x_i(t)) - f_i(\bar{x}_i)) \sum_{j=1}^n a_{ij} (f_j(x_j(t)) - f_j(\bar{x}_j)) \\ &+ \sum_{i=1}^n p_i (f_i(x_i(t)) - f_i(\bar{x}_i)) \sum_{j=1}^n b_{ij} (f_j(x_j(t - \tau)) - f_j(\bar{x}_j)) \\ &+ \sum_{j=1}^n \mu_j (f_j(x_j(t)) - f_j(\bar{x}_j))^2 - \sum_{j=1}^n \mu_j (f_j(x_j(t - \tau)) - f_j(\bar{x}_j))^2. \end{aligned}$$

Writing this more compactly, we have

$$\begin{aligned} \dot{V}(x_t) &= (f(x(t)) - f(\bar{x}))^T P A (f(x(t)) - f(\bar{x})) \\ &\quad + (f(x(t)) - f(\bar{x}))^T P B (f(x(t-\tau)) - f(\bar{x})) \\ &\quad + \frac{1}{2} (f(x(t)) - f(\bar{x}))^T Q (f(x(t)) - f(\bar{x})) \\ &\quad + \frac{1}{2} (f(x(t-\tau)) - f(\bar{x}))^T Q (f(x(t-\tau)) - f(\bar{x})). \end{aligned}$$

It follows immediately from (18) that

$$\dot{V}(x_t) \leq -\beta \sum_{i=1}^n ((f_i(x_i(t)) - f_i(\bar{x}_i))^2 + (f_i(x_i(t-\tau)) - f_i(\bar{x}_i))^2).$$

This completes the proof. \square

Comments.

1. The previous result shows that if a generalised Lotka–Volterra system has an interior equilibrium point and if there exist diagonal solutions $P \succ 0$, $Q \succ 0$ to (1) then the equilibrium is globally asymptotically stable independent of delay. Moreover, the stability is established using a simple *diagonal* Lyapunov–Krasovskii functional.
2. It should be noted that delay-independent stability conditions such as those given above are in general conservative when compared with delay-dependent conditions. However in many applications, particularly in Biology, it is not possible to know the value of the delay in advance and results giving delay-independent guarantees of stability are of potential use in such a scenario.
3. In the case where condition 1 of the theorem is not satisfied but condition 2 is, it is possible to show that the trajectories of the system are uniformly bounded. A detailed explanation of this is given below the next corollary.
4. If the system has an interior equilibrium point (condition 1), condition 2 will in general provide a conservative condition for delay-independent stability. However, in the situation considered in the following corollary, it can be shown that when A is Metzler, B is nonnegative and $c \gg 0$, condition 2 is a non-conservative condition for delay-independent stability.

We next describe the implication of this result for so-called *mutualistic systems* in which the matrix A is Metzler and B is nonnegative.

Corollary 5.1. *Consider the system (15). Assume that A is Metzler, B is nonnegative and that $c \gg 0$. Furthermore, assume that $A + B$ is Hurwitz. Then there exists an equilibrium point \bar{x} in $\text{int } \mathbb{R}_+^n$ which is globally asymptotically stable for all values of the delay τ .*

Proof. As $A + B$ is Metzler and Hurwitz, it follows from point (v) of [Proposition 1.1](#) that $(A + B)^{-1} \leq 0$. Hence, as $c \gg 0$ and $(A + B)^{-1} \neq 0$ we must have

$$-(A + B)^{-1}c \gg 0.$$

It now follows immediately from property (iv) of the functions f_i , $1 \leq i \leq n$, that there exists a unique $\bar{x} \in \text{int } \mathbb{R}_+^n$ satisfying

$$f(\bar{x}) = -(A + B)^{-1}c,$$

which implies that \bar{x} is an equilibrium of [\(15\)](#). Moreover, as $A+B$ is Metzler and Hurwitz, [Theorem 3.1](#) implies that there exist diagonal matrices $P \succ 0$, $Q \succ 0$ satisfying [\(1\)](#). The result now follows from [Theorem 5.1](#). \square

To finish off this section, we note that the existence of diagonal P , Q satisfying the Riccati inequality also implies uniform boundedness of the generalised system [\(15\)](#) in the case where there is not an interior equilibrium point.

The system [\(15\)](#) is uniformly ultimately bounded with the ultimate bound $R > 0$ if, for any $M > 0$, there exists $T = T(M) > 0$, such that $\|\mathbf{x}(t, \phi, t_0)\| \leq R$ for all $t_0 \geq 0$, $\|\phi\|_\tau < M$, $t \geq t_0 + T$.

Choose an interior point, not necessarily an equilibrium, $\bar{x} \in \text{int } \mathbb{R}_+^n$. Consider the Lyapunov–Krasovskii functional [\(17\)](#).

As in the proof of [Theorem 5.1](#), we obtain that $V(\phi) \rightarrow +\infty$ as $\phi(0)$ tends to the boundary of $\text{int } \mathbb{R}_+^n$.

A relatively straightforward calculation shows that for the derivative of the functional [\(17\)](#) with respect to the system [\(15\)](#) the estimate

$$\begin{aligned} \dot{V}(x_t) \leq & -\beta \sum_{i=1}^n ((f_i(x_i(t)) - f_i(\bar{x}_i))^2 + (f_i(x_i(t - \tau)) - f_i(\bar{x}_i))^2) \\ & + \eta \sum_{i=1}^n |f_i(x_i(t)) - f_i(\bar{x}_i)| \end{aligned} \tag{19}$$

holds, where β and η are positive constants.

This implies that there exists a number $L > 0$ such that $\dot{V}(x_t) < 0$ for $\|x(t)\| > L$, $x(t) \in \mathbb{R}_+^n$, which in turn implies that the system [\(15\)](#) is uniformly ultimately bounded in \mathbb{R}_+^n .

6. Concluding remarks

We have considered the general question of when a pair of matrices A, B is diagonally Riccati stable. Building on the result given in [\[13\]](#), we have derived a characterisation of diagonally Riccati stable pairs that echoes the classical result of Barker, Berman

and Plemmons for the Lyapunov equation. Applications to Metzler and Triangular matrices are presented and the relevance of diagonal Riccati stability to generalised Lotka–Volterra systems in population dynamics has been highlighted.

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